ANNALES
POLONICI MATHEMATICI
XLVI (1985)

Laguerre series and the Cauchy integral representation

by PETER RUSEV (Sofia, Bulgaria)

Franciszek Leja in memoriam

Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ with parameter $\alpha > -1$ are uniquely determined by the orthogonal property

$$\int_0^\infty t^\alpha \exp(-t) L_m^{(\alpha)}(t) L_n^{(\alpha)}(t)\, dt = J_n^{(\alpha)} \delta_{mn},$$

where $J_n^{(\alpha)} = \Gamma(n+\alpha+1)/\Gamma(n+1)$, and the assumption that the coefficient of z^n in $(-1)^n L_n^{(\alpha)}(z)$ is positive [2].

A series of the kind

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z) \tag{1}$$

we will call a *series in Laguerre polynomials* or briefly a *Laguerre series*. If

$$\lambda_0 = -\limsup (2\sqrt{n})^{-1} \ln |a_n| > 0,$$

then series (1) is absolutely uniformly convergent on every compact subset of the region $\Delta(\lambda_0)$: $\mathrm{Re}(-z)^{1/2} < \lambda_0$ and diverges at every point in $C - \overline{\Delta(\lambda_0)}$.

It is a well-known fact that the system $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ satisfies the linear recurrence equation of the second order

$$(n+1) y_{n+1} + (z-2n-\alpha-1) y_n + (n+\alpha) y_{n-1} = 0.$$

Another solution of the above equation is the system of complex functions

$$M_n^{(\alpha)}(z) = -\int_0^\infty \frac{t^\alpha \exp(-t) L_n^{(\alpha)}(t)}{t-z}\, dt, \qquad n = 0, 1, 2, \ldots,$$

holomorphic in the region $C-[0, +\infty)$. We call it the system of Laguerre functions of the second kind.

The two Laguerre systems mentioned above are also connected with a formula of the Christoffel–Darboux type, namely

(2) $$\frac{1}{\zeta-z}=\sum_{n=0}^{\nu}(J_n^{(\alpha)})^{-1}L_n^{(\alpha)}(z)M_n^{(\alpha)}(\zeta)+\frac{\Delta_{\nu+1}^{(\alpha)}(z,\zeta)}{\zeta-z},$$

where

$$\Delta_{\nu+1}^{(\alpha)}(z,\zeta)=\frac{\nu+1}{J_\nu^{(\alpha)}}\{L_\nu^{(\alpha)}(z)M_{\nu+1}^{(\alpha)}(\zeta)-L_{\nu+1}^{(\alpha)}(z)M_\nu^{(\alpha)}(\zeta)\}.$$

A complex function f holomorphic in a region of the kind $\Delta(\lambda_0)$ $(0<\lambda_0\leqslant+\infty)$ is said to have a Cauchy integral representation in that region if for every $\lambda\in(0,\lambda_0)$ and every $z\in\Delta(\lambda)$ the following equality holds:

(3) $$f(z)=\frac{1}{2\pi i}\int_{p(\lambda)}\frac{f(\zeta)}{\zeta-z}d\zeta,$$

where $p(\lambda)=\partial\Delta(\lambda)=\{\zeta:\ \mathrm{Re}(-\zeta)^{1/2}=\lambda\}$.

Every complex function f holomorphic in $\Delta(\lambda_0)$ and satisfying a certain "growth" condition has a Cauchy integral representation there. Further, Christoffel–Darboux formula (2) gives occassion for supposing that f might be expanded in $\Delta(\lambda_0)$ in a Laguerre series with coefficients

(4) $$a_n=\frac{1}{2\pi i J_n^{(\alpha)}}\int_{p(\lambda)}M_n^{(\alpha)}(\zeta)f(\zeta)d\zeta,\qquad n=0,1,2,\ldots$$

It will be seen that this is really the case and, moreover, that under an additional condition the converse is also true, i.e., if the function f has, in the region $\Delta(\lambda_0)$, a Laguerre expansion with coefficients given by (4), then it has a Cauchy integral representation in that region.

