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Stability and periodicity for linear differential equations with periodic coefficients(¹)

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Abstract. Various criteria are obtained which yield asymptotic stability or instability for the linear differential equation $L_n x = x^{(n)} + p_{n-1}x^{(n-1)} + \ldots + p_0 x = 0$, with $p_i(t)$ real periodic of period ω, in particular for the case $n = 3$.

1. Consider the n-th order linear differential operator $L_n x \equiv x^{(n)} + p_{n-1}(t)x^{(n-1)} + \ldots + p_0(t)x$, where the coefficients $p_i(t)$ are real, continuous for $t \in [0, \omega]$, and ω-periodic. We shall be concerned with establishing stability and instability criteria for the equation

$$L_n x = 0, \tag{1}$$

and in particular for the case $n = 3$, for which we shall also obtain solutions of (1) which satisfy certain (possibly non-linear) boundary conditions. The difficulty in establishing stability and instability criteria for (1) lies, of course, in the difficulty of determining the roots of the Floquet characteristic equation, i.e., the characteristic multipliers of (1), ([3], p. 78–80).

In [13] (see also the references therein) it is shown how the stability of (1) is related to periodic and multi-point de la Valleé Poussin-type boundary value problems, (BVPs), and this, in turn, is closely connected with the existence and sign of the corresponding Green functions. We discuss the case $n = 3$ in Section 3 below after some preliminary results in Section 2 concerning second order non-linear BVPs. For the case $n \geqslant 3$ we obtain the conclusion by discussing the relation to recent work of Hartman [7], Levin [11], and Coppel [4].

We recall that equation (1) is said to be *disconjugate on an interval I* of the real line in case no non-trivial solution of (1) has more than $n-1$

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zeros (counting multiplicity) on I, and that the BVP

$$L_n x = 0, \quad l_i x = 0,$$

where l_i is a linear functional on $C^{(n-i)}[a, b]$, $0 \leqslant i \leqslant n-1$, has a Green function $G(t, s)$ iff the only solution of the BVP is the trivial solution. In this case, the solution $u(t)$ of the BVP

$$L_n x = f, \quad l_i x = 0$$

may be written as

$$u(t) = \int_a^b G(t, s) f(s) ds.$$

We refer the reader to [11] for a recent survey of disconjugacy and oscillation properties of (1) and a fairly up-to-date bibliography.

2. For the case $n = 3$, the Ricatti equation associated with (1), obtained by the substitution $u = x'/x$, is

$$u'' = -3uu' - p_2 u' - u^3 - p_2 u^2 - p_1 u - p_0. \tag{2}$$

Disconjugacy criteria based on the relation between (1) and (2) have been developed in [9] and [6].

For convenience, we state below some results and definitions for the general second order non-linear differential equation

$$x'' = f(t, x, x'), \tag{3}$$

where f is continuous for $a \leqslant t \leqslant b$, $|x| + |x'| < +\infty$. (See [10], [5], and [12].)

A function $\alpha \in C^{(2)}[a, b]$ is called a *lower solution of* (3) *on* $[a, b]$ in case $\alpha'' \geqslant f(t, \alpha, \alpha')$ on $[a, b]$. Likewise, a function $\beta \in C^2[a, b]$ is called an *upper solution of* (3) *on* $[a, b]$ in case $\beta'' \leqslant f(t, \beta, \beta')$ on $[a, b]$.

The function $f(t, x, x')$ is said to *satisfy a Nagumo condition* on the set

$$E = \{(t, x):\ a \leqslant t \leqslant b,\ \alpha(t) \leqslant x \leqslant \beta(t)\}, \quad \alpha, \beta \in C[a, b]$$

in case there exists a positive continuous function $h(s)$, $0 \leqslant s < +\infty$ such that $|f(t, x, x')| \leqslant h(|x'|)$ for all $(t, x) \in E$ and all $|x'| < +\infty$, where

$$\int_\lambda^\infty \frac{s ds}{h(s)} > \max_{t \in I} \beta(t) \min_{t \in I} \alpha(t) -$$

with

$$\lambda = \max \left\{ \frac{\alpha|(b) - \beta(a)|}{b-a}, \frac{|\alpha(a) - \beta(b)|}{b-a} \right\}.$$

The Nagumo condition ensures that a family of bounded solutions of (3) have uniformly bounded derivatives and hence is sequentially compact.

