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C^r-solutions of a system of functional equations

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Abstract. The object of this paper is to study the system of functional equations

(1) $$\varphi_i(x) = h_i(x, \varphi_1[f_{i,1}(x)], \ldots, \varphi_n[f_{i,n}(x)]), \quad i = 1, \ldots, n,$$

where $f_{i,j}$, h_i denote the known functions and φ_i the unknown functions. There are given conditions for the existence and uniqueness of C^r-solutions of the system (1) and also there is proved a theorem on the continuous dependence of C^r-solutions of the system (1).

The purpose of the present paper is to prove some theorems concerning the existence and the continuous dependence of C^r-solutions of the system of functional equations

(1) $$\varphi_i(x) = h_i\big(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n[f_{in}(x)]\big) \quad (i = 1, 2, \ldots, n),$$

where f_{ij} and h_i are given functions and φ_i are unknown functions of one real variable x. This problem was investigated by B. Choczewski [2] and by J. Matkowski [6] in the case $i = 1$.

1. Let I be an interval $\langle 0, a)$, $a > 0$. We denote by $C^r[A]$ the class of functions which have continuous derivatives up to order r in A, $0 < r < \infty$ and by $S^r[I]$ the class of functions $f \in C^r[I]$ which fulfil the condition

$$0 < \frac{f(x)}{x} < 1 \quad \text{for } x \in I, x \neq 0.$$

We assume the following hypotheses:

(I) $$f_{ik} \in S^r[I],$$

(II) $$h_i \in C^r[\Omega], \quad \text{where } \Omega = I \times R^n, \qquad h_i(0, \ldots, 0) = 0$$

for $i, k = 1, 2, \ldots, n$.

We denote by $\Lambda[I]$ the class of functions $\varphi = \{\varphi_i\}$, $i = 1, \ldots, n$, defined in I and such that $\varphi_i(0) = 0$.

Let us define the functions h_{ik} by the recurrent relations:

$$h_{i1}(x, y_1^0, \ldots, y_n^0, y_1^1, \ldots, y_n^1) = \frac{\partial h_i}{\partial x}(x, y_1^0, \ldots, y_n^0) + \sum_{p=1}^{n} \frac{\partial h_i}{\partial y_p^0}(x, y_1^0, \ldots, y_n^0) y_p^1 f'_{ip}(x),$$

(2)

$$h_{ik+1}(x, y_1^0, \ldots, y_n^0, \ldots, y_1^{k+1}, \ldots, y_n^{k+1}) = \frac{\partial h_{ik}}{\partial x}(x, y_1^0, \ldots, y_n^0, \ldots, y_n^k) + \sum_{p=1}^{n}\sum_{l=0}^{k} \frac{\partial h_{ik}}{\partial y_p^l}(x, y_1^0, \ldots, y_n^0, \ldots, y_1^k, \ldots, y_n^k) y_p^{l+1} f'_{ip}(x),$$

$i = 1, 2, \ldots, n$, $k = 1, \ldots, r-1$, where $(x, y_1^0, \ldots, y_n^0, \ldots, y_1^s, \ldots, y_n^s) \in I \times R^{s+1}$.

We have the following two lemmas:

LEMMA 1. *If hypotheses* (I), (II) *are fulfilled, then the functions* h_{ik} *are defined and are of class* C^{r-k} *in* $\Omega \times R^{nk}$. *Moreover, we have for* $i = 1, \ldots, n$, $k = 1, \ldots, r$

(3) $$h_{ik}(x, y_1^0, \ldots, y_n^0, \ldots, y_1^k, \ldots, y_n^k) = g_{ik}(x, y_1^0, \ldots, y_n^0, \ldots, y_1^{k-1}, \ldots, y_n^{k-1}) + \sum_{p=1}^{n} \frac{\partial h_i}{\partial y_p^0}(x, y_1^0, \ldots, y_n^0) y_p^k [f'_{ip}(x)]^k,$$

where $g_{ik} \in C^{r-k}[\Omega \times R^{n(k-1)}]$.

LEMMA 2. *Let hypotheses* (I) *and* (II) *be fulfilled. If* $\varphi \in \Lambda[I]$ *is a* C^r-*solution of the system of equations* (1) *in* I, *then its derivatives* $\varphi_i^{(k)}$ *satisfy the equations*

$$\varphi_i^{(k)}(x) = h_{ik}\big(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n[f_{in}(x)], \ldots, \varphi_1^{(k)}[f_{i1}(x)], \ldots, \varphi_n^{(k)}[f_{in}(x)]\big),$$

$i = 1, \ldots, n$, $k = 1, \ldots, r$.

The proofs of these lemmas are similar to the proofs of analogous lemmas contained in [4].

Remark 1. Lemma 2 implies that if a function $\varphi \in \Lambda[I]$ is a C^r-solution of system (1) in I, then the values

(4) $$\eta_i^l = \varphi_i^{(l)}(0), \qquad i = 1, \ldots, n,\ l = 1, \ldots, r,$$
$$\eta_i^0 = \varphi_i(0) = 0, \qquad i = 1, \ldots, n,$$

fulfil the system of equations

(5) $$\eta_i^l = h_{il}(0, \ldots, 0, \eta_1^1, \ldots, \eta_n^1, \ldots, \eta_1^l, \ldots, \eta_n^l).$$

2. Now we suppose that $\varphi \in \Lambda[I] \cap C^r[I]$ is a solution of system (1). Let us write φ_i in the form

(6) $$\varphi_i(x) = P_i(x) + \gamma_i(x), \qquad i = 1, \ldots, n,$$

where

$$P_i(x) = \sum_{l=1}^{r} \frac{\varphi_i^{(l)}(0)}{l!} x^l \tag{7}$$

and $\gamma_i(x)$ is the rest in the form of Peano.

