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Imbedding of a space with an affine connection in the affine space

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1. Let us consider an n-dimensional hypersurface M^n in an affine N-dimensional space A^N. Let us attach to every point u of M^n an n-dimensional plane $S(u)$ and an $(N-n)$-dimensional plane $S^*(u)$ in such a manner that they have one common point. Galvani [1] has defined an induced connection on the hypersurface M^n by the projection of the n-plane $S(u+du)$ on the plane $S(u)$ parallel to $S^*(u)$. The torsion of the connection thus defined is, in general, different from zero.

Let us take an n-dimensional space L^n with an affine connection. The following problem arises: *for which N is it possible to imbed locally the space L^n in the space A^N so that the induced connection on the imbedded space will coincide with the connection of L^n?*

Let the forms ω^i, ω^i_k (1) define a connection of L^n and the forms θ^I, θ^I_K a connection on A^N. The forms ω and θ satisfy the well-known equations of structure

$$D\omega^i = [\omega^k \omega^i_k] + R^i_{kl}[\omega^k \omega^l]\,, \qquad D\omega^i_j = [\omega^k_j \omega^i_k] + R^i_{jkl}[\omega^k \omega^l]\,,$$
$$D\theta^I = [\theta^K \theta^I_K]\,, \qquad D\theta^I_L = [\theta^K_L \theta^I_K]\,.$$

The problem of imbedding is reduced to the problem of the existence of the solution of the Pfaff system $\theta^i = \omega^i$, $\theta^i_j = \omega^i_j$, the closed system of which is the following (see [2]):

$$\theta^i = \omega^i\,, \qquad \theta^i_j = \omega^i_j\,, \qquad [\theta^a_{j'} \theta^i_a] = R^i_{j'kl}[\omega^k \omega^l]\,, \tag{1}$$

where $\theta^a_0 = \theta^a$, $R^i_{0kl} = R^i_{kl}$.

Galvani [1] has shown that system (1) is involutive for $N = n^2$, while Rybnikov [3] has proved this for $N = (n^2+3n-2)/2$ (n odd) and

(1) The indices i, j, k, l run over the set of numbers $1, \ldots, n$, the indices j', k', over the set $0, 1, \ldots, n$, the indices I, K, L over the set $1, \ldots, N$ and the index a runs over the set $n+1, \ldots, N$. The set of values of other indices used are defined in the text.

$N = (n^2+4n-4)/2$ (n even); moreover (see [2]), he has proved that if $n = 3$, then $N = 7$. In this paper we reduce the dimension of A^N to $N = \frac{1}{2}\{n^2+2n-\frac{1}{2}[1-(-1)^n]\}$ but at the cost of some assumptions concerning the curvature tensor and torsion tensor we use Kaehler's method.

2. The characteristic forms of system (1) are the following: $n\omega^i$-forms, $(N-n)(2n+1)\theta^a_{j'}$-forms and θ^i_a-forms.

We have to find an integral hypersurface M^n on which $[\omega'\ldots\omega^n] \neq 0$. Thus on M^n we have

$$\theta^a_{k'} = l^a_{k'i}\omega^i\,, \qquad \theta^k_a = l^k_{ai}\omega^i\,, \tag{2}$$

where the coefficients $l^a_{k'i}$, l^k_{ai} are analytic functions of a point $u \in M^n$.

We construct a set of integral elements: at first $E_1(e_1)$, then $E_1(e_1, e_2)$, ..., and finally $E_n(e_1, \ldots, e_n)$. Any linear element is defined by the equations

$$\omega^1 = \ldots = \omega^{i-1} = 0\,, \quad \omega^i \neq 0\,, \quad \omega^{i+1} = \ldots = \omega^n = 0 \tag{3}$$

and equations (2).

The parameters $l^a_{k'1}$, l^k_{a1} which define the element e_1 may be chosen arbitrarily provided they are not all equal to zero.

Every linear element e_μ ($\mu = 2, \ldots, n$) is involutive to the elements $e_1, \ldots, e_{\mu-1}$. Hence the coefficients $l^a_{k'1}, l^k_{a1}; \ldots; l^a_{k'n}, l^k_{an}$ (for abbreviation denoted in the sequel by $l_1, \ldots, l_n$) satisfy the algebraic system

$$l^a_{k'\lambda}l^i_{a\mu} - l^i_{a\lambda}l^a_{k'\mu} = 2R^i_{k'\lambda\mu} \qquad (\lambda = 1, \ldots, \mu-1) \tag{F_μ}$$

following from (1), (2) and (3). The parameters l_μ in system (F_μ) we regard as unknown values and the parameters $l_1, \ldots, l_{\mu-1}$ as known ones. For a fixed index λ the part of system (F_μ) is denoted by ($\mathrm{F}_{\lambda\mu}$). Every system ($\mathrm{F}_{\lambda\mu}$) contains $n-1$ equations. System (F_n) contains $n_1 = n(n^2-1)$ equations.

