

On the absolute differentiation of geometric object fields

by IVAN KOLÁŘ (Brno)

The main purpose of the present paper is to develop an invariant algorithm (based on the works of É. Cartan) for higher order absolute differentiation of arbitrary geometric object fields.

The absolute (or covariant) differentiation was first studied for some special geometric object fields (mostly tensor fields). When Ehresmann introduced the general definition of a connection on a principal fibre bundle and when the general concept of a geometric object field was clarified within the framework of the theory of fibre bundles (a complete comparison with the classical definition of geometric objects in the sense of Wundheiler was given by Kucharzewski and Kuczma, [13], see also [12]), some authors treated the general problem of absolute differentiation. According to our opinion, there are three main points of view to this question: the axiomatic approach, [1], [15], a definition by Crittenden, [2], and, last but not least, a paper by Ehresmann, [5]. In this paper, Ehresmann outlines only the main ideas, but he clarifies the operations of higher order. A detailed development of his ideas (as given in [7]) shows that the higher order (i.e. iterated) absolute differentiation is closely related to the prolongations of a connection. This approach is essentially based on the concept of a semi-holonomic jet of higher order introduced in [3].

Since the present paper deals with the absolute differentiation of higher order, it is based on [5] and [7]. To show the connections with other points of view, we deduce that the definitions by Ehresmann (in the case of first order) and Crittenden are equivalent (the equivalence of the axiomatic approach with the definition by Crittenden was proved by Szybiak [15]). In what follows, we use essentially our results of [8] and [9]. In Section 1 it is shown that the invariant formula for the absolute differential of first order can be obtained by a simple specialization of Proposition 1 of [9]. In Section 2 we deduce a "recurrence formula", which is applied to the proof of our main result in the last section.

Our considerations are performed in the category C^∞ . The standard notation of the theory of jets is used throughout the paper, cf. [7].

I. On the absolute differential of the first order. Let $P(B, G; \pi)$ be a principal fibre bundle and let $E(B, F, G, P)$ be a fibre bundle associated with P . The elements of E are the equivalence classes $\{(u, s)\}$, $u \in P$, $s \in F$, with respect to the equivalence relation $(u, s) \sim (ug, g^{-1}(s))$. Hence every $u \in P_x$, $x \in B$, determines a diffeomorphism (denoted by the same symbol) $u: F \rightarrow E_x$, $s \mapsto \{(u, s)\}$. Thus we have a mapping of $P \boxtimes E$ (= the fibre product over B of P and E) into F , $(u, z) \mapsto u^{-1}(z)$, cf. [9]. The concept of a geometric object field is used in two equivalent forms, which we call the direct or the indirect form, respectively, [6]. A geometric object field in its direct form is a cross section $\sigma: B \rightarrow E$, while the indirect form of σ is the mapping

$$(1) \quad u \mapsto u^{-1}(\sigma(\pi(u)))$$

of P into F .

Let $\Phi = PP^{-1}$ be the groupoid associated with P , see e.g. [14]. Every element of Φ is of the form $\bar{u}u^{-1}$, $u, \bar{u} \in P$, and it can be regarded as a diffeomorphism $E_x \rightarrow E_{\bar{x}}$, $x = \pi(u)$, $\bar{x} = \pi(\bar{u})$, which is the composition of $u^{-1}: E_x \rightarrow F$ and $\bar{u}: F \rightarrow E_{\bar{x}}$, $\bar{u}u^{-1}(z) = \bar{u}(u^{-1}(z))$, $z \in E_x$. An element of connection (of first order) on Φ at $x \in B$ is a 1-jet at x of a local mapping $\lambda(y)$ of B into Φ of the form $\lambda(y) = \varrho(y)[\varrho(x)]^{-1}$, where $\varrho(y)$ is a local cross section of P . Denote by $Q^1(\Phi)$ the fibre bundle over B of all elements of connection on Φ ; a connection (of first order) on Φ is a cross section $C: B \rightarrow Q^1(\Phi)$.

If $\sigma: B \rightarrow E$ is a cross section, then $\lambda^{-1}(y)(\sigma(y))$ is a local mapping of B into E_x . According to Ehresmann [5], the value of the absolute differential $\nabla\sigma$ of σ at x with respect to C is defined by

$$(2) \quad (\nabla\sigma)(x) = j_x^1[\lambda^{-1}(y)(\sigma(y))] \in J_x^1(B, E_x).$$

Set $\mathcal{F}_0^1(E) = \bigcup_{x \in B} J_x^1(B, E_x)$, which is an associated fibre bundle of the symbol $(B, T_n^1(F), L_n^1 \times G, H^1(B) \boxtimes P)$. (If $u \in P_x, v \in H_x^1(B)$ and Y is an element of $\mathcal{F}_0^1(E)$ over x , then $Yv \in T_n^1(E_x)$ and $u^{-1}(Yv) \in T_n^1(F)$.) Hence $\nabla\sigma$ is a cross section of $\mathcal{F}_0^1(E)$.

Considering a connection Γ on P and a geometric object field in indirect form $\Sigma: P \rightarrow F$, Crittenden [2] defines the absolute differential $\nabla\Sigma$ of Σ with respect to Γ as follows. For every $u \in P$, $\nabla\Sigma$ is a linear map of $T_x(B)$ into $T_{\Sigma(u)}(F)$ determined by

$$(3) \quad \nabla\Sigma(X) = d\Sigma(\tilde{X}_u), \quad X \in T_x(B), \quad x = \pi(u),$$

where \tilde{X}_u denotes the horizontal lift of X at u . But this linear map is identified with a 1-jet of B into F with source x and target $\Sigma(u)$. If we further choose an element $v \in H_x^1(B)$, then the composition of $\nabla\Sigma$ and v is a 1-jet of \mathbf{R}^n into F with source 0, $n = \dim B$, i.e. an element of $T_n^1(F)$.