We define the functions

$$h_i^*(x, y_1^0, \ldots, y_n^0) \stackrel{\text{df}}{=} h_i(x, P_1[f_{i1}(x)] + y_1^0, \ldots, P_n[f_{in}(x)] + y_n^0) - P_i(x). \tag{8}$$

It is easily seen that these function belong to the class $C^r[\Omega]$. It follows from (6) and (8) that γ_i is a C^r-solution of the system

$$\gamma_i(x) = h_i^*(x, \gamma_1[f_{i1}(x)], \ldots, \gamma_n[f_{in}(x)]) \tag{9}$$

such that $\gamma_i(0) = 0$. Moreover, by (6) and (7) we get

$$\gamma_i'(0) = \ldots = \gamma_i^{(r)}(0) = 0. \tag{10}$$

We observe that if $\gamma = \{\gamma_i\}$, $i = 1, \ldots, n$, is a solution of system (9), then $\varphi = \{\varphi_i\}$, $i = 1, \ldots, n$, defined by (6), where

$$P_i(x) = \sum_{l=1}^{r} \frac{\eta_i^l}{l!} x^l$$

is a solution of system (1) and η_i^l are arbitrary fixed numbers. Thus we have the following

LEMMA 3. *The system* (1) *has a solution* $\varphi \in \Lambda[I] \cap C^r[I]$ *iff the system* (9) *has a solution in the class* $C^r[I]$.

Starting from the functions h_i^* instead of h_i we define functions h_{ik}^* and g_{ik}^* analogously as h_{ik} and g_{ik} and we can prove analogues of Lemmas 1 and 2. Now, by (10) and Lemma 2 it is seen that

$$h_{ik}^*(0, \ldots, 0) = 0, \quad k = 1, \ldots, r, \; i = 1, \ldots, n,$$

and by Lemma 1

$$g_{ik}^*(0, \ldots, 0) = 0, \quad k = 1, \ldots, r, \; i = 1, \ldots, n.$$

THEOREM 1. *Let hypotheses* (I) *and* (II) *be fulfilled. Suppose, further, that inequalities*

$$|f_{ik}'(x)| \leqslant 1, \quad i, k = 1, \ldots, n, \tag{11}$$

and

$$\left| \frac{\partial h_i}{\partial y_p^0}(x, y_1^0, \ldots, y_n^0)[f_{ip}'(x)]^r \right| \leqslant \vartheta_{ip}, \qquad \sum_{p=1}^{n} \vartheta_{ip} < 1, \tag{12}$$

$i = 1, \ldots, n$, *hold in a neighbourhood of zero. Then there exists at last one* C^r*-solution* $\varphi \in \Lambda[I]$ *of the system* (1) *in* I *fulfilling conditions* (4).

Proof. Without loss of generality we may assume that

$$h_{ik}(0,\ldots,0)=0 \quad \text{and} \quad g_{ik}(0,\ldots,0)=0, \qquad i=1,\ldots,n,\ k=1,\ldots,r. \tag{13}$$

Hence and by continuity of

$$g_{ir}(x, y_1^0, \ldots, y_n^0, \ldots, y_1^{r-1}, \ldots, y_n^{r-1}) \quad \text{and} \quad \frac{\partial h_i}{\partial y_p^0}(x, y_1^0, \ldots, y_n^0)[f'_{ip}(x)]^r$$

we have

$$\begin{gathered} |g_{ir}(x, y_1^0, \ldots, y_n^{r-1})| \leqslant \Big(1-\sum_{p=1}^{n}\vartheta_{ip}\Big)K \quad \text{in } \langle 0, c'\rangle \times \langle 0, d\rangle^{nr}, \quad \text{and} \\ \left|\frac{\partial h_i}{\partial y_p^0}(x, y_1^0, \ldots, y_n^0)[f'_{ip}(x)]^r - \frac{\partial h_i}{\partial y_p^0}(0, \ldots, 0)[f'_{ip}(0)]^r\right| \leqslant \Big(1-\sum_{p=1}^{n}\vartheta_{ip}\Big)K \end{gathered} \tag{14}$$

in $\langle 0, c'\rangle \times \langle 0, d\rangle^n$, where $c' > 0$ and $d > 0$ have been chosen in such a manner that inequalities (12) hold in $\langle 0, c'\rangle$ and $\langle 0, c'\rangle \times \langle 0, d\rangle^n$, respectively, and $K > 0$ is an arbitrary fixed number. Now we choose a c so that

$$0 < c \leqslant \min(c', 1, d/K) \tag{15}$$

and we define the set $D \subset \Omega \times R^{(r-1)n}$ as follows:

$$D = \{(x, y_1^0, \ldots, y_n^{r-1}):\ 0 \leqslant x \leqslant c,\ |y_p^l| \leqslant Kx,\ l = 0, 1, \ldots, r-1,\ p = 1, \ldots, n\}.$$

