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Generic submanifolds in almost Hermitian manifolds

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Abstract. By a generic submanifold in an almost complex manifold (M', J') we shall mean a real submanifold M of M' for which $\dim J' T_x M \cap T_x M$ is constant on M. The purpose of this paper is to make some observations and give some statements dealing with differential geometry of generic submanifolds.

1. Preliminaries. All the objects considered in this paper are assumed to be of class C^∞. Manifolds are assumed to be connected and paracompact.

Let (M', J', g') be an almost Hermitian manifold where J' is an almost complex structure and g' is the Riemannian metric tensor field on M' and let M be a real submanifold in M'. TM' and TM will denote the tangent bundles of M' and M, respectively. We also set:

$TM'|_M$ – the restriction of TM' to M, g – the restriction of g' to M, $\mathcal{N}$ – the normal bundle of TM in $TM'|_M$, p – the projection onto TM in $TM'|_M = TM \oplus \mathcal{N}$, n – the projection onto $\mathcal{N}$ in $TM'|_M = TM \oplus \mathcal{N}$, $P = p \circ J'|_{TM}$, $\psi = n \circ J'|_{TM}$, $\mathcal{D}_x = T_x M \cap J' T_x M$ for $x \in M$, $\mathcal{H}_x$ – the holomorphic extension of $T_x M$ in $T_x M'$, i.e., $\mathcal{H}_x = T_x M + J' T_x M$ for $x \in M$, $\mathcal{C}_x = \operatorname{im} P_x$ for $x \in M$, $\mathcal{C}_x^\perp$ – the orthogonal complement to $\mathcal{C}_x$ in $T_x M$, $\mathcal{D}_x^\perp$ – the orthogonal complement to $\mathcal{D}_x$ in $T_x M$, $\mathcal{D}_{0_x}$ – the orthogonal complement to $T_x M$ in $\mathcal{H}_x$, $\mathcal{N}\mathcal{H}_x$ – the orthogonal complement to $\mathcal{H}_x$ in $T_x M'$, ∇, ∇' – the Riemannian connections generated by g and g', respectively, D – the normal connection, i.e., the connection in $\mathcal{N}$ induced by ∇', α – the second fundamental form of M in M', A – the second fundamental tensor of M in M'.

We shall denote by $(\ ,\ )$ the metric tensor g' as well as g. The induced norms will be denoted by $\|\ \|$. It is easily seen that $\mathcal{N}\mathcal{H}_x \subset \mathcal{N}_x$, $\mathcal{D}_{0_x} \subset \mathcal{N}_x$, $\mathcal{N}_x = \mathcal{D}_{0_x} \oplus \mathcal{N}\mathcal{H}_x$, $\mathcal{D}_{0_x}$ and $\mathcal{N}\mathcal{H}_x$ are orthogonal, $\mathcal{D}_x$ is an orthogonal complement to $J'\mathcal{N}_x + \mathcal{N}_x$ in $T_x M'$, P is a $(1, 1)$ tensor field on M, ψ is an $\mathcal{N}$-valued 1-form on M. It is also easy to check that ψ_x is an epimorphism onto $\mathcal{D}_{0_x}$. In fact, if $\xi \in \mathcal{D}_{0_x}$, then $\xi \in \mathcal{H}_x$ so $\xi = J'X + Z$, where X and Z belong to $T_x M$. Hence $\xi = n\xi = nJ'X$. It is also obvious that $\ker \psi_x = \mathcal{D}_x$. Therefore, $\psi_x|_{\mathcal{D}_x^\perp}\colon \mathcal{D}_x^\perp \to \mathcal{D}_{0_x}$ is an isomorphism. Of course, $P|_{\mathcal{D}_x} = J'|_{\mathcal{D}_x}$. Since

$(pJ'X, Y) = -(X, pJ'Y)$, $(pJ'X, Y) = 0$ for $X \in T_x M$ and $Y \in \ker P_x$. It means that $\operatorname{im} P_x$ is orthogonal to $\ker P_x$ and consequently $\operatorname{im} P_x$ is an orthogonal complement to $\ker P_x$ in $T_x M$, i.e., $\mathcal{C}_x^{\perp} = \ker P_x$. If $X \in \mathcal{D}_x^{\perp}$ and $Y \in \mathcal{D}_x$, then $(pJ'X, Y) = -(X, J'Y) = 0$ so $P\mathcal{D}_x^{\perp} \subset \mathcal{D}_x^{\perp}$. Since $\mathcal{D}_x \subset \operatorname{im} P_x$, $\mathcal{C}_x^{\perp} \subset \mathcal{D}_x^{\perp}$. Let $X \in T_x M$. We have $(X, X) = (J'X, J'X) = (PX, PX) + (\psi X, \psi X)$. So $\|X\| = \|PX\|$ iff $\psi X = 0$. Therefore $\mathcal{D}_x = \{X \in T_x M;\ \|X\| = \|PX\|\}$. Notice also that $\mathcal{N}'\mathcal{H}_x$ and $\mathcal{D}_x$ are the greatest J'-invariant subspaces in $\mathcal{N}'_x$ and $T_x M$ respectively.

Denote by R and R' the Riemannian curvature tensor (of type (0,,4)) of M and M', respectively. Recall that if q and q' are J'-invariant planes in $T_x M'$, then the holomorphic bisectional curvature by q and q' is defined by

$$H'_B(q, q') = R'(X, J'X, Y, J'Y),$$

where X, X are unit vectors in q and q', respectively. We shall write $H'_B(X, Y)$ instead of $R'(X, JX', Y, JY')$. If M' is Kählerian, then by the Bianchi identity we have

$$H'_B(X, Y) = R'(X, Y, X, Y) + R'(X, J'Y, XJ'Y).$$

Recall also the Gauss equation

$$R'(W, Z, X, Y) = R(W, Z, X, Y) + (\alpha(X, Z), \alpha(Y, W)) - (\alpha(Y, Z), \alpha(X, W)),$$

and the relationship between the second fundamental form and the second fundamental tensor

$$(\alpha(X, Y), \xi) = (A_\xi X, Y).$$

Definition 1.1. A real submanifold M of an almost complex manifold (M', J') is called *generic* if $\dim \mathcal{D}_x$ is constant on M.