If $\alpha(t)$, $\beta(t)$ are lower and upper solutions of (3), respectively, with $\alpha(a) < \beta(a)$, $\alpha(b) < \beta(b)$, $\alpha(t) \leqslant \beta(t)$ on $[a, b]$, we denote by G the class of all continuous functions $g(x, x')$ defined on $[\alpha(a), \beta(a)] \times R$ which are non-decreasing in x' and satisfy $g(\alpha(a), \alpha'(a)) \geqslant 0$, $g(\beta(a), \beta'(a)) \leqslant 0$. Similarly, H will denote the class of all continuous functions $h(x, x')$ defined on $[\alpha(b), \beta(b)] \times R$ which are non-decreasing in x' and satisfy $h(\alpha(b), \alpha'(b)) \leqslant 0$, $h(\beta(b), \beta'(b)) \geqslant 0$.

For completeness, we now state the following theorems [5], [12].

THEOREM A [5]. *Let $\alpha(t), \beta(t)$ be lower and upper solutions of* (3), *respectively, with $\alpha(a) < \beta(a)$, $\alpha(b) < \beta(b)$, $\alpha(t) \leqslant \beta(t)$ on $[a, b]$. Assume $f(t, x, x')$ satisfies a Nagumo condition on the set E, and let $g(x, x') \in G$ and $h(x, x') \in H$. Then there exists a solution $x(t)$ of the BVP*

$$x'' = f(t, x, x'), \quad g(x(a), x'(a)) = 0 = h(x(b), x'(b)) \tag{4}$$

which satisfies $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ on $[a, b]$.

We shall also need the following theorem on periodic BVPs.

THEOREM B [12]. *Let $\alpha(t)$, $\beta(t)$ be lower and upper solutions of* (3), *respectively, with $\alpha(t) \leqslant \beta(t)$, $\alpha(a) = \alpha(b)$, $\alpha'(a) \geqslant \alpha'(b)$, $\beta(a) = \beta(b)$, $\beta'(a) \leqslant \beta'(b)$ and assume $f(t, x, x')$ satisfies a Nagumo condition on the set E. Then there exists a solution $x(t)$ of the BVP*

$$x'' = f(t, x, x'), \quad x(a) = x(b), \quad x'(a) = x'(b) \tag{5}$$

which satisfies $\alpha(t) \leqslant x(t) \leqslant \beta(t)$.

The existence of lower and upper solutions for (2) is discussed in [6] and [9].

3. For the equation

$$L_3 x \equiv x''' + p_2 x'' + p_1 x' + p_0 x = 0, \tag{6}$$

where the $p_i(t)$ are continuous and periodic of period $\omega > 0$, the Floquet characteristic equation is

$$\lambda^3 - A_2 \lambda^2 + A_1 \lambda - B = 0, \tag{7}$$

where A_1, A_2 are given in terms of the fundamental system of solutions of (6) u_1, u_2, u_3, with $u_i^{(j)}(0) = \delta_{ij}$, $0 \leqslant i, j \leqslant 2$ as follows:

$$A_2 = u_0(\omega) + u_1'(\omega) + u_2''(\omega),$$

$$A_1 = \begin{vmatrix} u_0(\omega) & u_1(\omega) \\ u_0'(\omega) & u_1'(\omega) \end{vmatrix} + \begin{vmatrix} u_0(\omega) & u_2(\omega) \\ u_0''(\omega) & u_2''(\omega) \end{vmatrix} + \begin{vmatrix} u_1'(\omega) & u_2'(\omega) \\ u_1''(\omega) & u_2''(\omega) \end{vmatrix}$$

and $B = \exp(-\omega J)$, $J = \dfrac{1}{\omega}\int\limits_0^\omega p_2(t)\,dt$.

We shall denote by $\lambda_1, \lambda_2, \lambda_3$ the roots of (7), i.e., the characteristic multipliers of (6).

In [13] various necessary and/or sufficient criteria are given for the stability and instability of (6). In particular, a necessary condition for asymptotic stability is $J > 0$ since we must have $B = \lambda_1\lambda_2\lambda_3 < 1$.