To a given $\varepsilon > 0$ we assign the numbers

$$\varepsilon_i' = \frac{\Big(1-\sum_{p=1}^{n}\vartheta_{ip}\Big)\varepsilon}{1+nK}, \qquad i = 1, \ldots, n.$$

The functions g_{ir} are uniformly continuous in D, f'_{ij} are uniformly continuous in $\langle 0, c\rangle$, and $\partial h_i/\partial y_p^0$ are uniformly continuous in

$$D' \overset{\text{df}}{=} \{(x, y_1^0, \ldots, y_n^0):\ 0 \leqslant x \leqslant c,\ |y_p^0| \leqslant Kx\}.$$

Hence we have

$$\begin{gathered} |g_{ir}(\bar{x}, \bar{y}_1^0, \ldots, \bar{y}_n^{r-1}) - g_{ir}(\bar{\bar{x}}, \bar{\bar{y}}_1^0, \ldots, \bar{\bar{y}}_n^{r-1})| \leqslant \varepsilon_i', \\ \left|\frac{\partial h_i}{\partial y_p^0}(\bar{x}, \bar{y}_1^0, \ldots, \bar{y}_n^0)[f'_{ip}(\bar{x})]^r - \frac{\partial h_i}{\partial y_p^0}(\bar{\bar{x}}, \bar{\bar{y}}_1^0, \ldots, \bar{\bar{y}}_n^0)[f'_{ip}(\bar{\bar{x}})]^r\right| \leqslant \varepsilon_i'. \end{gathered} \tag{16}$$

for $|\bar{x}-\bar{\bar{x}}| \leqslant \delta_1$, $|\bar{y}_p^l - \bar{\bar{y}}_p^l| \leqslant \delta_2$, $p = 1, \ldots, n$, $l = 0, 1, \ldots, r-1$, and $(\bar{x}, \bar{y}_1^0, \ldots, \bar{y}_n^{r-1})$, $(\bar{\bar{x}}, \bar{\bar{y}}_1^0, \ldots, \bar{\bar{y}}_n^{r-1}) \in D$.

Now we put

$$\delta \overset{\text{df}}{=} \min\left(\frac{\delta_2}{K},\ \delta_1\right) \tag{17}$$

and define T_ε as the set of all functions defined in $\langle 0, c\rangle$ fulfilling the condition

$$|\alpha(x)-\alpha(\bar{x})| \leqslant \varepsilon \quad \text{for } |x-\bar{x}| < \delta.$$

Next we define F as the space of functions u which are defined and are of class C^r in $\langle 0, c\rangle$. For $u \in F$ we define the norm:

$$\|u\| = \max\Big(\sup_{\langle 0,c\rangle} |u(x)|,\ \sup_{\langle 0,c\rangle} |u'(x)|, \ldots, \sup_{\langle 0,c\rangle} |u^{(r)}(x)|\Big). \tag{18}$$

Thus F is a normed vector space over the field of real numbers and the convergence of a sequence $u_n \in F$ is the uniform convergence of u_n, u_n', ..., $u_n^{(r)}$ in $\langle 0, c\rangle$. Hence it follows that F is a Banach space. Let $A_1 \subset F$ denote the class of functions fulfilling the following conditions:

$$\varphi(0) = \varphi'(0) = \ldots = \varphi^{(r)}(0) = 0, \tag{19}$$

$$|\varphi^{(r)}(x)| \leqslant K \quad \text{for } 0 \leqslant x \leqslant c, \tag{20}$$

$$\varphi^{(r)}(x) \in T_\varepsilon. \tag{21}$$

We assert that if $\varphi_1, \varphi_2 \in A_1$, then

$$\|\varphi_1 - \varphi_2\| = \sup_{\langle 0,c\rangle} |\varphi_1^{(r)}(x) - \varphi_2^{(r)}(x)|.$$

In fact, according to (19), (15) and the mean-value theorem we have

$$\sup_{\langle 0,c\rangle} |\varphi_1(x)-\varphi_2(x)| \leqslant \sup_{\langle 0,c\rangle} |\varphi_1'(x)-\varphi_2'(x)| \leqslant \ldots \leqslant \sup_{\langle 0,c\rangle} |\varphi_1^{(r)}(x)-\varphi_2^{(r)}(x)|. \tag{22}$$

For $\varphi \in A_1$, by (15), (19), (21) and the mean-value theorem we have also

$$|\varphi(x)| \leqslant Kx \quad \text{and} \quad |\varphi^{(k)}(x)| \leqslant Kx \tag{23}$$

for $0 \leqslant x \leqslant c$, $k = 1, \ldots, r-1$.

Now we define the transformation $\psi = P(\varphi)$ by the formula

$$\psi_i(x) = h_i\big(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n[f_{in}(x)]\big), \quad i = 1, \ldots, n, \tag{24}$$

$$(\varphi_1, \varphi_2, \ldots, \varphi_n) \in A_1^n \overset{\text{df}}{=} A.$$

We shall prove that A and ψ fulfil all asumptions of Schauder's theorem. The set A_1 is a compact and convex subset of the space F.