If M is generic, then $\mathcal{D} \stackrel{\mathrm{df}}{=} \bigcup_{x \in M} \mathcal{D}_x$, $\mathcal{D}^{\perp} \stackrel{\mathrm{df}}{=} \bigcup_{x \in M} \mathcal{D}_x^{\perp}$ are distributions on M and $\mathcal{N}'\mathcal{H} \stackrel{\mathrm{df}}{=} \bigcup_{x \in M} \mathcal{N}'\mathcal{H}_x$, $\mathcal{H} \stackrel{\mathrm{df}}{=} \bigcup_{x \in M} \mathcal{H}_x$, $\mathcal{D}_0 \stackrel{\mathrm{df}}{=} \bigcup_{x \in M} \mathcal{D}_{0_x}$ are vector subbundles of $TM'|_M$. The distribution $\mathcal{D}$ is parallel (with respect to ∇) iff the distribution $\mathcal{D}^{\perp}$ is parallel and iff the almost product structure $(\mathcal{D}, \mathcal{D}^{\perp})$ is parallel. The vector subbundle $\mathcal{N}'\mathcal{H}$ is parallel (with respect to D) iff $\mathcal{D}_0$ is parallel.

If M is generic and $\dim \mathcal{D} = 0$, then M is called *purely real*. If $\dim \mathcal{D} = \dim M$, then M is called *holomorphic*, A generic submanifold M is called *proper* if $\dim M \neq \dim \mathcal{D} \neq 0$. A generic submanifold M in an almost Hermitian manifold M' is called *CR-submanifold* (or *metric f-submanifold* [6]) if $\mathcal{D}_{0_x} = J'\mathcal{D}_x^{\perp}$ for any $x \in M$. A generic submanifold need not be a CR-submanifold [6]. A CR-submanifold is called *totally real* if it is purely real. We have

Lemma 1.1 ([5]). *The condition*: $\mathcal{D}_{0_x} = J'\mathcal{D}_x^{\perp}$ *for every* $x \in M$ *implies that* M *is generic.*

If M is a CR-submanifold, then $P|_{\mathscr{D}^\perp} = 0$. So $(\psi \circ P)(X) = 0$ for $X \in \mathscr{D}^\perp$. If $X \in \mathscr{D}$, then

$$(\psi \circ P)(X) = \psi J' X = nJ' J' X = -nX = 0.$$

Assume now that $\psi \circ P = 0$. For any $x \in T_x M$ we have $0 = (\psi \circ P)(X) = nJ' pJ' X$. Hence $J' pJ' X \in T_x M$ and consequently $pJ' X \in \mathscr{D}_x$. If $X \in \mathscr{D}_x^\perp$, then $pJ' X \in \mathscr{D}_x$ only if $pJ' X = 0$. Therefore, $J' X \in \mathscr{N}_x$ for $X \in \mathscr{D}_x^\perp$. Consequently $\psi \circ P = 0$ iff $J' \mathscr{D}_x^\perp = \mathscr{D}_{0_x}$ for every $x \in M$. By virtue of Lemma 1.1, M is a CR-submanifold iff $\psi \circ P = 0$.

A submanifold M is totally geodesic if $\alpha = 0$. A real submanifold M in an almost Hermitian manifold M' is called *mixed totally geodesic* if $\alpha(X, Y) = 0$ for any $X \in \mathscr{D}_x$, $Y \in \mathscr{D}_x^\perp$ and $x \in M$.

Remark 1.1. D. Blair and B-Y Chen proved in [1] that a CR-submanifold of a complex manifold has a natural structure of a CR-manifold. It should be remarked that a generic submanifold of a complex manifold carries a natural structure of a CR-manifold. Recall now the notion of a CR-manifold, [4].

Definition 1.2. A *CR-manifold* is a pair $(M, \mathscr{A}(M))$, where M is a real differentiable manifold and $\mathscr{A}(M)$ is a complex subbundle of $TM \otimes C$. The following conditions are satisfied:

(a) $\mathscr{A}(M) \cap \overline{\mathscr{A}(M)} = \{0\}$,

(b) $\mathscr{A}(M)$ is involutive, i.e., for any complex vector fields X and Y with values in $\mathscr{A}(M)$, $[X, Y]$ has values in $\mathscr{A}(M)$.

We have

Proposition 1.2. *A manifold M admits a structure of a CR-manifold if and only if there is an f-structure on M such that*

$$[fX, fY] - [X, Y] - f[X, fY] - f[fX, Y] = 0 \tag{1.1}$$

for any vector fields X, Y belonging to the distribution $\mathscr{D} = \operatorname{im} f$.

Proof. By virtue of Theorem 2 from [4], p. 280, we know that (a) from Definition 1.2 is equivalent to the existence of a distribution $\mathscr{D}$ on M and a field J of endomorphisms of $\mathscr{D}$ such that $J^2 = -\mathrm{id}_{\mathscr{D}}$. Moreover, $\mathscr{D} = \operatorname{re}(\mathscr{A}(M) + \overline{\mathscr{A}(M)})$ and $\mathscr{A}(M) = \{X - iJX, X \in \mathscr{D}\}$. If M is a CR-manifold, then we define

$$f(X) = \begin{cases} JX & \text{for } X \in \mathscr{D}, \\ 0 & \text{for } X \in \mathscr{D}^\perp, \end{cases}$$

where $\mathscr{D}^\perp$ is the orthogonal complement to $\mathscr{D}$ in TM. Conversely, if f is an f-structure on M, then we define $\mathscr{D}$ as $\operatorname{im} f$ and J as the restriction of f to $\mathscr{D}$.

Therefore, the condition (a) from Definition 1.2 is equivalent to the existence of an f-structure on M. We have

$$(1.2)\quad [X-iJX,\ Y-iJY] = [X,\ Y]-i[JX,\ Y]-i[X,\ JY]-[JX,\ JY] = ([X,\ Y]-[JX,\ JY])-i([JX,\ Y]+[X,\ JY]).$$

If M is a CR-manifold, then for vector fields X, Y belonging to $\mathcal{D}$ there is a vector field Z belonging to $\mathcal{D}$ and such that

$$[X,\ Y]-[JX,\ JY] = Z \quad \text{and} \quad [JX,\ Y]+[X,\ JY] = JZ,$$

i.e.,

$$TM \ni -[X,\ Y]+[JX,\ JY] = J([JX,\ Y]+[X,\ JY]) \in JTM.$$

It follows that

$$[fX, fY]-[X,\ Y]-f[fX,\ Y]-f[X, fY] = 0.$$

Conversely, if (1.1) holds then for any vector fields belonging to $\mathcal{D}$

$$[JX,\ JY]-[X,\ Y] = f([JX,\ Y]+[X,\ JY]).$$

It means that $[JX, JY]-[X, Y] \in \operatorname{im} f = \mathcal{D}$ and $[JX, Y]+[X, JY] = -([JX, J(JY)]-[X, JY]) \in \mathcal{D}$. Consequently

$$[JX,\ JY]-[X,\ Y] = J([JX,\ Y]+[X,\ JY]).$$

Putting this in (1.2), we conclude that M is involutive. The proof is complete.

