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An example related to boundedness of subharmonic functions

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Dedicated to the memory of Jacek Szarski

Abstract. It is known that if a subharmonic function $u(r, \Theta)$ is dominated by a function of the form $r^\mu f(\Theta)$ with f satisfying a certain condition, then $ur^{-\mu}$ is bounded. It will be shown that r^μ can not be replaced by a more rapidly growing function of r.

Let u be a subharmonic function on $\boldsymbol{R}^m$ ($m \geqslant 2$) and f be a non-negative Borel measurable function defined on S: $|x| = 1$. We denote the area of S by a_m and the area of a Borel set E on S by $|E|$. Set $r = |x|$ and $\Theta = x/|x|$ for $x \neq 0$. H. Yoshida [2] answered question 3.6 of Hayman [1] by proving that, if $u(x) \leqslant r^\mu f(\Theta)$ with some $\mu \geqslant 0$ on $R^m - \{0\}$, then $u(x) \leqslant \text{const}\cdot r^\mu$ for x with large $|x|$, provided f satisfies

$$\int_0^{a_m} t^{-(m-2)/(m-1)} \log^+ s_f(t)\,dt < \infty, \tag{1}$$

where $s_f(t)$ is the inverse function of the function $t = t(s) = |\{x \in S;\ f(x) \geqslant s\}|$. In this note we shall show that we cannot replace $r^\mu = \exp(\mu \log r)$ by $\exp\big((\log r)^a\big)$ for any $a > 1$ in Yoshida's result.

THEOREM. *Given $a > 1$, there exist a positive Borel measurable function f on S and a positive subharmonic function u on $\boldsymbol{R}^m$ such that* (1) *is satisfied, $u(x) \leqslant f(\Theta)\exp\big((\log r)^a\big)$ on $|x| \geqslant 1$ and $u(x_1, 0, \ldots, 0)/\exp\big((\log x_1)^a\big) \to \infty$ as $x_1 \to \infty$.*

Proof. For $x \neq 0$ we denote the angle between the x_1-axis and the vector x by θ. As $f(x)$ on S we take $\big(\pi/(2\theta)\big)^2$ if $\theta \leqslant \pi/2$ and take 1 if $\pi/2 \leqslant \theta \leqslant \pi$. Naturally $f \geqslant 1$. It is easy to check that (1) is satisfied. Set $a = 1+\varepsilon$ and consider

$$u_1(x) = (\log|x|)^\varepsilon \exp\big((\log|x|)^{1+\varepsilon} - \theta^2(\log|x|)^\varepsilon\big)$$

for x with $|x| \geqslant 1$. For simplicity write $g = g(x)$ for $u_1(x)(\log|x|)^{-\varepsilon}$. We have

$$\Delta u_1 = \frac{\partial^2 u_1}{\partial r^2} + (m-1)\frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2} + (m-2)\frac{\cot\theta}{r^2}\frac{\partial u_1}{\partial \theta}.$$

By an elementary computation we see that

$$\frac{r^2}{g}\,\Delta u_1 \geqslant (1+\varepsilon)^2(\log r)^{3\varepsilon}\big(1+o(1)\big)$$

if r is large. Hence there exists $r_0 > 1$ such that $\Delta u_1 > 0$ if $r > r_0$ so that u_1 is subharmonic there. Next we define

$$u_2(x) = (\log|x|)^{\varepsilon}\exp\left((\log|x|)^{1+\varepsilon} - \frac{\pi^2}{4}(\log|x|)^{\varepsilon}\right)$$

on $|x| \geqslant 1$. We can easily observe that $\Delta u_2 > 0$ if $|x|$ is large. We may assume that it is so on $|x| \geqslant r_0$. Set $u_3 = \max(u_1, u_2)$. It is subharmonic on $|x| > r_0$, and equal to u_1 if $\theta \leqslant \pi/2$ and to u_2 if $\pi/2 \leqslant \theta \leqslant \pi$.

Evidently $u_3(x_1, 0, \ldots, 0)/\exp\big((\log x_1)^{\alpha}\big) \to \infty$ as $x_1 \to \infty$. Let us see that

$$u_1(x) \leqslant \theta^{-2}\exp\big((\log|x|)^{\alpha}\big)$$

if $|x| \geqslant 1$. We may assume $\theta > 0$. The inequality $1 - te^{-t} > 0$ valid for any t yields

$$\theta^{-2} - (\log r)^{\varepsilon}\exp\big(-\theta^2(\log r)^{\varepsilon}\big) > 0$$

for any θ and $r \geqslant 1$ so that $u_1(x) \leqslant \theta^{-2}\exp\big((\log|x|)^{\alpha}\big)$ if $x \geqslant 1$. In particular,

$$u_2(x) = u_1(r, \pi/2) \leqslant (\pi/2)^{-2}\exp\big((\log r)^{\alpha}\big)$$

if $|x| \geqslant 1$, where u_1 is regarded as a function of r and θ. Therefore

$$u_3(x) \leqslant f(\Theta)\exp\big((\log r)^{\alpha}\big) \qquad \text{if } |x| \geqslant 1.$$

Finally we modify u_3 so as to obtain u subharmonic on $\boldsymbol{R}^m$. Choose $M > 1$ so that $u_3 < M$ on $1 \leqslant |x| \leqslant 2r_0$. We define u_4 by $\max(u_3, M)$ on $|x| > 1$ and by M on $|x| \leqslant 1$. We see that u_4 is positive subharmonic on $\boldsymbol{R}^m$, and $u_3 \leqslant u_4$ on $|x| > 1$ so that $u_4(x_1, 0, \ldots, 0)/\exp\big((\log x_1)^{\alpha}\big) \to \infty$ as $x_1 \to \infty$. It is easy to check that $u = u_4/M$ satisfies all the required conditions. Our theorem is now proved.

Remark. Let $f(\Theta) = (\pi/(2\theta))^{1/(2p)}$ if $\theta \leqslant \pi/2$ and $= 1$ if $\pi/2 \leqslant \theta \leqslant \pi$ for any $p > 0$. Then

$$\int_0^{a_m} t^{-(m-2)/(m-1)}\big(s_f(t)\big)^p\,dt < \infty;$$

(1) is therefore also satisfied. Consider

$$u_1(x) = (\log|x|)^{\varepsilon/(4p)}\exp\big((\log|x|)^{1+\varepsilon} - \theta^2(\log|x|)^{\varepsilon}\big)$$

for $\varepsilon > 0$ and $|x| \geqslant 1$. By modifying this function as above we obtain a positive subharmonic function u on $\boldsymbol{R}^m$ such that $u(x) \leqslant f(\Theta)\exp\big((\log r)^{1+\varepsilon}\big)$ on $|x| \geqslant 1$ and $u(x_1, 0, \ldots, 0)/\exp\big((\log x_1)^{1+\varepsilon}\big) \to \infty$ as $x_1 \to \infty$.

References

[1] W. K. Hayman, *Research problems in function theory*, The Athlone Press, London 1967.

[2] H. Yoshida, *A boundedness criterion of subharmonic functions*, J. London Math. Soc. (2), 24 (1981), p. 148–160.

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