From this point of view, $\nabla\Sigma$ can be interpreted as a mapping of $H^1(B) \boxtimes P$ into $T_n^1(F)$.

We have remarked in [7] that a connection on P can be considered as a G -invariant cross section of P into J^1P . Every connection C on Φ determines uniquely a connection Γ on P as follows. If $C(x) = j_x^1\lambda(y)$ and $u \in P_x$, then $\lambda(y)$ is uniquely expressed as $\lambda(y) = \varrho(y)u^{-1}$ and we put $\Gamma(u) = j_x^1\varrho(y)$; this connection Γ is said to be the *representative of C on P* , [7]. Now, the comparison of the definitions by Ehresmann and Crittenden is given by

LEMMA 1. *If the connection Γ on P represents the connection C on PP^{-1} and $\Sigma: P \rightarrow F$ is the indirect form of $\sigma: B \rightarrow E$, then the absolute differential $\nabla\Sigma$ of Σ with respect to Γ , according to Crittenden, is the indirect form of the absolute differential $\nabla\sigma$ of σ with respect to C , according to Ehresmann.*

Proof. Let $u \in P_x$ and $X \in T_x(B)$, $X = j_0^1\gamma(t)$, where γ is a curve on B . If $\Gamma(u) = j_x^1\varrho(y)$, then $\tilde{X}_u = j_0^1\varrho(\gamma(t))$ and

$$(4) \quad d\Sigma(\tilde{X}_u) = j_0^1[\varrho^{-1}(\gamma(t))(\sigma(\gamma(t)))]$$

by (1). On the other hand, $C(x) = j_x^1[\varrho(y)u^{-1}]$ and

$$(5) \quad u^{-1}(\nabla\sigma) = j_x^1[\varrho^{-1}(y)(\sigma(y))].$$

Comparing (4) and (5), we obtain Lemma 1.

We are now going to deduce the invariant formula for $\nabla\sigma$. In [8] we have defined the first prolongation $W^1(P)$ of P as the space of all 1-jets with source $(0, e)$ of the local isomorphisms of $\mathbf{R}^n \times G$ into P . (An analogous subject is also treated by Szybiak [16].) We have shown that $W^1(P)$ equals to the fibre product $H^1(B) \boxtimes J^1P$ and it has a natural structure of a principal fibre bundle over B with structure group G_n^1 , which is the semidirect product $L_n^1 \times T_n^1(G)$ of the groups L_n^1 and $T_n^1(G)$ with respect to the action $S \mapsto Sy$ of L_n^1 on $T_n^1(G)$, $y \in L_n^1$, $S \in T_n^1(G)$. We have also introduced the canonical $(\mathbf{R}^n \oplus \mathfrak{g})$ -valued form θ_1 of $W^1(P)$. There is a natural projection $\mu: W^1(P) \rightarrow H^1(B)$ and the diagram

$$(6) \quad \begin{array}{ccc} \mathbf{R}^n \oplus \mathfrak{g} & \xleftarrow{\theta_1} & T(W^1(P)) \\ \downarrow pr_1 & & \downarrow \mu_* \\ \mathbf{R}^n & \xleftarrow{\varphi_1} & T(H^1(B)) \end{array}$$

is commutative, provided φ_1 means the classical canonical form of $H^1(B)$.

Consider a connection $\Gamma: P \rightarrow J^1P$. Set $R(\Gamma) = H^1(B) \boxtimes \Gamma(P) \subset W^1(P)$ and denote by $\hat{\mu}$ the restriction of μ to $R(\Gamma)$, so that the projections $\hat{\mu}: R(\Gamma) \rightarrow H^1(B)$ and $\beta: R(\Gamma) \rightarrow P$ determine an identification τ of $R(\Gamma)$

and $H^1(B) \boxtimes P$, $\tau(U) = (\mu(U), \beta U)$. There is a natural injection $g \mapsto j_0^1 \hat{g}$ of G into $T_n^1(G)$, where $\hat{g}: \mathbf{R}^n \rightarrow G$ is the constant mapping $x \mapsto g$; in this sense we may write $G \subset T_n^1(G)$. Then $L_n^1 \times G$ is a subgroup of $G_n^1 = L_n^1 \overline{\times} T_n^1(G)$.

LEMMA 2. *The space $R(\Gamma)$ is a reduction of the principal fibre bundle $W^1(P)$ to $L_n^1 \times G \subset G_n^1$. Conversely, every reduction of $W^1(P)$ to $L_n^1 \times G$ determines a connection on P .*

Proof. An equivalent assertion for connections on the associated groupoid was deduced by Que [14].

PROPOSITION 1. *Let $\tilde{\theta}_1$ be the restriction of the canonical form of $W^1(P)$ to $R(\Gamma)$, let ω be the fundamental form of connection Γ and let φ_1 be the canonical form of $H^1(B)$. Then the diagram*

$$(7) \quad \begin{array}{ccc} \mathfrak{g} & \xleftarrow{\omega} & T(P) \\ \uparrow pr_2 & & \uparrow \beta_* \\ \mathbf{R}^n \oplus \mathfrak{g} & \xleftarrow{\tilde{\theta}_1} & T(R(\Gamma)) \\ \downarrow pr_1 & & \downarrow \hat{\mu}_* \\ \mathbf{R}^n & \xleftarrow{\varphi_1} & T(H^1(B)) \end{array}$$

is commutative.