We will say that *condition* W holds for $L_3 x = 0$ if for any $0 \leqslant a < \omega$ the BVP

$$(8) \quad \text{(i)} \quad L_3 x = 0, \quad \begin{cases} x(a) = x'(a) = x(a+\omega) = 0, \text{ or} \\ x(a) = x(a+\omega) = x'(a+\omega) = 0; \end{cases}$$

$$\text{(ii)} \quad L_3 x = 0, \quad x(a) = x(a+\omega) = x(a+2\omega) = 0$$

has Green functions.

Our first result is an existence theorem for solutions of a non-linear third order BVP in which the coefficients are not assumed to be periodic and which seems to be of independent interest.

THEOREM 1. *Consider the differential equation*

$$(9) \qquad l_3 x \equiv x''' + a_2 x'' + a_1 x' + a_0 x = 0,$$

where the a_i are continuous on $[a, b]$. Let $\alpha(t)$, $\beta(t)$ be lower and upper solutions, respectively, of the corresponding Ricatti equation

$$(10) \qquad u'' = -3uu' - a_2 u' - u^3 - a_2 u^2 - a_1 u - a_0$$

with $\alpha(t) < \beta(t)$ on $[a, b]$. Let $g(x)$, $h(x)$ be continuous functions on $[\alpha(a), \beta(a)]$, $[\alpha(b), \beta(b)]$, respectively, such that

$$(11) \qquad \begin{aligned} &\alpha'(a) + (\alpha(a))^2 \geqslant g(\alpha(a)), \qquad \alpha'(b) + (\alpha(b))^2 \leqslant h(\alpha(b)), \\ &\beta'(a) + (\beta(a))^2 \leqslant g(\beta(a)), \qquad \beta'(b) + (\beta(b))^2 \geqslant h(\beta(b)). \end{aligned}$$

Then $l_3 x = 0$ is disconjugate on $[a, b]$ and there exists a solution $x(t)$ of the form $x(t) = x(a)\exp\left(\int_a^t u(s)\,ds\right)$, with $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ on $[a, b]$ and satisfying

$$(12) \qquad \frac{x''(a)}{x(a)} = g\left(\frac{x'(a)}{x(a)}\right), \qquad \frac{x''(b)}{x(b)} = h\left(\frac{x'(b)}{x(b)}\right).$$

Proof. The fact that (9) is disconjugate on $[a, b]$ follows from Theorem 3.1 of [6], for example. Applying Theorem A above with $G(x, x') = x' + x^2 - g(x)$, $H(x, x') = x' + x^2 - h(x)$, and using (11), we obtain a solution $u(t)$ of (10) satisfying $u'(a) + (u(a))^2 = g(u(a))$, $u'(b) + (u(b))^2 = h(u(b))$. If $x(t)$ is defined by $x(t) \equiv x(a)\exp\left(\int_a^t u(s)\,ds\right)$, then calculation shows that $x(t)$ satisfies (12). This proves the theorem.

Remark. As an example, if c_1, c_2, d_1, d_2 are real constants satisfying $a'(a)+(a(a))^2 \geqslant c_1 a(a)+c_2$, $\beta'(a)+(\beta(a))^2 \leqslant c_1\beta(a)+c_2$, $a'(b)+a(b)^2 \leqslant \leqslant d_1(a(b))+d_2$, $\beta'(b)+(\beta(b))^2 \geqslant d_1\beta(b)+d_2$, and $a(t)<\beta(t)$ are lower and upper solutions of (12), then there is a positive solution $x(t)$ of $l_3x = 0$ satisfying

$$x''(a) = c_1x'(a)+c_2x(a), \quad x''(b) = d_1x'(b)+d_2x(b).$$

Other linear and non-linear boundary value problems may be solved similarly by the appropriate choices of the functions $g(x)$, $h(x)$ in Theorem A and by finding the appropriate lower and upper solutions of (2).