Let $\alpha>-1$, $0<\lambda_0\leqslant+\infty$ and $F(\alpha,\lambda_0)$ be the class of all complex functions f holomorphic in the region $\Delta(\lambda_0)$ and such that

(5) $$\int_{p(\lambda)}|\zeta|^{\alpha/2-1/4}|f(\zeta)|\,ds<+\infty$$

for every $\lambda\in(0,\lambda_0)$.

THEOREM. *A function $f\in F(\alpha,\lambda_0)$ has, in the region $\Delta(\lambda_0)$, a Laguerre expansion with coefficients* (4) *iff f has a Cauchy integral representation in $\Delta(\lambda_0)$.*

The proof is based mostly on the asymptotic properties of Laguerre polynomials and functions of the second kind.

As a consequence of Perron's formula [2], (8.22.3), and the inequality

[2], (7.6.11), it follows that there exist constants $A = A(z, \alpha)$ and $S = S(\alpha)$ such that

$$|L_n^{(\alpha)}(z)| \leqslant An^S \exp[2\mathrm{Re}(-z)^{1/2}\sqrt{n}]$$

for every $n \geqslant 1$. If $0 < \lambda < +\infty$, there exists a constant $M = M(\lambda, \alpha)$ with the property that

$$|M_n^{(\alpha)}(\zeta)| \leqslant M|\zeta|^{\alpha/2-1/4} n^{\alpha/2-1/4} \exp(-2\lambda\sqrt{n})$$

for every $\zeta \in p(\lambda)$ and n greater than some positive integer n_0 [1].

Using Stirling's formula, we find that there exist constants $K = K(z, \lambda, \alpha)$ and $\sigma = \sigma(\alpha)$ such that, if $\zeta = p(\lambda)$ and $n \geqslant n_0$,

$$(6)\quad |(J_n^{(\alpha)})^{-1} L_n^{(\alpha)}(z) M_n^{(\alpha)}(\zeta)| \leqslant Kn^\sigma |\zeta|^{\alpha/2-1/4} \exp\{-2[\lambda - \mathrm{Re}(-z)^{1/2}]\sqrt{n}\}.$$

Further, from Christoffel–Darboux formula (2) it follows that, if $0 < \lambda < +\infty$, $\zeta \in p(\lambda)$ and $z \in \Delta(\lambda)$, then

$$(7)\qquad \frac{1}{\zeta - z} = \sum_{n=0}^{\infty} (J_n^{(\alpha)})^{-1} L_n^{(\alpha)}(z) M_n^{(\alpha)}(\zeta).$$

Moreover, the series in (7) is uniformly convergent on every (finite) arc of $p(\lambda)$ provided that $z \in \Delta(\lambda)$ is fixed.

Let $f \in F(\alpha, \lambda_0)$ and suppose that f has a Cauchy integral representation in $\Delta(\lambda_0)$. If $z \in \Delta(\lambda_0)$, we choose $\lambda \in (\mathrm{Re}\, -(z)^{1/2}, \lambda_0)$, then multiply the series (7) by $f(\zeta)$ and integrate on $p(\lambda)$, which is possible since (5) holds. Therefore f has in $\Delta(\lambda_0)$ a representation by a Laguerre series with coefficients given by (4).

Conversely, let f satisfy (5) and have in $\Delta(\lambda_0)$ a representation by a Laguerre series with coefficients (4). Then, using (6) and (7), we find that for every $\lambda \in (0, \lambda_0)$ and every $z \in \Delta(\lambda)$ we have (3), i.e., f has a Cauchy integral representation in the region $\Delta(\lambda_0)$.

References

[1] P. Rusev, *An inequality for Laguerre functions of second kind*, C. R. Acad. Bulg. Sci. 30 (1977), 661–663 (in Russian).

[2] G. Szegö, *Orthogonal polynomials*, New York 1959.

Reçu par la Rédaction le 10.02.1984