From now on a CR-manifold we shall mean a manifold equipped with an f-structure satisfying (1.1).

COROLLARY 1.3. *A generic submanifold of a complex manifold is a CR-manifold.*

Proof. Let M be a generic submanifold of a complex manifold (M', J'). Since J' is integrable

$$[fX, fY]-[X,\ Y] = J'([fX,\ Y]+[X, fY])$$

for any vector fields X, Y belonging to $\mathcal{D}$, where f is the induced f-structure on M. But it means that $[fX, fY]-[X, Y]$ and $[fX, Y]+[X, fY]$ belong to $TM \cap J'TM = \mathcal{D}$. Hence $J'([fX, Y]+[X, fY]) = f[fX, Y]+f[X, fY]$, i.e., (1.1) holds good.

In paper [4] can be found the following assertion (Theorem 3, p. 281):

If M is a manifold satisfying (a) of Definition 1.2, then (b) is equivalent to the following

$$[X,\ Y]+J[JX,\ Y]+J[X,\ JY]-[JX,\ JY] = 0$$

for X, Y belonging to $\mathcal{D}$. But we cannot write $J[X, JY]$ since $[X, JY]$ need not belong to $\mathcal{D}$, i.e., the distribution $\mathcal{D}$ need not be integrable. Notice also that if M is a manifold equipped with an integrable f-structure, then it is a CR-manifold.

2. Structures induced on real submanifolds in almost Hermitian manifolds. For the 1-form ψ we put

$$(\nabla_X \psi) Y = D_X \psi Y - \psi \nabla_X Y$$

for any vector fields X and Y on M. The form ψ is said to be *parallel* if $\nabla\psi = 0$. Note that ψ and P are defined for arbitrary, not necessarily generic, real submanifolds in almost Hermitian manifolds. We shall prove the following

PROPOSITION 2.1. *Let M be a submanifold in an almost Hermitian manifold M'. If $\nabla\psi = 0$ or $\nabla P = 0$, then M is generic and the almost product structure $(\mathscr{S}, \mathscr{S}^\perp)$ is parallel with respect to ∇.*

Proof. Let $x, y \in M$ and let τ be a curve joining x and y. Let $X \in \mathscr{S}_x$. Denote by $\tilde{X}$ the vector field defined along τ, obtained by the parallel displacement of X along τ. Therefore, $\nabla_{\dot{\tau}} \tilde{X} = 0$, where $\dot{\tau}$ is the velocity vector field of τ. If $\nabla\psi = 0$, then we have

$$0 = (\nabla_{\dot{\tau}} \psi) \tilde{X} = D_{\dot{\tau}}(\psi \tilde{X}) - \psi(\nabla_{\dot{\tau}} \tilde{X}) = D_{\dot{\tau}}(\psi \tilde{X}).$$

Since the connection D is metric,

$$\dot{\tau}(\psi\tilde{X}, \psi\tilde{X}) = 2(D_{\dot{\tau}} \psi\tilde{X}, \psi\tilde{X}) = 0.$$

It means that $\|\psi\tilde{X}\|$ is constant along τ. But $(\psi\tilde{X})_x = 0$, so $(\psi\tilde{X})_y = 0$. Therefore, $\tau_y^x(\ker\psi_x) \subset \ker\psi_y$, i.e., $\tau_y^x(\mathscr{S}_x) \subset \mathscr{S}_y$, where τ_y^x denotes the parallel displacement along τ from x to y. By the same reason, $\tau_x^y(\mathscr{S}_y) \subset \mathscr{S}_x$. Consequently, $\tau_y^x(\mathscr{S}_x) = \mathscr{S}_y$.

Assume now that $\nabla P = 0$. In particular, we have $\nabla_{\dot{\tau}} P\tilde{X} = P\nabla_{\dot{\tau}} \tilde{X} = 0$. It means that $\tau_y^x(PX) = P\tau_y^x(X)$. The parallel displacement is an isometry and $X \in \mathscr{S}_x = \{Y \in T_x M\colon \|Y\| = \|PY\|\}$ so $\|\tau_y^x X\| = \|X\| = \|PX\| = \|\tau_y^x(PX)\| = \|P\tau_y^x(X)\|$. It means that $\tau_y^x(X) \in \mathscr{S}_y$. So $\tau_y^x(\mathscr{S}_x) \subset \mathscr{S}_y$ and like in the previous case we conclude that $\tau_y^x(\mathscr{S}_x) = \mathscr{S}_y$, i.e., $\dim \mathscr{S}_x$ is constant on M and $\mathscr{S}$ is parallel. The proof is complete.

Using the same method as in the above proof, we obtain

PROPOSITION 2.2. *Let M be a real submanifold in an almost Hermitian manifold M'. If $\nabla P = 0$, then $\mathscr{C} \stackrel{\mathrm{df}}{=} \bigcup_{x \in M} \mathscr{C}_x$, $\mathscr{C}^\perp \stackrel{\mathrm{df}}{=} \bigcup_{x \in M} \mathscr{C}_x^\perp$ are distributions parallel with respect to ∇.*

For any real submanifold in an almost Hermitian manifold we have the following equality

$$(\nabla'_X J') Y = (\nabla_X P) Y - K_1(X, Y) + (\nabla_X \psi) Y - K_2(X, Y),$$

where $K_1(X, Y) = pJ'\alpha(X, Y) + A_{\psi Y} X$ and $K_2(X, Y) = nJ'\alpha(X, Y) - \alpha(X, PY)$.