Let us arrange the unknown values l_n and the tensors on the right-hand side of (F_n) as follows

$$l^1_{an}, l^2_{an}, \ldots, l^n_{an}, l^a_{0n}, l^a_{1n}, \ldots, l^a_{nn};$$

$$R^1_{k'1n}, \ldots, R^1_{k',n-1,n}; \ldots; R^n_{k'1n}, \ldots, R^n_{k',n-1,n}$$

and let us write

$$\begin{pmatrix} l^a_{k'1} \\ l^a_{k'2} \\ \cdots \\ l^a_{k',n-1} \end{pmatrix} = L\,, \qquad \begin{pmatrix} -l^i_{ak\bullet} & O_1 & \ldots & O_1 \\ O_1 & -l^i_{ak\bullet} & \ldots & O_1 \\ \cdots & \cdots & \cdots & \cdots \\ O_1 & O_1 & \ldots & -l^i_{ak\bullet} \end{pmatrix} = L^i_{k\bullet}\,, \qquad \begin{pmatrix} L^i_1 \\ L^i_2 \\ \cdots \\ L^i_{n-1} \end{pmatrix} = L^i\,,$$

where $l^{\alpha}_{k'1}, \ldots, l^{\alpha}_{k',n-1}$ denotes the matrices with $N-n$ columns and $n+1$ rows and O_1 is the one-row zero-matrix with $N-n$ elements, $k^* = 1, \ldots, n-1$. Hence the matrix of (F_n) has the form

$$\begin{pmatrix} L & O & \ldots & O & L^1 \\ O & L & \ldots & O & L^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ O & O & \ldots & L & L^n \end{pmatrix}, \tag{4}$$

where O is the zero-matrix with $N-n$ columns and n^2-1 rows.

(F_n) has a solution different in general from zero if and only if the number of unknown values, i.e. $(N-n)(2n+1)$, is not less than n_1 and the rank of the matrix of (F_n) is equal to n_1. Rybnikov [2] has shown that $[n^2(n+2)/(2n+1)]+1$ is the least number for which the first condition is satisfied. It is unknown whether for that dimension of A^N the second condition is satisfied. Both conditions are fulfiled for $N = \{n^2 + 2n - \frac{1}{2}[1-(-1)^n]\}/2$.

If we represent the number $\left[\frac{n^2(n+2)}{(2n+1)}\right]+1$ as $\frac{[n^3+\varphi(n)]}{(2n+1)}+n$, where $-n+1 \leqslant \varphi(n) \leqslant n+1$ and $2n+1$ is the divisor of $n^3+\varphi(n)$, then it is easy to prove that

$$\frac{n^2+2n-\frac{1}{2}[1-(-1)^n]}{2} \geqslant \left[\frac{n^2(n+2)}{2n+1}\right]+1 .$$

For the initial n we have:

n	2	3	4	5	6	7	8
$\left[\frac{n^2(n+2)}{2n+1}\right]+1$	4	7	11	16	23	30	38
$\frac{n^2+2n-\frac{1}{2}[1-(-1)^n]}{2}$	4	7	12	17	24	31	40

It is easy to see that for $n = 2$ or 3 there exist such values of the parameters l_1 or l_1 and l_2 that the rank of matrix (4) is equal to $n_1 = 6$ or to $n_1 = 24$ for $N = 4$ or $N = 7$. The dimension of A^N in these two cases coincides with the dimension obtained by Rybnikov in [2] and [3] and for $n = 2$ with that obtained by Galvani in [1].

3. We now construct a set of integral elements.

The element e_1. We choose $l^i_{n+j,1}$ such that

$$\mathrm{r} l^i_{n+j,1} = n , \tag{5}$$

where $\mathrm{r} l^i_{n+j,1}$ denotes the rank of the matrix $l^i_{n+j,1}$; we take other l_1's equal to zero.

The element e_2. The parameters l_2 satisfy the system

$$(\mathrm{F}_2) \qquad -l^i_{n+j,1} l^{n+j}_{k'2} = 2R^i_{k'12} .$$

From (5) it follows that the determinant D_1 of (F_2) is different from zero and

$$(6) \qquad D_1 = -|l^i_{n+j,1}|^{n+1} .$$

If

$$(\mathrm{H}_2) \qquad \mathrm{r} R^i_{k'12} = n$$

then $\mathrm{r} l^{n+j}_{k'2} = n$. We choose the parameters $l^{2n+1}_{k'2}$ so that

$$(7) \qquad \mathrm{r} l^{n+1+j'}_{k'2} = n+1$$

and other l_2's are taken equal to zero.