THEOREM 2. *Consider the differential equation*

$$L_3x = x'''+p_2x''+p_1x'+p_0x = 0, \tag{6}$$

where the $p_i(t)$ are periodic of period $\omega > 0$. Let $a(t)$, $\beta(t)$ be lower and upper solutions of (2), *respectively, satisfying*

$$a(0) = a(\omega), \quad a'(0) \geqslant a'(\omega), \quad \beta(0) = \beta(\omega), \quad \beta'(0) \leqslant \beta'(\omega) \tag{13}$$

with $a(t)<\beta(t)$ on $[0, \omega]$. Then there exists a characteristic multiplier λ_1 of (6) *satisfying*

$$\exp\Big(\int_0^\omega a(s)\,ds\Big) \leqslant \lambda_1 \leqslant \exp\Big(\int_0^\omega \beta(s)\,ds\Big) \tag{14}$$

and equation (6) *is disconjugate on* $(-\infty, +\infty)$. *In particular*, (6) *has no complex characteristic multipliers.*

Proof. Theorem B implies the existence of a solution $u(t)$ of (2) with $u(0) = u(\omega)$, $u'(0) = u'(\omega)$, $a(t) \leqslant u(t) \leqslant \beta(t)$ on $[0, \omega]$. Therefore, $x(t) \equiv \equiv x(0)\exp\big(\int_0^t u(s)\,ds\big)$ is a solution of (6) with $x(\omega) = \mu x(0)$, $x'(\omega) = \mu x'(0)$, $x''(\omega) = \mu x''(0)$, and $\mu = \exp\big(\int_0^\omega u(s)\,ds\big) \equiv \lambda_1$ satisfies (14). Either $a(0) < u(0)$ or $u(0) < \beta(0)$; to be specific, assume $a(0) < u(0)$. Then $a(t)$, $u(t)$ may be extended periodically to the interval $I_n \equiv [-n\omega, n\omega]$ for each $n \geqslant 1$ and we may apply Theorem A above (with suitably chosen auxiliary functions) or Theorem 7.3 of [10] to infer the existence of a solution $v_n(t)$ of (2) with $\hat{a}(t) \leqslant v_n(t) < \hat{u}(t)$, $v_n(-n\omega) = v_n(n\omega) = a(0)$, where $\hat{a}$, $\hat{u}$ denote the periodic extensions of a, u. By Theorem 3.1 of [6], it follows that (6) is disconjugate on I_n for each $n \geqslant 1$ and therefore $L_3x = 0$ is disconjugate on $(-\infty, +\infty)$.

Remark. In the above proof it is not necessarily true that $\hat{a} \in C^{(2)}[I_n]$ but an analysis of the proof of Theorem 7.3 in [10], for example, shows that the theorem is nevertheless valid in the above case.

We may now obtain a sufficient condition for the asymptotic stability of (6).

THEOREM 3. *Let $J > 0$ and let there exist lower and upper solutions $a(t) < \beta(t)$ on $[0, \omega]$ satisfying* (13) *with $\int_0^\omega \beta(t)\,dt < 0$. Assume also that for each $\varrho \geqslant 0$ there exists a lower solution $a_\varrho(t)$ of* (2) *with $a_\varrho(t+\omega) = a_\varrho(t)$ and $\int_0^\omega a_\varrho(t)\,dt = \varrho\omega$ and such that a_ϱ is not a solution of* (2). *Then $L_3 x = 0$ is disconjugate on $(-\infty, +\infty)$ and is asymptotically stable.*

Proof. From Theorem (2) we infer the disconjugacy of (6) on $(-\infty, +\infty)$ and the existence of a characteristic multiplier $0 < \lambda_1 < 1$. Since there are no complex multipliers, it suffices to show that λ_2, λ_3 satisfy $0 < \lambda_2, \lambda_3 < 1$ (which, in this case, is equivalent to $B < \lambda_1, \lambda_2, \lambda_3 < 1$). Since (6) is disconjugate on $(-\infty, +\infty)$, we conclude that there are no negative characteristic multipliers of (6). Suppose there exists a characteristic multiplier, say $\lambda_2 \geqslant 1$. Then from Floquet theory, there exists a solution $y(t) = \psi(t)e^{\varrho t}$ of (6), $\psi(t+\omega) = \psi(t)$, $\psi(t) > 0$, and $\varrho = ln\lambda_2/\omega \geqslant 0$. Let $z(t) = \exp\left(\int_0^t a_\varrho(t)\,dt\right)$, which we may write as $z(t) \equiv \varphi(t)e^{\varrho t}$ and $\varphi(t) = \exp\left(\int_0^t \left(a_\varrho(t) - \varrho\right)dt\right)$. Let