Proof. According to [8], every $U \in W^1(P)$, $\beta U = u$, determines a linear isomorphism $\tilde{U}: \mathbf{R}^n \oplus \mathfrak{g} \rightarrow T_u(P)$ such that the restriction of \tilde{U} to $\{0\} \oplus \mathfrak{g}$ is the standard identification of \mathfrak{g} and $T_u(P_x)$. By definition, if $X \in T_U(W^1(P))$, then $\theta_1(X) = \tilde{U}^{-1}(\bar{X})$, where $\bar{X} = \beta_* X \in T_u(P)$. But $U \in R(\Gamma)$ implies that \tilde{U} transforms $\mathbf{R}^n \oplus \{0\}$ into $\Gamma(u)$ so that $pr_2 \tilde{\theta}_1(X)$, $X \in T_U(R(\Gamma))$, is the projection of $\tilde{U}^{-1}(\bar{X})$ into \mathfrak{g} in the direction of $\tilde{U}^{-1}(\Gamma(u))$. On the other hand, $\omega(\bar{X})$ is constructed by projecting \bar{X} into $T_u(P_x)$ in the direction of $\Gamma(u)$ and applying the standard identification of $T_u(P_x)$ and \mathfrak{g} . Hence the top quadrangle commutes. The commutativity of the bottom quadrangle follows immediately from (6), Q.E.D.

Consider a cross section $\sigma: B \rightarrow E$. Its first prolongation $j^1 \sigma$ is a cross section $B \rightarrow J^1 E$. Since $J^1 E$ is an associated bundle of the symbol $(B, T_n^1(F), G_n^1, W^1(P))$, [9], the indirect form of $j^1 \sigma$ is a mapping $\Sigma_1: W^1(P) \rightarrow T_n^1(F)$. On the other hand, connection Γ represents a connection C on PP^{-1} and the absolute differential $\nabla \sigma$ of σ with respect to C is a cross section of $\mathcal{F}_0^1(E)$, so that its indirect form is a mapping of $H^1(B) \boxtimes P$ into $T_n^1(F)$.

PROPOSITION 2. *For every $U \in R(\Gamma)$ it holds*

$$\Sigma_1(U) = (\nabla \Sigma)(\tau(U)).$$

Proof. Let U be 1-jet of a local isomorphism of $\mathbf{R}^n \times G$ into P determined by a pair (φ, ϱ) , where φ is a local diffeomorphism of \mathbf{R}^n into B and ϱ is a local cross section of P , $\varphi(0) = x$, $\varrho(x) = u$. Then

$$\Sigma(U) = j_0^1[\varrho^{-1}(\varphi(y))(\sigma(\varphi(y)))],$$

see [9]. On the other hand, $\tau(U) = (j_0^1\varphi, u)$ and $C(x) = j_x^1[\varrho(y)u^{-1}]$, so that

$$(\nabla\Sigma)(\tau(U)) = u^{-1}[(\nabla\Sigma)(j_0^1\varphi)] = u^{-1}j_0^1[u\varrho^{-1}(\varphi(y))(\sigma(\varphi(y)))] = \Sigma_1(U),$$

Q.E.D.

Consider some local coordinates y^p on F . Let $\bar{\omega}^\alpha$ be a basis of \mathfrak{g}^* and let

$$(8) \quad dy^p + \eta_a^p(y^q)\bar{\omega}^\alpha = 0, \quad p, q, \dots = 1, \dots, \dim F; \\ \alpha, \beta, \dots = 1, \dots, \dim G,$$

be the equations of the fundamental distribution on $G \times F$. In [9] we have deduced the following assertion. Let \bar{a}^p be the coordinate functions of a geometric object field $\sigma: B \rightarrow E$; then the coordinate functions a^p, a_i^p of $j^1\sigma$ on $W^1(P)$ satisfy $a^p = \beta^* \bar{a}^p$ and

$$(9) \quad da^p + \eta_a^p(a^q)\theta^\alpha = a_i^p\theta^i, \quad i, j, \dots = 1, \dots, n = \dim B,$$

where θ^i, θ^α are the components of the canonical form of $W^1(P)$.

PROPOSITION 3. Let \bar{a}^p be the coordinate functions of a geometric object field $\sigma: B \rightarrow E$, let $\bar{\omega}$ be the fundamental form of the connection Γ , let $\bar{\varphi}_1$ be the canonical form of $H^1(B)$ and let $p_1: H^1(B) \boxtimes P \rightarrow H^1(B)$, $p_2: H^1(B) \boxtimes \boxtimes P \rightarrow P$ be the product projections. Put $\omega = p_2^* \bar{\omega}$, $\varphi_1 = p_1^* \bar{\varphi}_1$ and denote by ω^α or φ^i the components of ω or φ_1 , respectively. Then the coordinate functions a^p, a_i^p of $\nabla\sigma$ on $H^1(B) \boxtimes P$ satisfy $a^p = p_2^* \bar{a}^p$ and

$$(10) \quad da^p + \eta_a^p(a^q)\omega^\alpha = a_i^p\varphi^i.$$

Proof. This is a direct consequence of (9) and of Propositions 1 and 2.

Remark 1. Equations (10) represent the invariant form of the corresponding local coordinate expressions deduced by Szybiak [15]. Indeed, consider a coordinate neighbourhood $D \subset B$ and a local coordinate system ξ on D with coordinates ξ^i . Then ξ determines a cross section $\xi: D \rightarrow H^1(B)$ and it holds $\xi^* \bar{\varphi}^i = d\xi^i$. Further, if $\varrho: D \rightarrow P$ is a cross section, then we have $\varrho^* \bar{\omega}^\alpha = \Gamma_i^\alpha(\xi) d\xi^i$. (Clearly, $\bar{\omega}^\alpha$ or $\bar{\varphi}^i$ are the components of $\bar{\omega}$ or $\bar{\varphi}_1$ respectively.) The pair (ξ, ϱ) is a cross section $D \rightarrow H^1(B) \boxtimes P$. Set $\tilde{a}^p = (\xi, \varrho)^* a^p$, $\tilde{a}_i^p = (\xi, \varrho)^* a_i^p$, then (10) implies

$$(11) \quad d\tilde{a}^p + \eta_a^p(\tilde{a}^q)\Gamma_i^\alpha d\xi^i = \tilde{a}_i^p d\xi^i,$$

which is formula (15) of [15].