The element e_3. The parameters l_3 satisfy system

$$(\mathrm{F}_3) \qquad \begin{aligned} -l^i_{n+j,1} l^{n+j}_{k'3} \qquad\qquad\qquad\qquad &= 2R^i_{k'13} , \\ -l^i_{n+j,2} l^{n+j}_{k'3} + l^i_{n+1+j',3} l^{n+1+j'}_{k'2} &= 2R^i_{k'23} . \end{aligned}$$

From (6) and (7) it follows that the determinant D_2 of (F_3) is different from zero and

$$(8) \qquad D_2 = D_1 |l^{n+1+j'}_{k'2}|^n .$$

Let us suppose that in the solution of (F_3) the parameters $l^i_{2n+1,3}$ are not all equal to zero, i.e.

$$(\mathrm{H}_3) \qquad \mathrm{r} l^i_{2n+1,3} = 1 ,$$

where $l^i_{2n+1,3}$ is a one-row matrix. Let us choose the parameters $l^i_{2n+2,3}, \ldots, l^i_{3n,3}$ in such a way that

$$(9) \qquad \mathrm{r} l^i_{2n+k,3} = n ,$$

and let us take other l_3's equal to zero.

The element e_4. The parameters l_4 satisfy the system

$$(\mathrm{F}_4) \qquad \begin{aligned} -l^i_{n+j,1} l^{n+j}_{k'4} \qquad\qquad\qquad\qquad\qquad\qquad\qquad &= 2R^i_{k'14} , \\ -l^i_{n+j,2} l^{n+j}_{k'4} + l^i_{n+1+j',4} l^{n+1+j'}_{k'2} \qquad\qquad\qquad &= 2R^i_{k'24} , \\ -l^i_{n+j,3} l^{n+j}_{k'4} + l^i_{n+1+j',4} l^{n+1+j'}_{k'3} - l^i_{2n+j,3} l^{2n+j}_{k'4} &= 2R^i_{k'34} . \end{aligned}$$

From (8) and (9) it follows that the determinant D_3 of (F_4) is different from zero and

$$D_3 = -D_2 |l^i_{2n+j,3}|^{n+1} .$$

Let us suppose that the solution of (F_4) satisfies the conditions

$$(\mathrm{H}_4) \qquad \mathrm{r} \| l^{2n+2}_{k'4} \ \ldots \ l^{3n}_{k'4} \| = n-1 ,$$

where $\|l_{k'4}^{2n+2} \; \dots \; l_{k'4}^{3n}\|$ denotes a matrix with $n-1$ columns and $n+1$ rows; let us choose the parameters $l_{k'4}^{3n+1}, l_{k'4}^{3n+2}$ so that

$$\mathrm{r}\, l_{k'4}^{2n+2+j'} = n+1;$$

we take other l_4's equal to zero.

By this procedure and by assumptions $(\mathrm{H}_2), (\mathrm{H}_3), \dots$ we can construct a set of linear elements in such a way that for any element

$$e_{2\nu} \quad (\nu = 1, \dots, \tfrac{1}{2}\{n-2+\tfrac{1}{2}[1-(-1)^n]\})$$

or

$$e_{2\nu+1} \quad (\nu = 1, \dots, \tfrac{1}{2}\{n-2+\tfrac{1}{2}[1-(-1)^n]\})$$

the determinant $D_{2\nu-1}$ or $D_{2\nu}$ of the system $(\mathrm{F}_{2\nu})$ or $(\mathrm{F}_{2\nu+1})$ respectively is different from zero and

$$D_{2\nu-1} = -D_{2\nu-2}\,|l^i_{\nu n+j,2\nu-1}|^{n+1} \tag{10}$$

or

$$D_{2\nu} = D_{2\nu-1}\,|l_{k',2\nu}^{\nu n+\nu+j'}|^n \,. \tag{11}$$

We suppose that the solution of $(\mathrm{F}_{2\nu})$ or $(\mathrm{F}_{2\nu+1})$ satisfies the conditions

$$(\mathrm{H}_{2\nu}) \qquad \mathrm{r}\,\|l_{k',2\nu}^{\nu n+\nu} \; \dots \; l_{k',2\nu}^{\nu n+n}\| = n-\nu+1\,,$$

where $(l_{k',2\nu}^{\nu n+\nu} \; \dots \; l_{k',2\nu}^{\nu n+n})$ denotes a matrix with $n-\nu+1$ columns and $n+1$ rows or

$$(\mathrm{H}_{2\nu+1}) \qquad \mathrm{r}\,\|l^i_{\nu n+\nu+1,2\nu+1} \; \dots \; l^i_{\nu n+2\nu,2\nu+1}\| = \nu\,,$$

where $(l^i_{\nu n+\nu+1,2\nu+1} \; \dots \; l^i_{\nu n+2\nu,2\nu+1})$ denotes a matrix with ν rows and n columns.