$$K = \max_{0 \leqslant t \leqslant \omega} \frac{\varphi(t)}{\psi(t)} \equiv \frac{\varphi(t_0)}{\psi(t_0)}$$

and define

$$v(t) \equiv Ky(t) - z(t).$$

We note that $v(t_0) = 0 = v(t_0+\omega) = v'(t_0+\omega)$ and $v(t) \geqslant 0$ for $t_0 \leqslant t \leqslant t_0+\omega$. Also $L_3(v(t)) = -L_3(z(t)) \equiv g(t) \leqslant 0$. Since (6) is disconjugate on $[t_0, t_0+\omega]$, it follows that the Green function $G_{12}(t, s)$ for the BVP $L_3 x = 0$, $x(t_0) = x(t_0+\omega) = x'(t_0+\omega) = 0$ is non-negative for $s, t \in [t_0, t_0+\omega]$; that is, condition W holds. (See [1], for example.) Therefore, $v(t) = \int_{t_0}^{t_0+\omega} G_{12}(t, s)g(s)\,ds \leqslant 0$, which implies $v(t) \equiv 0 \Rightarrow L_3(z(t)) \equiv 0$, a contradiction of our assumption that $a_\varrho(t)$ is not a solution of (2). Therefore, we conclude that all characteristic multipliers satisfy $0 < \lambda_i < 1$, $i = 1, 2, 3$. Hence, (6) is asymptotically stable.

As simple corollaries we have

COROLLARY 4. *Let $J > 0$ and let there exist $a(t)$, $\beta(t)$, lower and upper solutions of* (2) *on $[0, \omega]$, respectively, satisfying* (13) *and $a(t) < \beta(t)$ on $[0, \omega]$. If*

$$-\int_0^\omega p_2\,dt \leqslant \int_0^\omega a\,dt < \int_0^\omega \beta\,dt < 0, \tag{15}$$

then (6) *has at least two characteristic multipliers with $0 < \lambda_1, \lambda_2 < 1$.*

Proof. Inequality (15) ensures that there exists a characteristic multiplier λ_1 with $B < \lambda_1 < 1$. Hence, $B/\lambda_1 = \lambda_2\lambda_3 < 1$ so that one of λ_2, λ_3 satisfies $0 < \lambda_i < 1$.

COROLLARY 5. *Let condition* W *hold and assume further that for each* $\varrho \geqslant 0$ *there exists an* ω*-periodic lower solution* $a_\varrho(t)$ *of* (2) *with* $\int_0^\omega a_\varrho(t)\,dt = \omega\varrho$ *and such that* a_ϱ *is not a solution of* (2). *Then there exists at least one characteristic multiplier* λ_1 *with* $0 < \lambda_1 < 1$.

Proof. The hypotheses imply that there are no real characteristic multipliers satisfying $\lambda \geqslant 1$. Hence, since there must exist at least one real positive multiplier, the result follows.

Remark. If $\sigma(\varrho, t) \equiv \varrho^3 + p_2\varrho^2 + p_1\varrho + p_0$ satisfies $\sigma(\varrho, t) \geqslant 0$ (and $\not\equiv 0$) for all $\varrho \geqslant 0$, then we may take $a_\varrho(t) \equiv \varrho$ in the above theorem and corollary.

The next result gives a sufficient condition for instability for the case $J \geqslant 0$.

THEOREM 6. *Assume* $J \geqslant 0$ *and let there exist lower and upper solutions* $a(t)$, $\beta(t)$ *of* (2) *satisfying* (13) *with* $a(t) < \beta(t)$ *on* $[0, \omega]$. *Then* (6) *is unstable if either of the following conditions holds*:

(a) $\int_0^\omega \beta(t)\,dt < -\int_0^\omega p_2(t)\,dt = -\omega J,$

(b) $\int_0^\omega a(t)\,dt > 0.$

Proof. Condition (a) or (b) implies the existence of a characteristic multiplier λ_1 with $\lambda_1 < B$ or $\lambda_1 > 1$. Now if $\lambda_1 < B$, then $B/\lambda_1 = \lambda_2\lambda_3 > 1$ so that either λ_2 or λ_3 is > 1. Hence, in either case we have instability.