II. Recurrence formula. Assume that the direct product $H \times G$ of two Lie groups H and G acts on the left on a manifold A ; we shall write

$$(12) \quad (h, g)(a) = g(a)h^{-1}, \quad a \in A, h \in H, g \in G.$$

Let $Q(B, H)$ be a principal fibre bundle, let $F = F(B, A, H, Q)$ and let $\bar{q}: F \rightarrow B$ be the projection of this associated fibre bundle. According to (12), we shall write $h^{-1}v$ instead of the usual notation vh , $v \in Q$, $h \in H$. We introduce a left action of G on F by $g(\{(v, a)\}) = \{(v, g(a))\}$; this definition is correct since $g(\{(hv, (a)h^{-1})\}) = \{(hv, g(a)h^{-1})\} = \{(v, g(a))\}$. One sees also easily that this action of G on F is fibre-preserving.

Consider the associated fibre bundle $E(B, F, G, P) = (B, (B, A, H, Q), G, P)$ with projection $p: E \rightarrow B$. We define another projection $q: E \rightarrow B$ by $q(\{(u, s)\}) = \bar{q}(s)$, $s \in F$. Since the action of G on F is fibre-preserving, this definition is correct. Hence every $E_x = p^{-1}(x)$ is a fibered manifold (E_x, q_x, B) , where $q_x = q|_{E_x}$. Define

$$E_0 = \{z \in E; p(z) = q(z)\},$$

so that E_0 is an associated bundle of the symbol $(B, A, H \times G, Q \boxtimes P)$. Every $(v, u) \in Q_x \times P_x$ determines a mapping of E_{0x} onto A , which will be denoted by $z \mapsto u^{-1}(z)v$, $z \in E_{0x}$. According to [9], $J^1 E_0$ is an associated bundle $(B, T_n^1(A), (H \times G)_n^1, W^1(Q \boxtimes P))$. Introduce

$$\mathcal{S}_0^1 E = \bigcup_{x \in B} J_x^1 E_x \quad \text{or} \quad \mathcal{S}^1 E = \bigcup_{x \in B} J^1 E_x,$$

which is an associated bundle of the symbol $(B, T_n^1(A), H_n^1 \times G, W^1(Q) \boxtimes P)$ or $(B, J^1 F, G, P)$ respectively.

Consider a cross section $\sigma: B \rightarrow E_0$ and a connection C on PP^{-1} . Since $E_0 \subset E$ and E is a fibre bundle associated with P , we can construct the absolute differential $V\sigma$ of σ with respect to C , which is a cross section of $\mathcal{S}_0^1 E$. Further, $W^1(Q \boxtimes P) = H^1(B) \boxtimes J^1 Q \boxtimes J^1 P$ and the representative $\Gamma: P \rightarrow J^1 P$ of connection C determines a reduction $R(\Gamma) = H^1(B) \boxtimes J^1 Q \boxtimes \Gamma(P)$ of $W^1(Q \boxtimes P)$ to the group $H_n^1 \times G \subset (H \times G)_n^1$. Analogously to Section 1, there is an identification $\tau: R(\Gamma) \rightarrow W^1(Q) \boxtimes P$. Let $p_1: W^1(Q) \boxtimes P \rightarrow W^1(Q)$, $p_2: W^1(Q) \boxtimes P \rightarrow P$ be the product projections.

LEMMA 3. *Let $\tilde{\theta}_1$ be the restriction of the canonical form of $W^1(Q \boxtimes P)$ to $R(\Gamma)$, let ω be the fundamental form of connection Γ and let φ_1 be the canonical form of $W^1(Q)$. Then the diagram*

$$(13) \quad \begin{array}{ccc} \mathfrak{g} & \xleftarrow{\omega} & T(P) \\ \uparrow pr_3 & & \uparrow (p_2 \tau)_* \\ \mathbf{R}^n \oplus \mathfrak{h} \oplus \mathfrak{g} & \xleftarrow{\tilde{\theta}_1} & T(R(\Gamma)) \\ \downarrow (pr_1, pr_2) & & \downarrow (p_1 \tau)_* \\ \mathbf{R}^n \oplus \mathfrak{h} & \xleftarrow{\varphi_1} & T(W^1(Q)) \end{array}$$

is commutative.

Proof is a simple replica of the proof of Proposition 1.

The indirect form of $j^1\sigma$ is a mapping $\Sigma_1: W^1(Q \boxtimes P) \rightarrow T_n^1(A)$, while the indirect form of $\nabla\sigma$ is a mapping $\nabla\Sigma: W^1(Q) \boxtimes P \rightarrow T_n^1(A)$.

LEMMA 4. For every $U \in R(\Gamma)$ it holds

$$\Sigma_1(U) = (\nabla\Sigma)(\tau(U)).$$

Proof. Let U be the 1-jet of a local isomorphism of $\mathbf{R}^n \times H \times G$ into $Q \boxtimes P$ determined by a triple (φ, ψ, ϱ) , where φ is a local diffeomorphism of \mathbf{R}^n into B and ψ or ϱ is a local cross section of Q or P respectively, $\varphi(0) = x, \varrho(x) = u$. Then $\Sigma_1(U) = j_0^1[\varrho^{-1}(\varphi(y))(\sigma(\varphi(y)))\psi(\varphi(y))]$. On the other hand, $\tau(U) = (V, u)$, where V is the 1-jet of the local isomorphism of $\mathbf{R}^n \times H$ into Q determined by the pair (φ, ψ) . Since

$$C(x) = j_x^1[\varrho(y)u^{-1}],$$

it is

$$(\nabla\sigma)(x) = j_x^1[u\varrho^{-1}(y)(\sigma(y))]$$

and

$$(\nabla\Sigma)(V, u) = u^{-1}[(\nabla\sigma)(x)]V = j_0^1[\varrho^{-1}(\varphi(y))(\sigma(\varphi(y)))\psi(\varphi(y))] = \Sigma_1(U),$$