In fact

$$\begin{aligned}(\nabla'_X J')Y &= \nabla'_X J'Y - J'\nabla_X Y\\ &= \nabla'_X PY + \nabla'_X \psi Y - pJ'\nabla'_X Y - nJ'\nabla'_X Y\\ &= \nabla_X PY + \alpha(X, PY) + D_X \psi Y - A_{\psi Y} X - P\nabla_X Y -\\ &\quad - pJ'\alpha(X, Y) - \psi(\nabla_X Y) - nJ'\alpha(X, Y)\\ &= (\nabla_X P)Y - A_{\psi Y} X - pJ'\alpha(X, Y) + (\nabla_X \psi)Y - nJ'\alpha(X, Y) + \alpha(X, PY).\end{aligned}$$

K_1 is a (1, 2)-tensor field on M, K_2 is an $\mathcal{N}$-valued 2-form (not necessarily antisymmetric) on M. We shall call K_1 and K_2 *Hermitian fundamental forms* of M in M'. If M' is Kählerian, then $(\nabla_X P)Y = K_1(X, Y)$ and $(\nabla_X \psi)Y = K_2(X, Y)$. Hence if M' is Kählerian, $\nabla P = 0$ iff $K_1 = 0$ and $\nabla\psi = 0$ iff $K_2 = 0$. By Proposition 2.1 we also obtain

COROLLARY 2.3. *If M is a real submanifold in a Kählerian manifold M', then $K_1 = 0$ or $K_2 = 0$ implies that M is generic.*

Remark 2.4. The Hermitian fundamental form K_1 of a real submanifold M in an almost Hermitian manifold M' is symmetric if and only if it is zero. In fact, α is symmetric so K_1 is symmetric iff $A_{\psi Z} Y = A_{\psi Y} Z$ for any tangent vectors Y, Z. On the other hand, $(pJ'\alpha(X, Y), Z) = (J'\alpha(X, Y), Z) = -(\alpha(X, Y), nJ'Z) = -(A_{\psi Z} Y, X)$ and $(A_{\psi Y} X, Z) = (A_{\psi Y} Z, X)$. It means that $K_1 = 0$ iff $A_{\psi Y} Z = A_{\psi Z} Y$ for any tangent vectors Y, Z.

Remark 2.5. Assume that M is a CR-submanifold in an almost Hermitian manifold M' and K_2 is symmetric. Then $\alpha(X, PY) = \alpha(Y, PX)$ for any tangent vectors X and Y, so $\alpha(X, J'Y) = 0$ for $X \in \mathcal{D}^{\perp}$ and $Y \in \mathcal{D}$. It follows that M is mixed totally geodesic. If $X, Y \in \mathcal{D}$ then $\alpha(X, J'Y) = \alpha(J'X, Y)$. Assume now that M is mixed totally geodesic and $\alpha(X, J'Y) = \alpha(J'X, Y)$ for any $X, Y \in \mathcal{D}$. Then $\alpha(X, PY) = \alpha(X, J'Y) = \alpha(J'X, Y) = \alpha(PX, Y)$ for $X, Y \in \mathcal{D}$ and $\alpha(X, PY) = 0 = \alpha(PX, Y)$ for $X \in \mathcal{D}^{\perp}$ and $Y \in \mathcal{D}$. Clearly $\alpha(X, PY) = 0 = \alpha(PX, Y)$ for $X, Y \in \mathcal{D}^{\perp}$. If M' is Kählerian, the condition $\alpha(X, J'Y) = \alpha(J'X, Y)$ for $X, Y \in \mathcal{D}$ is equivalent to the fact that the distribution $\mathcal{D}$ is integrable. It follows that in the case where M' is Kählerian K_2 is symmetric if and only if M is mixed foliate, i.e., the distribution $\mathcal{D}$ is involutive and M is mixed totally geodesic (this definition is taken from [3]).

Assume now that both fundamental Hermitian forms vanish. Since $K_1 = 0$, $(J'\alpha(X, Y) + A_{\psi Y} X, Z) = 0$ for any X, Y, $Z \in TM$. It means that $\alpha(X, Y) \in \mathcal{N}\cap\mathcal{H}$ for any $Y \in \mathcal{D}$. By the fact that $K_2 = 0$ we obtain $nJ'\alpha(X, Y) = \alpha(X, J'Y)$ for any $Y \in \mathcal{D}$. But $\alpha(X, Y) \in \mathcal{N}\cap\mathcal{H}$, so $J'\alpha(X, Y) = nJ'\alpha(X, Y)$. We have obtained the following

PROPOSITION 2.6. *Let M be a real submanifold in an almost Hermitian manifold. If both fundamental Hermitian forms vanish, then* $\alpha(X, J'Y)$

$= J'\alpha(X, Y)$ *for any* $Y \in \mathcal{D}$, $X \in TM$. *In particular,* $\alpha(X, Y) \in \mathcal{N}\mathcal{H}$, *provided* X *or* Y *belongs to* $\mathcal{D}$.

Suppose now that M is generic and M' is almost Hermitian. We have a naturally defined f-structure on M, i.e.,

$$fX = \begin{cases} 0 & \text{for } X \in \mathcal{D}^{\perp}, \\ JX & \text{for } X \in \mathcal{D}. \end{cases}$$

It is easy to check that f is an f-structure, i.e., $f^3 + f = 0$. f will be called the *induced f-structure* on M. We also put

$$F(\xi) = \begin{cases} 0 & \text{for } \xi \in \mathcal{D}_0, \\ J'\xi & \text{for } \xi \in \mathcal{N}\mathcal{H}. \end{cases}$$

Since $F^3 + F = 0$ on $\mathcal{N}$, F will be called the *induced F-structure* in the normal bundle. F will be said to be *parallel* if $\mathcal{D}_X F\xi = FD_X \xi$ for any normal vector field ξ and $X \in TM$.

As for the parallelity of the induced f-structure on M we have

PROPOSITION 2.7 ([5]). *Let M be a generic submanifold in a Kählerian manifold M'. The induced f-structure on M is parallel if and only if*

$$\alpha(X, J'Y) = J'\alpha(X, Y) \quad \textit{for } Y \in \mathcal{D} \textit{ and } X \in TM \tag{2.1}$$

(equivalently $\alpha(X, Y) \in \mathcal{N}\mathcal{H}$ *provided* X *or* Y *belongs to* $\mathcal{D}$*).*

Remark 2.8. Let M be a generic submanifold in a Kählerian manifold M'. Then $\nabla f = 0$ iff the distribution $\mathcal{D}$ (equiv. $(\mathcal{D}, \mathcal{D}^{\perp})$) is parallel, $DF = 0$ iff $\mathcal{D}_0$ (equiv. $(\mathcal{D}_0, \mathcal{N}\mathcal{H})$) is parallel. In fact, let $Y \in \mathcal{D}$. Since M is Kählerian, $J'\nabla_X Y + J'\alpha(X, Y) = \nabla_X J'Y + \alpha(X, J'Y)$.