Let $(\varphi_1, \ldots, \varphi_n) \in A$. Differentiating (24) k times we obtain

$$\psi_i^{(k)}(x) = h_{ik}\big(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n^{(k)}[f_{in}(x)]\big), \tag{25}$$

$k = 1, \ldots, r$, $i = 1, \ldots, n$.

According to Lemma 2 the functions $\varphi_i^{(r)}$ are continuous in $\langle 0, c\rangle$ and consequently $\psi_i \in F$. Putting $x = 0$ in (24) and (25) we obtain by (II), (13) and (19)

$$\psi_i(0) = h_i(0, \ldots, 0) = 0, \qquad i = 1, \ldots, n,$$
$$\psi_i^{(k)}(0) = h_{ik}(0, \ldots, 0) = 0, \qquad k = 1, \ldots, r,\ i = 1, \ldots, n.$$

Thus ψ_i fulfils condition (19).

By (24) and (3) we have

$$|\psi_i^{(r)}(x)| \leqslant \big|g_{ir}\big(\bar{x}, \varphi_1[f_{i1}(x)], \ldots, \varphi_n^{(r-1)}[f_{in}(x)]\big)\big| + $$
$$+ \sum_{p=1}^{n} \left| \frac{\partial h_i}{\partial y_p^0}\big(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n[f_{in}(x)]\big)[f_{ip}'(x)]^r \varphi_p^{(r)}[f_{ip}(x)]\right|.$$

In view of (I), (23) and (15) we have

$$|\varphi_i[f_{ji}(x)]| \leqslant Kx \leqslant d \quad \text{and} \quad |\varphi_i^{(k)}[f_{ji}(x)]| \leqslant Kx \leqslant d,$$

$k = 1, \ldots, r-1$, $i = 1, \ldots, n$.

Continuing our estimation of $\psi_i^{(r)}$ we have by (23), (12) and (14)

$$|\psi_i^{(r)}(x)| \leqslant \Big(1 - \sum_{p=1}^{n} \vartheta_{ip}\Big) K + \sum_{p=1}^{n} \vartheta_{ip} K = K,$$

what means that $\psi_i^{(r)}$ fulfil inequality (20).

Let us take an arbitrary $\varepsilon > 0$ and let $|\bar{x} - \bar{\bar{x}}| < \delta$, where $\delta(\varepsilon)$ is given by (17). By (25) and (3) we have

$$|\psi_i^{(r)}(\bar{x}) - \psi_i^{(r)}(\bar{\bar{x}})| \leqslant \big|g_{ir}\big(\bar{x}, \varphi_1[f_{i1}(\bar{x})], \ldots, \varphi_n^{(r-1)}[f_{in}(\bar{x})]\big) -$$
$$- g_{ir}\big(\bar{\bar{x}}, \varphi_1[f_{i1}(\bar{\bar{x}})], \ldots, \varphi_n^{(r-1)}[f_{in}(\bar{\bar{x}})]\big)\big| +$$
$$+ \sum_{p=1}^{n} \Bigg\{ \left| \frac{\partial h_i}{\partial y_p^0}\big(\bar{x}, \varphi_1[f_{i1}(\bar{x})], \ldots, \varphi_n[f_{in}(\bar{x})]\big)[f_{ip}'(\bar{x})]^r \big(\varphi_p^{(r)}[f_{ip}(\bar{x})] - \varphi_p^{(r)}[f_{ip}(\bar{\bar{x}})]\big)\right| +$$
$$+ \Bigg|\varphi_p^{(r)}[f_{ip}(\bar{\bar{x}})] \Bigg(\frac{\partial h_i}{\partial y_p^0}\big(\bar{x}, \varphi_1[f_{i1}(\bar{x})], \ldots, \varphi_n[f_{in}(\bar{x})]\big)[f_{ip}'(\bar{x})]^r -$$
$$- \frac{\partial h_i}{\partial y_p^0}\big(\bar{\bar{x}}, \varphi_1[f_{i1}(\bar{\bar{x}})], \ldots, \varphi_n[f_{in}(\bar{\bar{x}})]\big)[f_{ip}'(\bar{\bar{x}})]^r\Bigg)\Bigg|\Bigg\}.$$

By the mean-value theorem and (21), (23), (15), (11) we have

$$|\varphi_p[f_{ip}(\bar{x})] - \varphi_p[f_{ip}(\bar{\bar{x}})]| \leqslant |\varphi_p'(t_p)|\ |f_{ip}(\bar{x}) - f_{ip}(\bar{\bar{x}})| \leqslant K|\bar{x} - \bar{\bar{x}}| \leqslant \delta_2,$$
$$|\varphi_p^{(l)}[f_{ip}(\bar{x})] - \varphi_p^{(l)}[f_{ip}(\bar{\bar{x}})]| \leqslant |\varphi_p^{(l+1)}(t_p)|\ |f_{ip}(\bar{x}) - f_{ip}(\bar{\bar{x}})| \leqslant K|\bar{x} - \bar{\bar{x}}| \leqslant \delta_2$$

for $l = 1, \ldots, r-1$, $p = 1, \ldots, n$, and also

$$|\varphi_p^{(r)}[f_{ip}(\bar{x})] - \varphi_p^{(r)}[f_{ip}(\bar{\bar{x}})]| \leqslant \varepsilon.$$

Continuing our estimation we obtain from (16), (12) and (20)

$$|\psi_i^{(r)}(\bar{x})-\psi_i^{(r)}(\bar{\bar{x}})| \leqslant \varepsilon_i' + \sum_{p=1}^{n} (\varepsilon\vartheta_{ip}+K\varepsilon_i') = \varepsilon,$$

i. e. ψ_i fulfil condition (21). This completes the proof of the inclusion $\psi_i(A) \subset A_1$.