We choose the parameters

$$l_{k',2\nu}^{\nu n+n+1}, \dots, l_{k',2\nu}^{\nu n+n+\nu}$$

or

$$l^i_{\nu n+2\nu+1,2\nu+1}, \dots, l^i_{\nu n+2n,2\nu+1}$$

so that

$$\mathrm{r}\, l_{k',2\nu}^{\nu n+\nu+j'} = n+1 \tag{12}$$

or

$$\mathrm{r}\, l^i_{\nu n+n+k,2\nu+1} = n; \tag{13}$$

we take other $l_{2\nu}$'s or $l_{2\nu+1}$'s equal to zero except $l_{k',n-1}^{\frac{1}{2}n(n+1)}$ or $l^i_{\frac{1}{2}n(n+1),n-1}$ which we take equal to 1.

The element e_n. The parameters l_n for n even satisfy the system

$$(\mathbf{F}_n)\quad \begin{cases} & (\mathbf{F}_{1n}), \ldots, (\mathbf{F}_{n-2,n}), \\ (\mathbf{F}_{n-1,n}): & -l^i_{n+j,n-1} l^{n+j}_{k'n} + \ldots - l^i_{\frac{1}{2}n^2+j,n+1} l^{\frac{1}{2}n^2+j}_{k'n} = 2R^i_{k',n-1,n} - l^i_{\frac{1}{2}n(n+1),n}. \end{cases}$$

It follows from (11) and (13) for $\nu = (n-2)/2$ that the determinant D_{n-1} of $(\mathbf{F}_n)$ is different from zero and

$$D_{n-1} = -D_{n-1} |l^i_{\frac{1}{2}n^2+j,n-1}|^{n+1} \quad (n \text{ even}). \tag{14}$$

The parameters l_n for n odd satisfy the system

$$(\mathbf{F}_n)\quad \begin{cases} & (\mathbf{F}_{n1}), \ldots, (\mathbf{F}_{n-2,n}), \\ (\mathbf{F}_{n-1,n}): & -l^i_{n+j,n-1} l^{n+j}_{k'n} + \ldots + l^i_{\frac{1}{2}(n^2-1)+j',n} l^{\frac{1}{2}(n^2-1)+j'}_{k',n-1} = 2R^i_{k',n-1,n} + l^{\frac{1}{2}n(n+1)}_{n'n}. \end{cases}$$

It follows from (10) and (12) for $\nu = (n-1)/2$ that the determinant D_{n-1} of $(\mathbf{F}_n)$ is different from zero and

$$D_{n-1} = D_{n-2} \left| l^{\frac{1}{2}(n^2-1)+j'}_{k',n-1} \right|^n \quad (n \text{ odd}). \tag{15}$$

In $(\mathbf{F}_n)$ for n even and for n odd the parameters $l^i_{\frac{1}{2}n(n+1),n}$ (n even) and $l^{\frac{1}{2}n(n+1)}_{k'n}$ (n odd) are arbitrary values. Therefore, if necessery, we take them so that $(\mathbf{F}_n)$ is not a uniform system. Hence the solution of $(\mathbf{F}_n)$ for n even and for n odd is not the zero solution.

4. Our construction may be performed as above if the dimension N of A^n is not smaller than the greatest index $\frac{1}{2}n^2+n$ in (14) or the greatest index $\frac{1}{2}(n^2-1)+n$ in (15) respectively.

The curvature and torsion tensors of a space with an affine connection are called affine curvature tensor and affine torsion tensor.

Hypotheses $(\mathbf{H}_3), \ldots, (\mathbf{H}_{n-1})$ evidently like $(\mathbf{H}_2)$, are quality restrictions imposed on curvature and torsion tensors. Therefore the above construction is performable for almost every space (in the sense of measure in the space of all affine curvature tensors and affine torsion tensors) with an affine connection.

Hence we get the following

THEOREM. *If an anlytic space L^n with an affine connection is given, then L^n may be, in general, locally imbedded into the affine space A^N of dimension $N = \frac{1}{2}\{n^2+2n-\frac{1}{2}[1-(-1)^n]\}$ in such a way that the connection induced on the imbedded space in the sense of Galvani coincides with the given connection.*

We suppose that this theorem is also true without the words "in general" and we hope that the theorem thus obtained follows from the above and that the proof may be obtained without using the methods of the involutive Pfaff system.

References

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