If the distribution $\mathcal{D}$ is parallel, then $J'\nabla_X Y \in \mathcal{D}$ and $\nabla_X J'Y \in \mathcal{D}$. But $J'\alpha(X, Y)$, $\alpha(X, J'Y) \in \mathcal{N} + J'\mathcal{N}$, hence $J'\alpha(X, Y) = \alpha(X, J'Y)$. By Proposition 2.7, $\nabla f = 0$. Suppose now that the distribution $\mathcal{N}\mathcal{H}$ is parallel. Let $\xi \in \mathcal{N}\mathcal{H}$. We have

$$\begin{aligned} D_X F\xi = D_X J'\xi &= \nabla'_X J'\xi + A_{J'\xi} X = J'\nabla'_X \xi + A_{J'\xi} X \\ &= J'D_X \xi - J'A_\xi X + A_{J'\xi} X = FD_X \xi - J'A_\xi X + A_{J'\xi} X. \end{aligned}$$

Since $D_X F\xi$, $FD_X \xi \in \mathcal{N}\mathcal{H}$, $J'A_\xi X$ and $A_{J'\xi} X \in TM + J'TM = \mathcal{H}$ so $J'A_\xi X = A_{J'\xi} X$ and $D_X F\xi = FD_X \xi$. The inverse implications are trivial and hold even if M' is only almost Hermitian.

Now we can prove

PROPOSITION 2.9. *Let M be a generic submanifold in a Kählerian manifold M'. The induced F-structure in the normal bundle is parallel if and only if*

$$\alpha(X, Y) \in \mathcal{D}_0 \tag{2.2}$$

provided X or Y belongs to $\mathcal{D}^{\perp}$.

Proof. At first we shall prove that condition (2.2) is sufficient. If $X \in TM$, $Y \in \mathcal{D}^\perp$ and $\xi \in \mathcal{N}\mathcal{H}$, then, by (2.2), $(A_\xi X, Y) = (\alpha(X, Y), \xi) = 0$. Hence $A_\xi X \in \mathcal{D}$ for any $X \in TM$, $\xi \in \mathcal{N}\mathcal{H}$. Now we shall prove that the vector subbundle $\mathcal{N}\mathcal{H}$ is parallel with respect to D. It is sufficient to show that $(D_X \xi, J'Y) = 0$ for any vector fields $X \in TM$, $Y \in \mathcal{D}^\perp$ and $\xi \in \mathcal{N}\mathcal{H}$. But for such vector fields $A_\xi X \in \mathcal{D}$ and $J'Y \in \mathcal{D}^\perp \oplus \mathcal{D}_0$. It follows that $(A_\xi X, J'Y) = 0$. Therefore

$$(D_X \xi, J'Y) = -(\nabla'_X J'\xi, Y) = -(D_X J'\xi, Y) + (A_{J'\xi} X, Y).$$

Since $Y \in \mathcal{D}^\perp$ and $J'\xi \in \mathcal{N}\mathcal{H}$, $(A_{J'\xi} X, Y) = 0$. Moreover, $D_X J'\xi \in \mathcal{N}$ and $Y \in \mathcal{D}^\perp \subset TM$ so $(D_X J'\xi, Y) = 0$. Hence $(D_X \xi, J'Y) = 0$.

Assume now that the induced F-structure in the normal bundle is parallel. We have

$$D_X F\xi = \nabla'_X J'\xi + A_{J'\xi} X, \qquad FD_X \xi = J'D_X \xi = J'\nabla'_X \xi + J'A_\xi X$$

for such vector fields $A_\xi X \in \mathcal{D}$ and $J'Y \in \mathcal{D}^\perp \oplus \mathcal{D}_0$. It follows that $(A_\xi X, J'Y)$

$$A_{J'\xi} X = J'A_\xi X.$$

But it means that $A_\xi X \in TM \cap J'TM = \mathcal{D}$. Let $Y \in \mathcal{D}^\perp$. Then for any $\xi \in \mathcal{N}\mathcal{H}$, $0 = (A_\xi X, Y) = (\alpha(X, Y), \xi)$. Consequently, $\alpha(X, Y) \in \mathcal{D}_0$. The proof is complete.

Corollary 2.10. *With the same assumptions as in Proposition 2.9, we have: The induced F-structure in the normal bundle is parallel if and only if one of the following conditions is fulfilled:*

(1) $$A_{J'\xi} X = J'A_\xi X \quad \textit{for } \xi \in \mathcal{N}\mathcal{H} \textit{ and } X \in TM,$$

(2) $$A_\xi X \in D \quad \textit{for } \xi \in \mathcal{N}\mathcal{H} \textit{ and } X \in TM.$$

By virtue of the fact that a totally geodesic submanifold in a Kählerian manifold is generic and by Propositions 2.7 and 2.9 we obtain

Corollary 2.11. *Let M be a totally geodesic submanifold in a Kählerian manifold. Then induced f-structure on M and the induced F-structure in the normal bundle are parallel.*

We also have

Corollary 2.12. *Let M be a purely real submanifold in a Kählerian manifold M'. Then the induced F-structure in the normal bundle is parallel iff one of the following statements holds:*

(1) $$\alpha(X, Y) \in \mathcal{D}_0 \quad \textit{for any } X, Y \in TM,$$

(2) $$A_\xi X = 0 \quad \textit{for any } \xi \in \mathcal{N}\mathcal{H}.$$

Now we shall prove

Proposition 2.13. *Let M' be an almost Hermitian manifold and let M be* *a real submanifold in M'. If $\nabla\psi = 0$, then $\mathcal{N}\mathcal{H}$ is parallel with respect to D. If $\nabla P = 0$, then $\nabla f = 0$.*

Proof. Assume that $\nabla\psi = 0$. Let ξ be a normal vector field on M belonging to $\mathcal{N}\mathcal{H}$. Then $(\xi, J'Z) = 0$ for any tangent vector field Z. Hence $0 = X(\xi, J'Z) = X(\xi, \psi Z)$ for any tangent vector field X. But

$$X(\xi, \psi Z) = (D_X \xi, \psi Z) + (\xi, D_X \psi Z) = (D_X \xi, \psi Z) + (\xi, \psi \nabla_X Z).$$

Since $\xi \in \mathcal{N}\mathcal{H}$ and ψ has values in $\mathcal{S}_0$, $(\xi, \psi\nabla_X Z) = 0$. Hence $(D_X\xi, \psi Z) = 0$. Z has been choosen arbitrarily, ψ is onto $\mathcal{S}_0$ so $D_X\xi \in \mathcal{N}\mathcal{H}$. Assume now that $\nabla P = 0$. We know (by Proposition 2.1) that the almost product structure $(\mathcal{S}, \mathcal{S}^\perp)$ is parallel. With the aim of proving our assertion it is sufficient to show that $\nabla_X fY = f\nabla_X Y$ for $Y \in \mathcal{S}$. But if $Z \in \mathcal{S}$, then $fZ = J'Z = PZ$. So

$$\nabla_X fY = \nabla_X PY = P\nabla_X Y = f\nabla_X Y.$$

This completes the proof.