Let $\varphi_{sl} \in A_1$, $s = 1, \ldots, n$, $l = 1, 2, 3, \ldots$, tends to φ_{s0} (in the sence of convergence in F). Let us write

$$\psi_{sl}(x) = h_s\big(x, \varphi_{1l}[f_{s1}(x)], \ldots, \varphi_{nl}[f_{sn}(x)]\big),$$

$$\psi_{s0}(x) = h_s\big(x, \varphi_{10}[f_{s1}(x)], \ldots, \varphi_{n0}[f_{sn}(x)]\big).$$

Since h_s, $s = 1, \ldots, n$, are functions of class C^r, thus the sequence ψ_{sl} tends to ψ_{s0}. On account of the Schauder's theorem there exists a function $\varphi \in A$ satisfying the system of equations (1) in $\langle 0, c\rangle$.
This solution can be extended onto the whole interval I in the same manner as in [4] (Theorem 4.2, p. 89).

J. Matkowski has proved in [7] the following

THEOREM 2. *Let* (I) *and* (II) *be fulfilled. If*

$$\left|\frac{\partial h_i}{\partial y_p^0}(x, y_1^0, \ldots, y_n^0)[f_{ip}'(x)]^r\right| \leqslant \vartheta_{ip} \quad \textit{in a certain neighbourhood of zero and}$$

(26)
$$\sum_{i=1}^{n} \vartheta_{ip} < 1 \quad \textit{for } p = 1, \ldots, n,$$

then for any system of η_i^l, $i = 1, \ldots, n$, $l = 0, 1, \ldots, r$, *fulfilling* (5) *there exists at most one* $C^r[I]$*-solution of system* (1) *fulfilling the conditions* $\varphi_i^{(l)}(0) = \eta_i^l$.

3. In this section we shall prove a certain theorem on the continuous dependence for the sequence of systems of equations

$$\varphi_i(x) = \underset{p}{h_i}\big(x, \varphi_1[\underset{p}{f_{i1}}(x)], \ldots, \varphi_n[\underset{p}{f_{in}}(x)]\big), \tag{1'}$$

$i = 1, \ldots, n$, $p = 0, 1, 2, \ldots$

We assume that

(I') $\quad \underset{p}{f_{ik}} \in S^r[I], \quad I = \langle 0, a\rangle, a > 0;$

(II') $\quad \underset{p}{h_i} \in C^r[\Omega], \quad \Omega = I \times R^n, \quad \underset{p}{h_i}(0, \ldots, 0) = 0;$

(III) $\underset{p}{f_{ik}}$ tends to $\underset{0}{f_{ik}}$ in I, $\underset{p}{h_i}$ tends to $\underset{0}{h_i}$ in Ω, together with derivatives up to the order r uniformly on compact sets.

For arbitrary p we can define the sequence $\underset{p}{h}_{ik}$ in the same way as in Section 1. Thus for arbitrary p Lemmas 1 and 2 hold. By induction we have

LEMMA 4. *Let hypotheses* (I′), (II′) *and* (III) *be fulfilled. The sequence* $\underset{p}{h}_{ik}$ *tends to* $\underset{0}{h}_{ik}$ *in* $\Omega \times R^{nk}$, $\underset{p}{g}_{ik}$ *tends to* $\underset{0}{g}_{ik}$ *in* $\Omega \times R^{n(k-1)}$, *uniformly on compact sets.*

It follows from Lemma 2 that if $\underset{p}{\varphi} = \{\underset{p}{\varphi_i}\}$, $i = 1, \ldots, n$, functions belonging to $\Lambda[I]$ are $C^r[I]$-solutions of the sequence of systems (1′), then

$$\underset{p}{\eta_i^k} = \underset{p}{\varphi_i^{(k)}}(0), \quad i = 1, \ldots, n, \ k = 1, \ldots, r, \tag{4′}$$

satisfy the system of equations

$$\underset{p}{\eta_i^k} = \underset{p}{h}_{ik}(0, \ldots, 0, \underset{p}{\eta_1^1}, \ldots, \underset{p}{\eta_n^k}). \tag{5′}$$

Hence and by (3) we obtain

$$\left(E - \frac{\partial \underset{p}{h_i}}{\partial y_j^0}(0, \ldots, 0)[\underset{p}{f'_{ij}}(0)]^k\right)\underset{p}{\eta^k} = \underset{p}{g_k}(0, \ldots, 0, \underset{p}{\eta_1^1}, \ldots, \underset{p}{\eta_n^{k-1}});$$

where $k = 1, \ldots, r$, $i, j = 1, \ldots, n$, E is the unit matrix and

$$\underset{p}{\eta^k} = \begin{bmatrix} \underset{p}{\eta_1^k} \\ \vdots \\ \underset{p}{\eta_n^k} \end{bmatrix}, \quad \underset{p}{g_k} = \begin{bmatrix} \underset{p}{g_{1k}} \\ \vdots \\ \underset{p}{g_{nk}} \end{bmatrix}.$$