Remark. If neither $a(t)$ nor $\beta(t)$ is a solution of (2), then the strict inequalities in conditions (a) and (b) can be replaced by $\leqslant$ and $\geqslant$. Similar remarks apply in Theorem 3 and Corollary 4.

The next result is a generalization of a result of [13].

THEOREM 7. *Let* $J \geqslant 0$ *and let condition* W *hold. Assume that for each* $\varrho \in [-J, 0]$ *there exists an* ω*-periodic lower solution* a_ϱ *of* (2) *with* $\int_0^\omega a_\varrho(t)\,dt = \omega\varrho$ *such that* a_ϱ *is not a solution of* (2). *Then* (6) *is unstable.*

Proof. The proof is similar to Theorem 3. Suppose that $B \leqslant \lambda \leqslant 1$ for some characteristic multiplier λ and let $\varrho = \ln\lambda/\omega \in [-J, 0]$. Then we have $y(t) = \psi(t)e^{\varrho t}$, $\psi(t+\omega) = \psi(t)$, $L_3 y = 0$ and $z(t) = \exp\int_0^t a_\varrho(t)\,dt \equiv \varphi(t)e^{\varrho t}$. Again, if $K = \max_{0\leqslant t\leqslant\omega} \frac{\varphi(t)}{\psi(t)} = \frac{\varphi(t_0)}{\psi(t_0)}$ we conclude that for $v(t) \equiv Ky(t)-z(t)$ we have $v(t_0) = v(t_0+\omega) = v'(t_0+\omega) = 0$, $L_3 v \leqslant 0$, and $v(t) \geqslant 0$, which because of W yields a contradiction as in Theorem 3.

For the case $J = 0$, it is easily seen that $L_3 x = 0$ is stable iff $\lambda_1 = 1$ and λ_2, λ_3 are complex, $|\lambda_2| = |\lambda_3| = 1$. Therefore, we have

THEOREM 8. *Let $J = 0$ and assume that there exist $\alpha(t)$, $\beta(t)$, lower and upper solutions of* (2), *respectively, satisfying* (13), *with $\alpha(t) < \beta(t)$ on $[0, \omega]$. Then* (6) *is unstable.*

Proof. The hypotheses imply that (6) is disconjugate on $(-\infty, +\infty)$ and hence has no negative or complex multipliers. Therefore, since $B = 1 = \lambda_1\lambda_2\lambda_3$, it follows that $\lambda_i > 1$ for some characteristic multiplier. (Otherwise, if $\lambda_i = 1$, $i = 1, 2, 3$, then any solution which vanishes once is oscillatory, which would contradict the disconjugacy of (6).)

For the case $J < 0$, there always exists at least one unstable solution. By techniques similar to those above, one can obtain criteria which imply that (6) has at least two unstable independent solutions. For example, if there exist ω-periodic lower and upper solutions $\alpha(t)$, $\beta(t)$ satisfying (13) and $\alpha(t) < \beta(t)$ on $[0, \omega]$ and if

$$-\int_0^\omega p_2(t)\,dt > \int_0^\omega \beta(t)\,dt \geqslant \int_0^\omega \alpha(t)\,dt > 0,$$

then we may conclude the existence of a characteristic multiplier λ_1 with $1 < \lambda_1 < B$. Therefore, $\lambda_2\lambda_3 = B/\lambda_1 > 1$ so that either $\lambda_2 > 1$ or $\lambda_3 > 1$. Again, strict inequalities may be replaced by $>$ if α, β are not solution.

Other criteria may be obtained similarly.

Remark. The above techniques yield criteria under which 1 is not a characteristic multiplier of (6) and therefore they give sufficient conditions for the unique solvability of the periodic BVP

$$(16) \qquad L_3 x = f, \quad x(0)-x(\omega) = x'(0)-x'(\omega) = x''(\omega)-x''(\omega) = 0.$$

For example, in [8] it is shown that if $a(t), f(t)$ are continuous and ω-periodic, if $a(t) \not\equiv 0$ and does not change sign and if $x''' + a(t)x = 0$ is disconjugate on $[0, \omega]$, then the BVP

$$(17) \qquad x''' + a(t)x = f, \quad x(0)-x(\omega) = x'(0)-x'(\omega) = x''(0)-x''(\omega) = 0$$

has a unique ω-periodic solution. For $a(t) \geqslant 0$, this result can be deduced from Theorem 7. For $a(t) \leqslant 0$, it is easy to show directly that the BVP (17) (with $f \equiv 0$) has only the trivial solution. Therefore, the conclusion follows immediately.

