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A general description of the Bergman projection

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Abstract. Let $D \subset C^N$ be the union of domains D_i, $i = 1, 2, \ldots, m$. We shall apply a theorem of I. Halperin to describe the Bergman projection $P\colon L^2(D) \to L^2 H(D)$ in terms of the Bergman projections in domains D_i. This yields a more constructive description of distances related to the Bergman function. A number of related recent results is reported.

1. Orthogonal projections. The following theorem is due to I. Halperin [2]. A very elegant proof, given by Amemiya and Ando, can be found in Helgason book [3], p. 55.

Theorem 1. *Let H be a Hilbert space, and P_i the orthogonal projection onto the closed linear subspace F_i, $i = 1, 2, \ldots, m$. Then for every $f \in H$*

$$\lim_{n \to \infty} (P_m P_{m-1} \ldots P_1)^n f = Pf,$$

where P denotes the orthogonal projection onto $F = \bigcap_{i=1}^{m} F_i$.

From the above result easily follows that the sequence of alternating projections

$$f_1 = P_1 f, \quad f_2 = P_2 f_1, \quad \ldots, \quad f_m = P_m f_{m-1}, f_{m+1} = P_1 f_m,$$

converges in H to Pf. We shall also need a well-known theorem:

Theorem 2. *Let H be a Hilbert space, and $P_j\colon H \to F_j$, $j = 1, 2, \ldots$ a sequence of orthogonal projections, such that $F_1 \supset F_2 \supset F_3 \supset \ldots$ Then for every $f \in H$*

$$\lim_{j \to \infty} P_j f = Pf,$$

where P denotes the orthogonal projection onto $F = \bigcap_{j=1}^{\infty} F_j$.

For the proof see M. H. Stone [14], p. 74.

2. Consequences for the Bergman projection. We shall consider the following cases

1° $D \subset C^N$ is a finite union of domains $D_i \subset C^N$, $i = 1, 2, \ldots, m$;

2° $D \subset C^N$ is the union of an increasing sequence of domains $G_j \subset C^N$, $j = 1, 2, \ldots$

In the first case we can prove

THEOREM 3. *Let $D \subset C^N$ be the union of domains $D_i \subset C^N$, $i = 1, 2, \ldots, m$. In the Hilbert space $H = L^2(D)$, consider for each i the closed linear subspace F_i of functions which are holomorphic in D_i and arbitrary in $D \setminus D_i$. Let Q_i: $L^2(D_i) \to L^2H(D_i)$ be the Bergman projection in D_i. Then the orthogonal projection P_i: $H \to F_i$ is given by*

$$(P_i f)(z) = \begin{cases} (Q_i f_{|D_i})(z), & z \in D_i, \\ f(z), & z \in D \setminus D_i, \end{cases} \tag{1}$$

and for every $f \in L^2(D)$

$$\lim_{n \to \infty} (P_m P_{m-1} \ldots P_1)^n f = Pf, \tag{2}$$

where P is the Bergman projection in D.

Proof. The right-hand side of (1) is obviously in F_i. Let us denote it by g. The difference $f - g$ is in $F_i^{\perp}$. Indeed, it vanishes on $D \setminus D_i$, and on D_i it is orthogonal to $L^2H(D_i)$. Hence, for every $h \in F_i$,

$$\int_D (f-g)\bar{h} = \int_{D \setminus D_i} (f-g)\bar{h} + \int_{D_i} (f-g)\bar{h} = 0.$$

This proves (1). By Halperin's theorem (2) holds where P is defined as the orthogonal projection onto $F = \bigcap_{i=1}^{m} F_i$. Since $F = L^2H(D)$, we see that P is the Bergman projection in D. ■

In the second case we can state

THEOREM 4. *Let $D \subset C^N$ be the union of an increasing sequence of domains $G_j \subset C^N$, $j = 1, 2, \ldots$ Denote by F_j the closed linear subspace in $H = L^2(D)$ consisting of functions which are holomorphic in G_j, and arbitrary in $D \setminus G_j$. Let P_j: $H \to F_j$ be the corresponding orthogonal projection. Then for every $f \in L^2(D)$*

$$\lim_{j \to \infty} P_j f = Pf, \tag{3}$$

where P is the Bergman projection in D.

Proof. Note that subspaces F_j, $j = 1, 2, \ldots$, form a decreasing sequence. By Theorem 2 we have (3) with P defined as the orthogonal projection onto $F = \bigcap_{j=1}^{\infty} F_j$. Since $F = L^2H(D)$, we see that P is the Bergman projection in D. ■

Theorems 3 and 4 show that "in principle" we can determine the Bergman projection in an arbitrary domain $D \subset C^N$. Indeed, it is easy to represent D as a union of an increasing sequence of domains G_j, where every G_j is a finite union of open balls. The Bergman projection in each ball is known explicite. Therefore by Theorem 3 we can determine the Bergman projection in every G_j. Then using Theorem 4 we can determine the Bergman projection in D. (It may be convenient sometimes to replace balls by other domains for which the Bergman function is known explicite.)

The Bergman projection P: $L^2(D) \to L^2 H(D)$ can be applied to determine the Bergman function $K_D(z, t)$, $z, t \in D$. (A standard reference for the Bergman function is [1].) For an arbitrary point $t \in D$ we can find a ball $B \subset D$ with center at t. Then $\varphi = (\mathrm{vol}\, B)^{-1} \chi_B \in L^2(D)$ has the reproducing property

$$\int_D f(z) \overline{\varphi(z)}\, dm(z) = f(t), \quad f \in L^2 H(D).$$

It follows that $P\varphi$ has the same reproducing property. Since $P\varphi \in L^2 H(D)$, we conclude that $P\varphi = K_D(\cdot, t)$. If we know that a sequence of operators P_n in $L^2(D)$ converges pointwise to P we can conclude that the sequence $\varphi_n = P_n \varphi$ converges in $L^2(D)$ to $K_D(\cdot, t)$. Conversely, the Bergman function determines the Bergman projection by the formula

$$(Pf)(t) = \int_D f(z) \overline{K_D(z, t)}\, dm(z).$$

It should be noted that Theorem 4 is essentially equivalent to a theorem of I. P. Ramadanov, proved by different method in [7].