Hence we have the simple

LEMMA 5. *Let hypotheses* (I′) *and* (II′) *be fulfilled. If* (11), (12), (26) *hold, then for every* p *the following statements hold true:*

(a) *the system of equations* (1′) *has exactly one solution* $\varphi \in \Lambda[I] \cap C^r[I]$ *iff all characteristic rotts of the matrix*

$$\underset{p}{A_k} = \left(\frac{\partial \underset{p}{h_i}}{\partial y_j^0}(0, \ldots, 0)[\underset{p}{f'_{ij}}(0)]^k\right), \quad i, j = 1, \ldots, n, \tag{27}$$

are different from 1 *for* $k = 1, \ldots, r$;

(b) *if for some* k *there exists a characteristic root equal* 1, *then the system of equations* (1′) *has a* C^r*-solution iff the ranks of the matrix* $\underset{p}{A_k}$ *and of the complement matrix are equal. In this case the system* (1′) *has an* $(n-l_1) \cdot \ldots$

$\ldots \cdot (n-l_r)$*-parameter family of* $C^r[I]$*-solutions, where* l_k *denotes the rank of the matrix* $(E - \underset{p}{A}_k)$.

LEMMA 6. *Let hypotheses* (I′), (II′), (III) *and condition* (27) *be fulfilled. If* $\underset{p}{\varphi} = \{\underset{p}{\varphi}_i\}$, $i = 1, \ldots, n$, *form a solution belonging to* $\Lambda[I] \cap C^r[I]$ *for* $p = 0, 1, 2, \ldots$ *of the sequence of systems* (1′), *then*

$$\lim_{p\to\infty} \underset{p}{\varphi}_i^{(k)}(0) = \underset{0}{\varphi}_i^{(k)}(0)$$

for $k = 1, \ldots, r$, $i = 1, \ldots, n$.

LEMMA 7. *If* A *is a compact metric space and the sequence of transformations* $\underset{p}{T}$ *fulfils the conditions*

1° $\underset{p}{T}(A) \subset A$,

2° $\underset{p}{T}$ *is continuous in* A *for* $p = 0, 1, 2, \ldots$,

3° $\underset{p}{T}(\varphi) = \varphi$ *if and only if* $\varphi = \underset{p}{\varphi}$ *and* $\underset{p}{\varphi} \in A$, $p = 0, 1, 2, \ldots$,

4° $\underset{p}{T}$ *tends to* $\underset{0}{T}$ *uniformly in* A,

then $\underset{p}{\varphi}$ *is convergent and* $\lim_{p\to\infty} \underset{p}{\varphi} = \underset{0}{\varphi}$.

The proof of this lemma may be found in [6].

THEOREM 3. *Let hypotheses* (I′), (II′), (III) *be fulfilled. If* (11), (12), (26) *and* (27) *hold for* $p = 0, 1, 2, \ldots$, *then there exists an interval* $\langle 0, c\rangle$ *such that for arbitrary* p *there exists exactly one* C^r*-solution* $\underset{p}{\varphi}$ *of system* (1′). *Moreover,* $\underset{p}{\varphi}$ *tends to* $\underset{0}{\varphi}$ *together with derivatives up to order* r, *uniformly in this interval.*

Proof. Let T_ε, F, A_1 and A be defined as in the proof of Theorem 1. We define the norm in the set $\overline{F} \overset{\mathrm{df}}{=} F^n$ as follows:

$$\|(\varphi_1, \ldots, \varphi_n)\| \overset{\mathrm{df}}{=} \|\varphi_1\| + \|\varphi_2\| + \ldots + \|\varphi_n\|. \tag{28}$$

Now we define the sequence of transformations $\underset{p}{T}$:

$$\underset{p}{T} = (\underset{p}{\psi}_1, \ldots, \underset{p}{\psi}_n),$$

where $\underset{p}{\psi}_i$ are defined by the formula

$$\underset{p}{\psi}_i(x) = \underset{p}{h}_i\big(x, \varphi_1[\underset{p}{f}_{i1}(x)], \ldots, \varphi_n[\underset{p}{f}_{in}(x)]\big).$$

We shall show that A and the sequence of transformations $\underset{p}{T}$ fulfil the hypotheses of Lemma 7. Similarly as in the proof of Theorem 1 we can show that 1° and 2° are fulfilled. Now, Theorem 2 and Lemma 5 imply condition 3°. To prove 4° it is sufficient to show that for a given $\varepsilon > 0$ there exists an N such that for $p \geqslant N$ and for every $\varphi \in A$ we have

$$\|\underset{p}{T}[\varphi] - \underset{0}{T}[\varphi]\| \leqslant \varepsilon .$$