Q. E. D.

Let y^b be some local coordinates on A , let $\tilde{\omega}^*$ be a basis of \mathfrak{h}^* and let

$$(14) \quad dy^b + \eta_a^b(y^c)\tilde{\omega}^a + \eta_x^b(y^c)\tilde{\omega}^x = 0, \quad \begin{matrix} b, c = 1, \dots, \dim A, \\ x = 1, \dots, \dim H, \end{matrix}$$

be the equations of the fundamental distribution on $(H \times G) \times A$. According to [9], if \bar{a}^b are the coordinate functions on $Q \boxtimes P$ of a geometric object field $\sigma: B \rightarrow E_0$, then the coordinate functions of $j^1\sigma$ on $W^1(Q \boxtimes P)$ satisfy $a^b = \beta^* \bar{a}^b$ and

$$(15) \quad da^b + \eta_a^b(a^c)\theta^a + \eta_x^b(a^c)\theta^x = a_i^b \theta^i,$$

where $\theta^i, \theta^x, \theta^a$ are the components of the canonical form of $W^1(Q \boxtimes P)$.

LEMMA 5. Let $\bar{\omega}$ be the fundamental form of Γ and let $\bar{\varphi}_1$ be the canonical form of $W^1(Q)$. Put $\omega = p_2^* \bar{\omega}, \varphi_1 = p_1^* \bar{\varphi}_1$ and denote by ω^a or φ^i, φ^x the components of ω or φ_1 respectively. Then the coordinate functions a^b, a_i^b of $\nabla\sigma$ on $W^1(Q) \boxtimes P$ satisfy $a^b = p_2^* \bar{a}^b$ and

$$(16) \quad da^b + \eta_a^b(a^c)\omega^a + \eta_x^b(a^c)\varphi^x = a_i^b \varphi^i.$$

Proof. This is a direct consequence of (15) and of Lemmas 3 and 4.

III. Absolute differentials of higher order. We introduce

$$\mathcal{F}^1(E) = \bigcup_{x \in E} J^1(B, E_x),$$

which is an associated bundle of the symbol $(B, J^1(B, F), G, P)$. Since $\mathcal{F}_0^1(E) \subset \mathcal{F}^1(E)$ and $\mathcal{F}^1(E)$ is associated with P , we can construct the

absolute differential of the cross section $V\sigma: B \rightarrow \mathcal{F}^1(E)$ with respect to C , which will be called the *absolute differential* of second order of σ with respect to C and will be denoted by $V^2\sigma = V(V\sigma)$. Roughly speaking, the absolute differential of order r of σ with respect to C is defined by induction $V^r\sigma = V(V^{r-1}\sigma)$. Using the results of [5] and [7], we can describe this recurrence procedure in more details as follows. Set

$$\bar{\mathcal{F}}_0^r(E) = \bigcup_{x \in B} \bar{J}_x^r(B, E_x) \quad \text{or} \quad \bar{\mathcal{F}}^r(E) = \bigcup_{x \in B} \bar{J}^r(B, E_x),$$

which is an associated fibre bundle of the symbol $(B, \bar{T}_n^r(F), L_n \times G, H^r(B) \boxtimes P)$ or $(B, \bar{J}^r(B, F), G, P)$, respectively. (We recall that $\bar{J}^r(B, E_x)$ denotes the space of all semiholonomic r -jets of B into E_x , see [3].) Assume by induction, that we have constructed the absolute differential $V^{r-1}\sigma$ as a cross section of $\bar{\mathcal{F}}_0^{r-1}(E)$. Since $\bar{\mathcal{F}}_0^{r-1}(E) \subset \bar{\mathcal{F}}^{r-1}(E)$ and $\bar{\mathcal{F}}^{r-1}(E)$ is associated with P , we can construct the absolute differential $V(V^{r-1}\sigma)$ of $V^{r-1}\sigma$ with respect to C . According to [5] and [7], $V(V^{r-1}\sigma) = V^r\sigma$ is a cross section of $\bar{\mathcal{F}}_0^r(E)$.

The local coordinates y^p on F are prolonged to some local coordinates $y^p, y_i^p, \dots, y_{i_1 \dots i_r}^p$ on $\bar{T}_n^r(F)$, [4]. The analogous local coordinates on $\bar{T}_n^{r-1}(F)$ will be denoted by $\bar{y}^p, \dots, \bar{y}_{i_1 \dots i_{r-1}}$. The coordinate functions of $V^r\sigma: B \rightarrow \bar{\mathcal{F}}_0^r(E)$ can be treated by a recurrent algorithm whose first step is described in Proposition 3. Assume by induction that we have applied this algorithm $(r-1)$ -times and that we have deduced the equations of the fundamental distribution on $(L_n^{r-1} \times G) \times \bar{T}_n^{r-1}(F)$ in the form

$$\begin{aligned} d\bar{y}^p + \eta_a^p(\bar{y}^a)\hat{\omega}^a &= 0, \\ d\bar{y}_i^p + \Phi_i^p(\bar{y}^a, \bar{y}_j^a, \hat{\omega}^a, \hat{\omega}_j^k) &= 0, \\ &\vdots \\ d\bar{y}_{i_1 \dots i_{r-1}}^p + \Phi_{i_1 \dots i_{r-1}}^p(\bar{y}^a, \dots, \bar{y}_{i_1 \dots i_{r-1}}^a, \hat{\omega}^a, \hat{\omega}_j^k, \dots, \hat{\omega}_{j_1 \dots j_{r-1}}^k) &= 0, \end{aligned} \tag{17}$$

where $\hat{\omega}^a$ is a basis of \mathfrak{g}^* and $\hat{\omega}_j^k, \dots, \hat{\omega}_{j_1 \dots j_{r-1}}^k$ is the natural basis of \mathfrak{l}_n^{r-1} . Let $\mu: H^r(B) \boxtimes P \rightarrow H^{r-1}(B) \boxtimes P$ be the natural (jet) projection and let p_1 and p_2 be the product projections of $H^r(B) \boxtimes P$.