We leave to the reader the formulation of additional criteria based on these techniques.

It is to be noted that if in the equation

$$(6) \qquad L_3 x = x''' + p_2 x'' + p_1 x' + p_0 x = 0$$

the $p_i(t)$ are not periodic, then one can infer the existence of a bounded solution of (6) if there exist lower and upper solutions $\alpha(t) \leqslant \beta(t)$ of (2) on $(-\infty, +\infty)$ with $\alpha, \beta \in L_1(-\infty, +\infty)$. Furthermore, $L_3 x = 0$ is disconjugate on $(-\infty, +\infty)$. Examples are easily contructed to illustrate these remarks.

Finally, for the equation

$$(1') \qquad L_n x = x^{(n)} + p_{n-1} x^{(n-1)} + \ldots + p_0 x = 0,$$

where the P_i's are periodic of period ω, if the roots of the equation

$$\sigma_n(\varrho, t) = \varrho^n + p_{n-1}\varrho^{n-1} + \ldots + p_0 = 0$$

are all real and separated by constants,

$$\mu_0 \leqslant \varrho_1(t) \leqslant \mu_1 \leqslant \varrho_2(t) \leqslant \ldots \leqslant \varrho_n(t) \leqslant \mu_n$$

with $\mu_0 < \mu_1 < \ldots < \mu_n$, then the results of [4] (see also [7] and [11]), imply that (1) is disconjugate on $[0, +\infty)$ and that there exists a fundamental system of positive solutions $y_1, \ldots, y_n$ of (1) satisfying $\mu_0 \leqslant \frac{y_1'}{y_1} \leqslant \mu_1 \leqslant \frac{y_2'}{y_2} \leqslant \ldots \leqslant \frac{y_n'}{y_n} \leqslant \mu_n$. (For the case $n = 3$, we note that μ_1, μ_2 are lower and upper solutions, respectively, of (2).) It follows therefore that the characteristic multipliers $\lambda_1, \ldots, \lambda_n$ of (1) satisfy $\mu_{i-1} \leqslant \frac{\ln \lambda_i}{\omega} \leqslant \mu_i$, $1 \leqslant i \leqslant n$. Hence, if $\mu_i < 0$, $0 \leqslant i \leqslant k$ and $0 < \mu_i$, $k+1 \leqslant i \leqslant n$ for some $0 \leqslant k \leqslant n$, then (1) has k independent solutions which tend to zero and at least $n-(k+1)$ independent unstable solutions.

Examples. 1. If $-m \leqslant p_0(t) \leqslant 0$ (but $\not\equiv 0$), if $p_1(t) \leqslant 0$, and if there exists a $\delta > 0$ with $p_2(t) \geqslant \frac{m+\delta^3}{\delta^2}$ on $[0, \omega]$, then $L_3 x = 0$ is disconjugate on $(-\infty, +\infty)$ and there exist at least two characteristic multipliers with $0 < \lambda_1, \lambda_2 < 1$ by Corollary 4. (In this case, $\{-\varepsilon, 0\}$ are lower and upper solutions, respectively, of (2).) It follows also that $\lambda_3 > 1$ by the preceding remarks concerning the separation of roots of $\sigma(\varrho, t)$ by constants.

2. If p_0, p_1, p_2 are such that $p_0(t) \geqslant 0$ on $(-\infty, +\infty)$ and if there exists a $\delta > 0$ such that

$$\frac{6t^2-6t-2\delta+1}{t^2+\delta} \leqslant [(2t-1)p_2(t)-(t^2+\delta)p_1(t)-(t^2+\delta)^2 p_0(t)],$$

then $L_3 x = 0$ is disconjugate on $(-\infty, +\infty)$ and has a bounded solution. (In this case, $0, \dfrac{1}{t^2+\delta}$ are lower and upper solutions of (2), respectively.)

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