As a corollary from Propositions 2.1, 2.2, 2.7, 2.9, 2.13, we obtain

THEOREM 2.14. *Let M be a real submanifold in an almost Hermitian manifold M'.*

(1) *If $\nabla\psi = 0$, then M is generic and the almost product structure $(\mathcal{S}, \mathcal{S}^\perp)$ is parallel with respect to ∇. The vector subbundles $\mathcal{N}\mathcal{H}$ and $\mathcal{S}_0$ are parallel with respect to D.*

(2) *If M' is Kählerian and $\nabla\psi = 0$, then M is generic, $\nabla f = 0$, and $DF = 0$.*

(3) *If $\nabla P = 0$, then M is generic, the induced f-structure on M and the almost product structure $(\mathcal{C}, \mathcal{C}^\perp)$ are parallel.*

(4) *If M' is Kählerian and $K_1 = 0$, then M is generic and the bundles $\mathcal{S}$, $\mathcal{S}^\perp$, $\mathcal{C}$, $\mathcal{C}^\perp$ are parallel. If $K_2 = 0$, then M is generic and the bundles $\mathcal{S}$, $\mathcal{S}^\perp$, $\mathcal{S}_0$, $\mathcal{N}\mathcal{H}$ are parallel. If $K_1 = 0$, then $\nabla P = 0$ and $\nabla f = 0$. If $K_2 = 0$, then $\nabla\psi = 0$, $\nabla f = 0$ and $DF = 0$.*

(5) *If M' is Kählerian and M is totally geodesic, then $\nabla\psi = 0$, $\nabla P = 0$, $\nabla f = 0$, $DF = 0$ the almost product structures $(\mathcal{S}, \mathcal{S}^\perp)$, $(\mathcal{C}, \mathcal{C}^\perp)$ are parallel and the subbundles $\mathcal{S}_0$, $\mathcal{N}\mathcal{H}$ are parallel.*

We shall say that a generic submanifold in an almost Hermitian manifold satisfies the condition:

(A) if the distribution $\mathcal{S}$ is integrable and its leaves are totally geodesic in M,

(B) if the distribution $\mathcal{S}^\perp$ is integrable and its leaves are totally geodesic in M.

Assume now that M' is Kählerian. B-Y Chen proved, [2], that if M satisfies A, then $\alpha(X, Y) \in \mathcal{N}\mathcal{H}$ for any X, $Y \in \mathcal{S}$. B-Y Chen remarked also (without giving a counterexample) that the converse is not true in the case of an arbitrary generic submanifold. But the converse seems to me to be true. Assume that $\alpha(X, Y) \in \mathcal{N}\mathcal{H}$ for any $X, Y \in \mathcal{D}$. Let X and Y be vector fields belonging to $\mathcal{D}$. Since M' is Kählerian $\alpha(X, J'Y) + \nabla_X J'Y = J'\alpha(X, Y) + J'\nabla_X Y$. But $\alpha(X, J'Y)$, $J'\alpha(X, Y) \in \mathcal{N}\mathcal{H}$ and $J'\nabla_X Y$, $\nabla_X J'Y \in \mathcal{H}$ so $\nabla_X J'Y = J'\nabla_X Y$. It means that $\nabla_X Y \in \mathcal{D}$ for any vector fields X, Y belonging to $\mathcal{D}$. Therefore $[X, Y] = \nabla_X Y - \nabla_Y X \in \mathcal{D}$ if only X, Y belong to $\mathcal{D}$. Consequently $\mathcal{D}$ is integrable. Let M_x be the maximal integral submanifold of $\mathcal{D}$ passing through a point x and let X, Y be vector fields on M_x. They belong to $\mathcal{D}$, so $\nabla_X Y \in \mathcal{D}$, i.e., $\nabla_X Y \in TM_x$. This means that the leaves of $\mathcal{D}$ are totally geodesic in M. It is also easy to prove the converse of Lemma 3.5 from [2], which states that if M' is Kählerian and M satisfies B, then $\alpha(X, Y) \in \mathcal{N}\mathcal{H}$ for any $X \in \mathcal{D}^\perp$ and $Y \in \mathcal{D}$. Assume $\alpha(X, Y) \in \mathcal{N}\mathcal{H}$ for any $X \in \mathcal{D}^\perp$ and $Y \in \mathcal{D}$. Let X, Y be vector fields on M belonging to $\mathcal{D}^\perp$ and $\mathcal{D}$ respectively. Since $\nabla_X J'Y + \alpha(X, J'Y) = J'\nabla_X Y + J'\alpha(X, Y)$ and $\alpha(X, J'Y)$, $J'\alpha(X, Y) \in \mathcal{N}\mathcal{H}$, $\nabla_X J'Y$, $J'\nabla_X Y \in \mathcal{H}$ so $\nabla_X J'Y = J'\nabla_X Y$. Hence $\nabla_X Y \in \mathcal{D}$. Let Z be a vector field on M belonging to $\mathcal{D}^\perp$. We have $0 = X(Y, Z) = (\nabla_X Y, Z) + (\nabla_X Z, Y)$. Since $\nabla_X Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, $(\nabla_X Y, Z) = 0$ so $(\nabla_X Z, Y) = 0$. Consequently $\nabla_X Z \in \mathcal{D}$ for any $X, Z \in \mathcal{D}^\perp$ and like in the previous case we obtain that $\mathcal{D}^\perp$ is involutive and its leaves are totally geodesic in M. So we have

PROPOSITION 2.15. *Let M be a generic submanifold in a Kählerian manifold M'. M satisfies* (A) *iff $\alpha(X, Y) \in \mathcal{N}\mathcal{H}$ (equivalently $\alpha(X, J'Y) = J'\alpha(X, Y)$) for any $X, Y \in \mathcal{D}$, M satisfies* (B) *iff $\alpha(X, Y) \in \mathcal{N}\mathcal{H}$ (equivalently $\alpha(X, J'Y) = J'\alpha(X, Y)$) for any $X \in \mathcal{D}^\perp$ and $Y \in \mathcal{D}$.*

COROLLARY 2.16. *Let M' be a Kählerian manifold and let M be a generic submanifold in M' satisfying* (A) *and* (B). *Then $\nabla f = 0$.*

Proof. It follows from Propositions 2.15 and 2.7.