From (28), (21) and Lemma 2 we have

$$\begin{aligned}
&\|\underset{p}{T}[\varphi_1, \ldots, \varphi_n] - \underset{0}{T}[\varphi_1, \ldots, \varphi_n]\| \\
&= \sum_{l=1}^{n} \|\underset{p}{\psi_l}[\varphi_1, \ldots, \varphi_n] - \underset{0}{\psi_l}[\varphi_1, \ldots, \varphi_n]\| \\
&= \sum_{l=1}^{n} \sup_{\langle 0,c\rangle} \big|\underset{p}{h_{lr}}\big(x, \varphi_1[\underset{p}{f_{l1}}(x)], \ldots, \varphi_n^{(r)}[\underset{p}{f_{ln}}(x)]\big) - \underset{0}{h_{lr}}\big(x, \varphi_1[\underset{0}{f_{l_1}}(x)], \ldots, \varphi_n^{(r)}[\underset{0}{f_{ln}}(x)]\big)\big| \\
&\leqslant \sum_{l=1}^{n} \Big\{\sup_{\langle 0,c\rangle} \big|\underset{p}{h_{lr}}\big(x, \varphi_1[\underset{p}{f_{l1}}(x)], \ldots, \varphi_n^{(r)}[\underset{p}{f_{ln}}(x)]\big) - \underset{0}{h_{lr}}\big(x, \varphi_1[\underset{p}{f_{l1}}(x)], \ldots, \varphi_n^{(r)}[\underset{p}{f_{ln}}(x)]\big)\big| + \\
&\quad + \sup_{\langle 0,c\rangle} \big|\underset{0}{h_{lr}}\big(x, \varphi_1[\underset{p}{f_{l1}}(x)], \ldots, \varphi_n^{(r)}[\underset{p}{f_{ln}}(x)]\big) - \underset{0}{h_{lr}}\big(x, \varphi_1[\underset{p}{f_{l1}}(x)], \ldots, \varphi_n^{(r)}[\underset{0}{f_{ln}}(x)]\big)\big|\Big\}.
\end{aligned}$$

According to Lemma 4 the sequence $\underset{p}{h_{lr}}$ tends to $\underset{0}{h_{lr}}$ uniformly in $D \times \langle 0, K\rangle$. Thus there exists p_0 such that for $p \geqslant p_0$ and $x \in \langle 0, c\rangle$ we have

$$\big|\underset{p}{h_{lr}}\big(x, \varphi_1[\underset{p}{f_{l1}}(x)], \ldots, \varphi_n^{(r)}[\underset{p}{f_{ln}}(x)]\big) - \underset{0}{h_{lr}}\big(x, \varphi_1[\underset{p}{f_{l1}}(x)], \ldots, \varphi_n^{(r)}[\underset{p}{f_{ln}}(x)]\big)\big| \leqslant \varepsilon/2n .$$

It follows from the uniform continuity of $\varphi_i^{(k)}$ in $\langle 0, c\rangle$ that there exists a p_1 such that for $p \geqslant p_1$ and $i = 1, \ldots, n$, $k = 0, 1, \ldots, r$

$$|\varphi_i^{(k)}[\underset{p}{f_{li}}(x)] - \varphi_i^{(k)}[\underset{0}{f_{li}}(x)]| \leqslant \delta .$$

Continuing our estimation we have

$$\|\underset{p}{T}[\varphi_1, \ldots, \varphi_n] - \underset{0}{T}[\varphi_1, \ldots, \varphi_n]\| < \varepsilon$$

for $p \geqslant \max(p_0, p_1)$, $0 \leqslant x \leqslant c$, $(\varphi_1, \ldots, \varphi_n) \in A$. Thus 4° is proved.

Now, Lemma 6 and (6) complete the proof.

4. In this section we consider the special case of a system of equations (1), namely:

$$\varphi_i(x) = h_i\big(x, \varphi_1[f(x)], \ldots, \varphi_n[f(x)]\big), \quad i = 1, \ldots, n. \tag{29}$$

We shall prove that in this case the assumption (11) is superfluous (see Theorem 1).

We have the following theorem:

THEOREM 4. *Let hypotheses* (I), (II) *and condition* (12) *be fulfilled. Then there exists at most one C^r-solution of system* (29) *belonging to $\Lambda[I]$ and fulfilling conditions* (4).

Proof. It follows from hypotheses (I) that $|f'(0)| \leqslant 1$. Theorem 1 guarantees the existence of a C^r-solution in the case $|f'(0)| < 1$, thus it is sufficient to consider the case $|f'(0)| = 1$. Now, by (12) we obtain

$$\sum_{l=1}^{n} \left| \frac{\partial h_i}{\partial y_l^0}(0, \ldots, 0) \right| < 1, \quad i = 1, \ldots, n.$$

Let

$$\vartheta_{ij} \stackrel{\text{df}}{=} \left| \frac{\partial h_i}{\partial y_l^0}(0, \ldots, 0) \right| + \varepsilon',$$

where $\varepsilon' > 0$ is chosen so that

$$\sum_{j=1}^{n} \vartheta_{ij} < 1, \quad i = 1, \ldots, n. \tag{30}$$

From the continuity of $\partial h_i / \partial y_j^0$ and by the mean-value theorem there exists a neighbourhood of zero V such that for $(x, \bar{y}_1^0, \ldots, \bar{y}_n^0)$, $(x, \bar{\bar{y}}_1^0, \ldots, \bar{\bar{y}}_n^0) \in V$ we have

$$|h_i(x, \bar{y}_1^0, \ldots, \bar{y}_n^0) - h_i(x, \bar{\bar{y}}_1^0, \ldots, \bar{\bar{y}}_n^0)| \leqslant \sum_{l=1}^{n} \vartheta_{il} |\bar{y}_l^0 - \bar{\bar{y}}_l^0|.$$