PROPOSITION 4. Let $\bar{a}^p, \dots, \bar{a}_{i_1 \dots i_{r-1}}^p$ be the coordinate functions on $H^{r-1}(B) \boxtimes P$ of $V^{r-1}\sigma$, let $\bar{\omega}$ be the connection form of Γ and let $\bar{\varphi}_r$ be the canonical form of $H^r(B)$. Put $\omega = p_2^*\bar{\omega}$, $\varphi_r = p_1^*\bar{\varphi}_r$ and denote by ω^a or $\varphi^i, \varphi_j^i, \dots, \varphi_{j_1 \dots j_{r-1}}^i$ the components of ω or φ_r respectively. Then the coordinate functions $a^p, \dots, a_{i_1 \dots i_r}^p$ of $V^r\sigma$ on $H^r(B) \boxtimes P$ satisfy $a^p = \mu^*\bar{a}^p, \dots, a_{i_1 \dots i_{r-1}}^p = \mu^*\bar{a}_{i_1 \dots i_{r-1}}^p$ and

$$\begin{aligned} da^p + \eta_a^p(a^a)\omega^a &= a_i^p\varphi^i, \\ da_i^p + \Phi_i^p(a^a, a_i^a, \omega^a, \varphi_j^k) &= a_{ij}^p\varphi^j, \\ &\vdots \\ da_{i_1 \dots i_{r-1}}^p + \Phi_{i_1 \dots i_{r-1}}^p(a^a, \dots, a_{j_1 \dots j_{r-1}}^a, \omega^a, \varphi_j^k, \dots, \varphi_{j_1 \dots j_{r-1}}^k) &= a_{i_1 \dots i_{r-1}j}^p\varphi^j. \end{aligned} \tag{18}$$

Proof. If one considers the natural action of the direct product $L_n^{r-1} \times G$ on $\bar{T}_n^{r-1}(F)$, then $\bar{\mathcal{F}}_0^{r-1}(E) = (B, \bar{T}_n^{r-1}(F), L_n^{r-1} \times G, H^{r-1}(B) \boxtimes P)$ satisfies the assumptions of Section 2 provided one substitutes

$$\left(\begin{array}{cccccc} A & , & H & , & Q & , & E_0 & , & E \\ \bar{T}_n^{r-1}(F) & , & L_n^{r-1} & , & H^{r-1}(B) & , & \bar{\mathcal{F}}_0^{r-1}(E) & , & \bar{\mathcal{F}}^{r-1}(E) \end{array} \right).$$

Further, $\mathcal{S}^1(\bar{\mathcal{F}}^{r-1}(E)) = \bigcup_{x \in B} J^1(\bar{J}^{r-1}E_x)$ and $\mathcal{S}_0^1(\bar{\mathcal{F}}^{r-1}(E)) = \bigcup_{x \in B} J_x^1(\bar{J}^{r-1}E_x) = (B, T_n^1(\bar{T}_n^{r-1}(F)), (L_n^{r-1})_n^1 \times G, W^1(H^{r-1}(B)) \boxtimes P)$. Let $\bar{\omega} = \pi_2^* \bar{\omega}$ and $\theta_1 = \pi_1^* \theta_1$, where $\bar{\theta}_1$ is the canonical form of $W^1(H^{r-1}(B))$ and π_1, π_2 are the product projections of $W^1(H^{r-1}(B)) \boxtimes P$. Let $\bar{\omega}^a$ or $\theta^i, \theta_j^i, \dots, \theta_{j_1 \dots j_{r-1}}^i$ be the components of $\bar{\omega}$ or θ_1 , respectively. By Lemma 5, if $\tilde{a}^p, \dots, \tilde{a}_{i_1 \dots i_{r-1}}^p, b_i^p, \dots, b_{i_1 \dots i_r}^p$ are the coordinate functions of $V(V^{r-1}\sigma)$ on $W^1(H^{r-1}(B)) \boxtimes P$, then it holds

$$\begin{aligned} d\tilde{a}^p + \eta_a^b(\tilde{a}^a)\bar{\omega}^a &= b_i^p \theta^i, \\ (19) \quad d\tilde{a}_i^p + \Phi_i^q(\tilde{a}^q, \tilde{a}_i^q, \bar{\omega}^a, \theta_j^i) &= b_{ij}^p \theta^j, \\ &\vdots \\ d\tilde{a}_{i_1 \dots i_{r-1}}^p + \Phi_{i_1 \dots i_{r-1}}^q(\tilde{a}^q, \dots, \tilde{a}_{j_1 \dots j_{r-1}}^q, \bar{\omega}^a, \theta_j^i, \dots, \theta_{j_1 \dots j_{r-1}}^i) &= b_{i_1 \dots i_{r-1} j}^p \theta^j. \end{aligned}$$

According to [5] and [7], the values of $V^r\sigma$ lie in $\bar{\mathcal{F}}_0^r(E) = \bigcup_{x \in B} \bar{J}_x^r E_x \subset \mathcal{S}_0^1(\bar{\mathcal{F}}^{r-1}(E))$. This is characterized by $\tilde{a}_i^p = b_i^p, \dots, \tilde{a}_{i_1 \dots i_{r-1}}^p = b_{i_1 \dots i_{r-1}}^p$. Further we have $H^r(B) \subset W^1(H^{r-1}(B))$ and the restriction of the canonical form of $W^1(H^{r-1}(B))$ to $T(H^r(B))$ is the canonical form of $H^r(B)$. Thus, restricting all quantities of (19) to $H^r(B) \boxtimes P$, we obtain (18), Q.E.D.