The results of this section indicate a part of the borderland in the theory of operators in Hilbert space, and the theory of several complex variables, which, in the authors opinion, will be important for the future development of both the subjects.

3. The orthogonality condition. Unfortunately we are so far not able to obtain new expressions for the Bergman function in a closed form using the procedure of Theorems 3 and 4. Nevertheless, since the procedure is both constructive and general, we shall study it for its own sake. As a very simple example we can consider the decomposition of C as a union of domains D_1, D_2 such that the sets $\mathrm{int}\, D_1^c$, $\mathrm{int}\, D_2^c$ are nonvoid. We shall say that D_1 and D_2 satisfy the *orthogonality condition* if

$$(4) \qquad \int_{D_1 \cap D_2} h_1 \bar{h}_2 = 0 \quad \text{for every } h_1 \in L^2 H(D_1) \text{ and } h_2 \in L^2 H(D_2).$$

It was observed in [13] that all computations required in Theorem 3 can be carried out explicitely in two simple cases: 1° domains bounded by concentric circles (D_2 is a disc and D_1 is the exterior of a smaller concentric disc); 2° domains bounded by parallel lines (D_1 and D_2 are halfplanes, which intersect along a strip). Since the subspaces F_1, F_2 in $L^2(C)$ (notation of Theorem 3) intersect only at 0, we may introduce the angle $\alpha \in [0, \frac{1}{2}\pi]$ such that

$$\cos\alpha = \sup\left\{\frac{|\langle f_1, f_2\rangle|}{\|f_1\|\cdot\|f_2\|};\ f_i \in F_i\setminus\{0\},\ i = 1, 2\right\}. \tag{5}$$

Let us associate with every domain $G \subset C^N$ and every measurable subset $T \subset G$ the biholomorphic invariant

$$p_G(T) = \sup\left\{\frac{\|h\|_T}{\|h\|_G};\ h \in L^2 H(G)\setminus\{0\}\right\}.$$

Obviously, $p_G(T) \in [0, 1]$. Write $p_1 = p_{D_1}(D_1\setminus D_2)$, $p_2 = p_{D_2}(D_2\setminus D_1)$. It was proved in [9] that (4) implies $\cos\alpha = \max(p_1, p_2)$. It follows that $\alpha > 0$ if domains are bounded by concentric circles, and $\alpha = 0$ if domains are bounded by parallel lines. The situation where $\alpha = 0$ is "not favourable" for Theorem 3. (This can be understood by looking at Halperins theorem in a finite dimensional space.) It is therefore remarkable that the convergence in case 2° is still satisfactory (comparable with $\sum_{n=1}^{\infty} n^{-2}$).

4. Distances related to the Bergman function. The invariant distance

$$\varrho_D(z, t) = \left(1 - \left(\frac{K_D(z, t)\,K_D(t, z)}{K_D(z, z)\,K_D(t, t)}\right)^{1/2}\right)^{1/2} \tag{6}$$

explicitely expressed by the Bergman function in a domain $D \subset\subset C^N$ was discovered by the author and investigated in [10]. (See also [1], p. 193.) The direct connection of ϱ_D with the Bergman function K_D is important for applications ([10], [11], p. 31). (See also [11], p. 25, Theorem III.15, and [6].) In a less explicite form, the distance ϱ_D has appeared already in [4]. In a special case of the unit disc the distance ϱ_D (in still another form) was considered by Tsuji [15].

Expression (6) defines the distance function also for some unbounded domains. A complete characterization of such domains for $n > 1$ remains an open problem. For $n = 1$ it is known [12] that ϱ_D defines a distance function in D if and only if $C\setminus D$ is not polar. (See also [9].)

The results of Section 2 can be used to determine ϱ_D ($D \subset\subset C$ for simplicity) in the following way:

COROLLARY 1. *Assume that ψ_n, $n = 1, 2, \ldots$, converges in $L^2(D)$ to $K_D(\cdot, z)$, and that φ_n, $n = 1, 2, \ldots$, converges in $L^2(D)$ to $K_D(\cdot, t)$. Then*

$$\varrho_D(z, t) = \lim_{n\to\infty}\left(1 - \frac{|\langle\psi_n, \varphi_n\rangle|}{\|\psi_n\|\cdot\|\varphi_n\|}\right)^{1/2}.$$

Let us remark, that ϱ_D is related to the Bergman metric

$$ds^2 = \sum_{j,k=1}^{N} \left(\frac{\partial^2 \log K_D(z, z)}{\partial z_j \partial \bar{z}_k} dz_j \otimes d\bar{z}_k + \frac{\partial^2 \log K_D(z, z)}{\partial \bar{z}_j \partial z_k} d\bar{z}_j \otimes dz_k \right)$$

in the following way. Denote by $l_B(\gamma)$ the length of a C^1 curve γ: $[a, b] \to D$ with respect to the Bergman metric. Let us introduce the ϱ_D-length of γ by the formula

$$l_{\varrho_D}(\gamma) = \sup \sum_{i=1}^{n-1} \varrho_D(\gamma(t_i), \gamma(t_{i+1})),$$

where $a = t_0 < t_1 < \ldots < t_n = b$ are arbitrary points on the segment $[a, b]$. It was proved recently [5] that $l_B(\gamma) = 2l_{\varrho_D}(\gamma)$.

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