In the case where M' is Kählerian and M is a submanifold in M' we can write scheme 1.

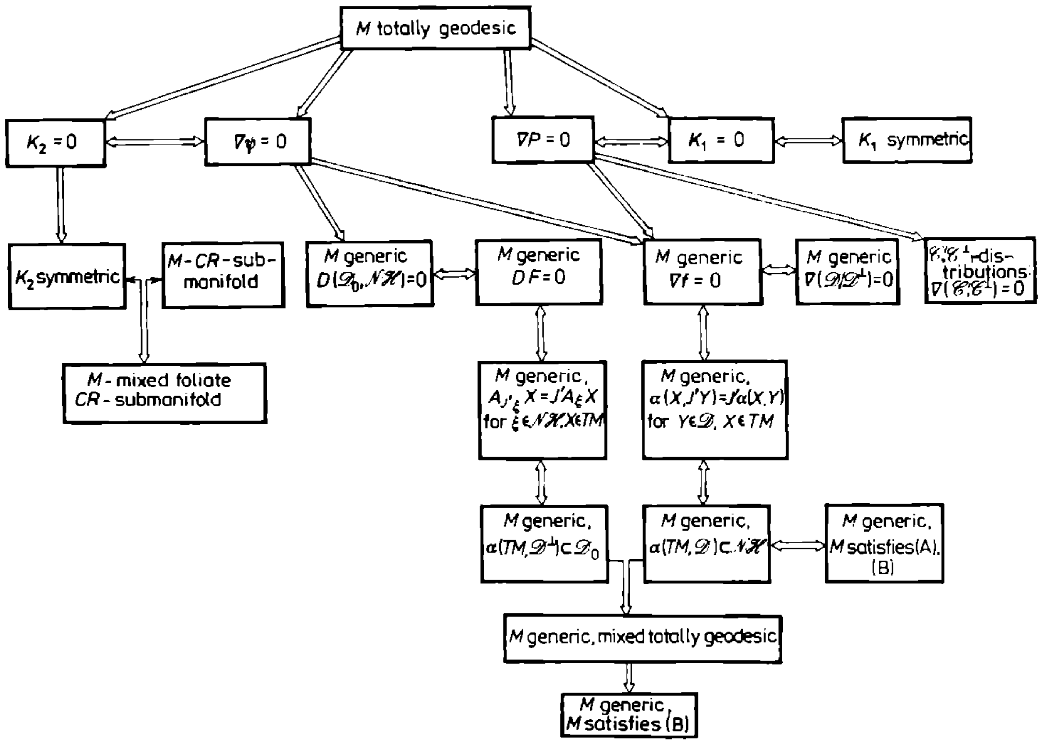
3. Generic submanifolds in some special Kählerian manifolds. Let M be a generic submanifold in an almost Hermitian manifold M'. Suppose, moreover, that M is proper, i.e., both distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are non-trivial. Let X and Y be unit vectors belonging to $\mathcal{D}^\perp$ and $\mathcal{D}$ respectively. By the Gauss equation,

$$R'(X, Y, X, Y) = R(X, Y, X, Y) + (\alpha(X, Y), \alpha(X, Y)) - (\alpha(X, X), \alpha(Y, Y))$$

and

$$R'(X, J'Y, X, J'Y) = R(X, J'Y, X, J'Y) + (\alpha(X, J'Y), \alpha(X, J'Y)) - (\alpha(X, X), \alpha(J'Y, J'Y)).$$

Scheme 1



If M' is Kählerian and $\nabla f - 0$ (equivalently the almost product structure $(\mathscr{D}, \mathscr{D}^{\perp})$ is parallel), then

$$H'_B(X, Y) = 2\|\alpha(X, Y)\|^2. \tag{3.1}$$

In fact, by virtue of Proposition 2.7 $\alpha(X, J'Y) = J'\alpha(X, Y)$ and $\alpha(J'Y, J'Y) = -\alpha(Y, Y)$. The formula above follows now from

LEMMA 3.1 ([5]). *Let $T = (T_1, T_2)$ be an almost product structure on a Riemannian manifold M. If T is parallel with respect to the Riemannian connection ∇ on M, then $R(X, Y, Z, W) = 0$ if two of vectors belong to two different distributions of T.*

In paper [2] the following definition was given.

DEFINITION 3.1. A real submanifold M in a Kählerian manifold M' is called a *generic product* if it is locally the Riemannian product of holomorphic submanifold M^T and a purely real submanifold $M^{\perp}$ of M'.

M is a generic product if and only if the almost product structure $(\mathscr{D}, \mathscr{D}^{\perp})$ is parallel, i.e., iff $\nabla f = 0$. Hence formula (3.1) was discovered by B-Y Chen; see Lemma 6.2 [2].

By formula (3.1) and Theorem 2.14, (2) and (3), we obtain

COROLLARY 3.2. *Let M be a real submanifold in a Kählerian manifold M', passing through a point in which bisectional holomorphic curvature of M' is negative. If $\nabla\psi = 0$ (equivalently $K_2 = 0$) or $\nabla P = 0$ (equivalently $K_1 = 0$), then M is holomorphic or purely real.*

This statements generalizes Proposition 5.9 and partialy Theorem 5.3 from [2].

In the case where M' is only almost Hermitian we have

COROLLARY 3.3. *Let M be a generic submanifold in an almost Hermitian manifold M', passing through a point in which Riemannian sectional curvature of M' is negative. Suppose, moreover, that the almost product structure $(\mathscr{D}, \mathscr{D}^{\perp})$ is parallel and both Hermitian fundamental forms vanish. Then M is holomorphic or purely real.*

Proof. If $K_1 = 0$ and $K_2 = 0$, then by Proposition 2.6 $\alpha(X, J'Y) = J'\alpha(X, Y)$ for any $Y \in \mathscr{D}$. Suppose that the almost product structure $(\mathscr{D}, \mathscr{D}^{\perp})$ is non-trivial.