It remains to show an algorithm for finding the equations of the fundamental distribution on $(L_n^r \times G) \times \bar{T}_n^r(F)$.

PROPOSITION 5. Using (17), write formally the relations

$$\begin{aligned} dy^p + \eta_a^p(y^a)\omega^a &= y_i^p \varphi^i, \\ (20) \quad dy_i^p + \Phi_i^q(y^q, y_j^q, \omega^a, \varphi_j^k) &= y_{ij}^p \varphi^j, \\ &\vdots \\ dy_{i_1 \dots i_{r-1}}^p + \Phi_{i_1 \dots i_{r-1}}^q(y^q, \dots, y_{j_1 \dots j_{r-1}}^q, \omega^a, \varphi_j^k, \dots, \varphi_{j_1 \dots j_{r-1}}^k) &= y_{i_1 \dots i_{r-1} j}^p \varphi^j. \end{aligned}$$

Applying the exterior differentiation to the last row of (20) and replacing: (a) $d\varphi^j, \dots, d\varphi_{j_1 \dots j_{r-1}}^k$, according to the structure equations of φ_r , (b) $d\omega^a$ by $\frac{1}{2}c_{\beta\gamma}^a \omega^\beta \wedge \omega^\gamma$, according to the structure equations of G , (c) $dy^p, \dots, dy_{i_1 \dots i_{r-1}}^p$, according to (20), we obtain an expression of the form

$$[dy_{i_1 \dots i_r}^p + \Phi_{i_1 \dots i_r}^q(y^q, \dots, y_{j_1 \dots j_r}^q, \omega^a, \varphi_j^k, \dots, \varphi_{j_1 \dots j_r}^k)] \wedge \varphi^{i_r} = 0.$$

Then the equations of the fundamental distribution on $(L_n^r \times G) \times \bar{T}_n^r(F)$ are

$$(21) \quad \begin{aligned} dy^p + \eta_a^p(y^a) \hat{\omega}^a &= 0, \\ dy_i^p + \Phi_i^p(y^a, y_j^a, \hat{\omega}^a, \bar{\omega}_j^k) &= 0, \\ &\vdots \\ dy_{i_1 \dots i_r}^p + \Phi_{i_1 \dots i_r}^p(y^a, \dots, y_{j_1 \dots j_r}^a, \hat{\omega}^a, \bar{\omega}_j^k, \dots, \bar{\omega}_{j_1 \dots j_r}^k) &= 0, \end{aligned}$$

where $\bar{\omega}_j^k, \dots, \bar{\omega}_{j_1 \dots j_r}^k$ is the natural basis of \mathbb{L}_n^* .

Proof. (In this proof, we shall use freely the notation of [9].) Using [9], we first deduce an algorithm for finding the equations of the fundamental distribution on $G_n^r \times \bar{T}_n^r(F)$. Since $\tilde{J}^r E$ or $J^1(\tilde{J}^{r-1} E)$ can be considered as an associated bundle of the symbol $(B, \tilde{T}_n^r(F), \tilde{G}_n^r, \tilde{W}^r(P))$ or $(B, T_n^1(\tilde{T}_n^{r-1}(F)), (\tilde{G}_n^{r-1})_n^1, W^1(\tilde{W}^{r-1}(P)))$ respectively and it holds $J^1(\tilde{J}^{r-1}(E)) = \tilde{J}^r E$, the results of [9] imply directly an algorithm for finding the equations of the fundamental distribution on $\tilde{G}_n^r \times \tilde{T}_n^r(F)$. Then we obtain the equations of the fundamental distribution on $G_n^r \times \bar{T}_n^r(F)$ applying the inclusions $G_n^r \subset \tilde{G}_n^r$ and $\bar{T}_n^r(F) \subset \tilde{T}_n^r(F)$. This gives the following algorithm. Let

$$(22) \quad \begin{aligned} d\bar{y}^p + \eta_a^p(\bar{y}^a) \bar{\omega}^a &= 0, \\ d\bar{y}_i^p + \Psi_i^p(\bar{y}^a, \bar{y}_j^a, \bar{\omega}^a, \bar{\omega}_i^\lambda) &= 0, \\ &\vdots \\ d\bar{y}_{i_1 \dots i_{r-1}}^p + \Psi_{i_1 \dots i_{r-1}}^p(\bar{y}^a, \dots, \bar{y}_{j_1 \dots j_{r-1}}^a, \bar{\omega}^a, \bar{\omega}_j^\lambda, \dots, \bar{\omega}_{j_1 \dots j_{r-1}}^\lambda) &= 0 \end{aligned}$$

be the equations of the fundamental distribution on $G_n^{r-1} \times \bar{T}_n^{r-1}(F)$. Write formally the relations

$$(23) \quad \begin{aligned} dy^p + \eta_a^p(y^a) \theta^a &= y_i^p \theta^i, \\ dy_i^p + \Psi_i^p(y^a, y_j^a, \theta^a, \theta_i^\lambda) &= y_{ij}^p \theta^j, \\ &\vdots \\ dy_{i_1 \dots i_{r-1}}^p + \Psi_{i_1 \dots i_{r-1}}^p(y^a, \dots, y_{j_1 \dots j_{r-1}}^a, \theta^a, \theta_j^\lambda, \dots, \theta_{j_1 \dots j_{r-1}}^\lambda) &= y_{i_1 \dots i_{r-1} j}^p \theta^j. \end{aligned}$$