Since $\vartheta_{il} > 0$ and inequalities (30) hold, thus from Lemma 1 of [5] it follows that $\vartheta_{il}^{\varkappa} > 0$, where

$$\vartheta_{il}^{1} = \begin{cases} \vartheta_{il}, & i \neq l, \\ 1 - \vartheta_{il}, & i = l, \end{cases}$$

and

$$\vartheta_{il}^{\varkappa+1} = \begin{cases} \vartheta_{11}^{\varkappa} \vartheta_{i+1\,l+1}^{\varkappa} + \vartheta_{i+1\,1}^{\varkappa} \vartheta_{1\,l+1}^{\varkappa}, & i \neq l, \\ \vartheta_{11}^{\varkappa} \vartheta_{i+1\,l+1}^{\varkappa} - \vartheta_{i+1\,1}^{\varkappa} \vartheta_{1\,l+1}^{\varkappa}, & i = l, \end{cases}$$

$\varkappa = 1, \ldots, n-1$, $i, l = 1, \ldots, n-\varkappa$. Now, by Theorem 1 of [1] (see also [3], Theorem 4) there exists exactly one continuous solution of the system (29) in I. We shall prove that this solution is of class $C^r[I]$. For this purpose we introduce the sequence of systems of equations

$$\varphi_i(x) = h_i\big(x, \varphi_1[f_k(x)], \ldots, \varphi_n[f_k(x)]\big), \quad i = 1, \ldots, n, \tag{31}$$

where $f_k(x) = t_k f(x)$, $0 < t_k < 1$, $\lim_{k\to\infty} t_k = t_0 = 1$.

In view of Theorem 1 and Theorem 1 of [1] for $k = 1, 2, \ldots$, the system of equations (31) has exactly one $C^r[I]$-solution $\varphi_k = \{\varphi_{ik}\}$, $i = 1, \ldots, n$, belonging to $\Lambda[I]$ since $|f_k'(0)| = t_k < 1$. Thus we have

$$\varphi_{ik}(x) = h_i\big(x, \varphi_{1k}[f_k(x)], \ldots, \varphi_{nk}[f_k(x)]\big).$$

Differentiating both sides of this equality we have

$$\varphi_{ik}'(x) = \frac{\partial h_i}{\partial x}\big(x, \varphi_{1k}[f_k(x)], \ldots, \varphi_{nk}[f_k(x)]\big) + \\ + \sum_{j=1}^{n} \frac{\partial h_i}{\partial y_j}\big(x, \varphi_{1k}[f_k(x)], \ldots, \varphi_{nk}[f_k(x)]\big)\varphi_{jk}'[f_k(x)]f_k'(x).$$

This means that $\Phi_{ik}(x) = \varphi_{ik}'(x)$ satisfies the system

$$\Phi_{ik}(x) = H_{ik}\big(x, \Phi_{1k}[f_k(x)], \ldots, \Phi_{nk}[f_k(x)]\big), \tag{32}$$

where

$$H_{ik}(x, v_1, \ldots, v_n) \overset{\text{df}}{=} \frac{\partial h_i}{\partial x}\big(x, \varphi_{1k}[f_k(x)], \ldots, \varphi_{nk}[f_k(x)]\big) + \\ + \sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j}\big(x, \varphi_{1k}[f_k(x)], \ldots, \varphi_{nk}[f_k(x)]\big)f_k'(x)v_j.$$

Since the sequence f_k tends to f_0 uniformly on every compact subset of I, the sequence φ_{ik} tends to φ_{i0} on every compact subset of I (see [1], Theorem 5). Hence it follows that H_{ik} uniformly converges to the function

$$H_{i0}(x, v_1, \ldots, v_n) = \frac{\partial h_i}{\partial x}\big(x, \varphi_1[f(x)], \ldots, \varphi_n[f(x)]\big) + \\ + \sum_{j=1}^{n} \frac{\partial h_i}{\partial y_j}\big(x, \varphi_1[f(x)], \ldots, \varphi_n[f(x)]\big)f'(x)v_j$$

on every compact subset of $I \times R^n$. For $k = 0, 1, 2, \ldots$ we have

$$\sum_{j=1}^{n}\left|\frac{\partial H_{ik}}{\partial v_j}(0, \ldots, 0)\right| = \sum_{j=1}^{n}\left|\frac{\partial h_i}{\partial y_j}(0, \ldots, 0)f_k'(0)\right| \\ = \sum_{j=1}^{n} t_k\left|\frac{\partial h_i}{\partial y_j}(0, \ldots, 0)\right| < 1.$$

Therefore (similary as above) $\Phi_{ik} = \varphi_{ik}'$ is the unique solution of the system of equations (32). Applying again the theorem on the continuous dependence of continuous solutions (Theorem 5 of [1]) we obtain $\lim\limits_{k\to\infty} \varphi_{ik}' = \Phi_{i0}$ uniformly on every compact subset of I. Hence it follows that φ_i is of class $C^1[I]$ and $\varphi_i' = \Phi_{i0}$ in I, because $\lim\limits_{k\to\infty}\varphi_{ik} = \varphi_i$. Repeating this procedure r times we obtain $\varphi_i \in C^r[I]$. This completes the proof.

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