Let $0 \neq X \in \mathscr{D}^{\perp}$, $0 \neq Y \in \mathscr{D}$. Then

$$R'(X, J'Y, X, J'Y) = R(X, J'Y, X, J'Y) + (\alpha(X, Y), \alpha(X, Y)) + \\ + (\alpha(X, X), \alpha(Y, Y)).$$

So

$$R'(X, Y, X, Y) + R'(X, J'Y, X, J'Y) \\ = R(X, Y, X, Y) + R(X, J'Y, X, J'Y) + 2\|\alpha(X, Y)\|^2.$$

Since $\nabla(\mathscr{D}, \mathscr{D}^{\perp}) = 0$, $R(X, Y, X, Y) = R(X, J'Y, X, J'Y) = 0$. So

$$R'(X, Y, X, Y) + R'(X, J'Y, X, J'Y) = 2\|\alpha(X, Y)\|^2.$$

By the hypothesis on the curvature of M' the left-hand side is negative. Hence a contradiction which completes the proof.

If M' is Kählerian, we have also

COROLLARY 3.4. *Let M be a generic mixed totally geodesic submanifold in a Kählerian manifold M'. Assume that M passes through a point of M' in which bisectional holomorphic curvature of M' is non-zero for any pair of J'-invariant planes. If $\nabla f = 0$, then M is holomorphic or purely real.*

As a corollary of given assertions we obtain

THEOREM 3.5. *Let M be a Kählerian manifold and let M be a real submanifold in M', passing through a point in which M' has positive or negative Riemannian sectional curvature.*

(1) *If M is generic, the induced f-structure on M is parallel and $\nabla' \mathscr{H} = 0$, then M is holomorphic or purely real.*

(2) *If* $\nabla P = 0$ *and* $\mathcal{N}\mathcal{H} = 0$, *then* M *is holomorphic or purely real.*

(3) *If* M *is generic,* $\nabla f = 0$ *and* $DF = 0$, *then* M *is holomorphic or purely real.*

(4) *If* $\nabla\psi = 0$ *(equivalently* $K_2 = 0$*), then* M *is holomorphic or purely real.*

(5) *If* $\nabla P = 0$ *and* M *is mixed totally geodesic, then* M *is holomorphic or purely real.*

(6) *If* M *is totally geodesic, then* M *is holomorphic, totally real or purely real such that* $J'TM \cap \mathcal{N} = 0$.

(7) *If* M *is generic satisfying* (A) *and* M *is mixed totally geodesic, then* M *is holomorphic or purely real.*

Proof. Let x be a point of M in which M' has positive or negative Riemannian sectional curvature for any plane in $T_x M$.

(1) By Proposition 2.7 $\alpha(X, Y) = 0$ if only X or Y belongs to $\mathcal{D}_x$. If M is proper generic and $0 \neq X \in \mathcal{D}_x$, $0 \neq Y \in \mathcal{D}_x^{\perp}$, then by the Gauss equation we have $R'(X, Y, X, Y) = R(X, Y, X, Y)$. But $R'(X, Y, X, Y) \neq 0$ and by the assumption $\nabla f = 0$, $R(X, Y, X, Y) = 0$. It means that M must be holomorphic or purely real.

(2) It follows from (1) and Proposition 2.13.

(3) By Propositions 2.7 and 2.9 we have $\alpha(X, Y) \in \mathcal{N}\mathcal{H}_x$ if X or Y belongs to $\mathcal{D}_x$ and $\alpha(X, Y) \in \mathcal{D}_0$ if X or Y belongs to $\mathcal{D}_x^{\perp}$. It means that M is mixed totally geodesic. The assertion follows now from Corollary 3.4.

(4) This assertion follows from (3) and Proposition 2.14, (2).

(5) This is a consequence of Proposition 2.14, (3) and Corollary 3.4.

(6) Since M is totally geodesic $\nabla\psi = 0$. So M is holomorphic or purely real. Since $\nabla P = 0$, the almost product structure $(\mathcal{C}, \mathcal{C}^{\perp})$ is parallel. Suppose that $(\mathcal{C}, \mathcal{C}^{\perp})$ is non-trivial. Let $0 \neq X \in \mathcal{C}_x$. Then, by Lemma 3.1, $R(X, Y, X, Y) = 0$. On the other hand, $R'(X, Y, X, Y) = R(X, Y, X, Y)$. The vectors X and Y are orthogonal because the distributions $\mathcal{C}$, $\mathcal{C}^{\perp}$ are orthogonal. Hence $R'(X, Y, X, Y) \neq 0$. It means that $(\mathcal{C}, \mathcal{C}^{\perp})$ must be trivial. If $\mathcal{C}^{\perp} = TM$, then M is totally real. If $\mathcal{C}^{\perp} = 0$, then $J'TM \cap \mathcal{N} = 0$.

(7) If M is mixed totally geodesic, then M satisfies (B) and by Corollary 2.16 $\nabla f = 0$. Therefore (7) follows from Corollary 3.4. The proof is complete.

Remarks. The statements (2), (4) and (7) of the theorem above generalize Theorems 5.5, 5.3 and Corollary 5.1 from [2] in the sense that in the statements from [2] M' is assumed to be a complex space form. Similarly, in Proposition 5.4 from [2] it is sufficient to assume that M passes through a point of M' in which the sectional Riemannian curvature of M' is positive. It follows from Lemma 5.6 from [2], which states that if M is generic submanifold in a Kählerian manifold M', the distribution $\mathcal{D}$ is integrable and $\alpha(\mathcal{D}, \mathcal{D}^{\perp}) \subset \mathcal{N}\mathcal{H}$, then

$$H'_B(X, Z) = R(X, J'X, PZ, Z) + 2\|\alpha(X, Z)\|^2 + \\ + \|\nabla_X PZ\|^2 - \|P\nabla_X Z\|^2 - \|A_{\psi Z} X\|^2$$

for any unit vectors X in $\mathscr{D}$ and Z in $\mathscr{D}^\perp$. In fact assume that M is a proper generic submanifold in a Kählerian manifold M', M passes through a point in which the Riemannian sectional curvature of M' is positive, $\mathscr{D}$ is integrable and M is mixed totally geodesic. Let $Z \in \mathscr{D}^\perp$ be a unit vector such that $J'Z$ is normal to M, i.e., $Z \in \ker P$. By Lemma 5.6 from [2] we obtain

$$H'_B(X, Z) = -\|P\nabla_X Z\|^2 - \|A_{\psi Z} X\|^2$$

for any unit vector X. By the assumption on the curvature the left-hand side of this equality is positive. It means that there is no non-zero vector Z such that $J'Z$ is normal to M, i.e., $\ker P = 0$.

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