Applying the exterior differentiation to the last row of (23) and replacing $d\theta^\lambda, \dots, d\theta_{i_1 \dots i_{r-1}}^\lambda$, according to the structure equations of θ_r and $dy^p, \dots, dy_{i_1 \dots i_{r-1}}^p$, according to (23), we obtain an expression of the form

$$[dy_{i_1 \dots i_r}^p + \Psi_{i_1 \dots i_r}^p(y^a, \dots, y_{j_1 \dots j_r}^a, \theta^a, \theta_j^\lambda, \dots, \theta_{j_1 \dots j_r}^\lambda)] \wedge \theta^{i_r} = 0.$$

Then the equations of the fundamental distribution on $G_n^r \times \bar{T}_n^r(F)$ are

$$(24) \quad \begin{aligned} dy^p + \eta_a^p(y^a) \omega^a &= 0, \\ dy_i^p + \Psi_i^p(y^a, y_j^a, \omega^a, \omega_j^\lambda) &= 0, \\ &\vdots \\ dy_{i_1 \dots i_r}^p + \Psi_{i_1 \dots i_r}^p(y^a, \dots, y_{j_1 \dots j_r}^a, \omega^a, \omega_j^\lambda, \dots, \omega_{j_1 \dots j_r}^\lambda) &= 0. \end{aligned}$$

Furthermore, there is a natural injection $g \mapsto j_0^r \hat{g}$ of G into $T_n^r(G)$, where $\hat{g}: \mathbf{R}^n \rightarrow G$ is the constant mapping $x \mapsto g$; in this sense we may write $G \subset T_n^r(G)$. Then $L_n^r \times G$ is a subgroup of $G_n^r = L_n^r \bar{\times} T_n^r(G)$ and one sees easily that its differential equations are

$$\omega_i^r = 0, \dots, \omega_{i_1 \dots i_r}^r = 0, \quad r = n+1, \dots, n + \dim G.$$

Summarizing all these results into a direct algorithm, we obtain Proposition 5.

Remark 2. For $r = 1$, we obtain the following equations of the fundamental distribution on $(L_n^1 \times G) \times T_n^1(F)$

$$(25) \quad \begin{aligned} dy^p + \eta_a^p(y^a) \tilde{\omega}^a &= 0, \\ dy_i^p + \frac{\partial \eta_a^p}{\partial y^a} y_i^a \tilde{\omega}^a - y_j^p \tilde{\omega}_i^j &= 0. \end{aligned}$$

Hence the invariant formula for the second absolute differential is

$$(26) \quad da^p + \eta_a^p(a^a) \omega^a = a_i^p \varphi^i,$$

$$(27) \quad da_i^p + \partial_a \eta_a^p(a^r) a_i^a \omega^a - a_j^p \varphi_i^j = a_{ij}^p \varphi^j.$$

Remark 3. We shall evaluate the condition for a_{ij}^p to be symmetric in both subscripts. We have

$$\begin{aligned} d\omega^a &= \frac{1}{2} c_{\beta\gamma}^a \omega^\beta \wedge \omega^\gamma + R_{ij}^a \varphi^i \wedge \varphi^j, \\ d\varphi^i &= \varphi^j \wedge \varphi_j^i, \end{aligned}$$

where $R_{ij}^a \varphi^i \wedge \varphi^j$ is the curvature form of Γ . The exterior differentiation of (26) yields

$$[da_i^p + \partial_a \eta_a^p(a^r) a_i^a \omega^a - a_j^p \varphi_i^j + R_{ij}^a \eta_a^p(a^a) \varphi^j] \wedge \varphi^i = 0.$$

Comparing with (27), we find that a_{ij}^p are symmetric in both subscripts if and only if

$$(28) \quad \eta_a^p(a^a) R_{ij}^a \varphi^i \wedge \varphi^j = 0.$$

Since $\eta_a^p(y^a) \omega^a = 0$ are the differential equations of the stability group of an element of F , (28) can be explained geometrically as follows. Let G_x be the group of all isomorphisms of E_x , let $H_x \subset G_x$ be the stability group of $\sigma(x) \in E_x$ and let \mathfrak{g}_x or \mathfrak{h}_x be the corresponding Lie algebra, cf. [10] or [11]. The curvature form $\Omega(x)$ of Γ at x can be considered as an element of $\mathfrak{g}_x \otimes \Lambda^2 T_x^*(B)$ and (28) implies that $(V^2 \sigma)(x)$ is holonomic if and only if the projection of $\Omega(x)$ into $(\mathfrak{g}_x/\mathfrak{h}_x) \otimes \Lambda^2 T_x^*(B)$ vanishes. We have established this result in a quite different way in [10], Proposition 1.

References

- [1] J. Aczél und S. Gołąb, *Funktionalgleichungen der Theorie der geometrischen Objekte*, Warszawa 1961.
- [2] R. Crittenden, *Covariant differentiation*, Quart. J. Math. Oxford Ser. 13 (1962), p. 285-298.
- [3] C. Ehresmann, *Extension du calcul des jets aux jets non-holonomes*, CRAS Paris 239 (1954), p. 1762-1764.
- [4] — *Applications de la notion de jet non holonome*, CRAS Paris 240 (1954), p. 397-399.
- [5] — *Sur les connexions d'ordre supérieur*, Atti del V° Congresso del l'Unione Mathematica Italiana, 1955, Roma Cremonese, p. 344-346.
- [6] I. Kolář, *Geometric objects of submanifolds of a space with fundamental Lie pseudogroup*, Comment. Math. Univ. Carolinae 11 (1970), p. 227-234.
- [7] — *On the torsion of spaces with connection*, Czechoslovak Math. J. 21 (96) (1971), p. 124-136.
- [8] — *Canonical forms on the prolongations of principle fibre bundles*, Rev. Roumaine Math. Pures Appl. 16 (1971), p. 1091-1106.
- [9] — *On the prolongations of geometric object fields*, An. Şti. Univ. "Al. I. Cuza" Iaşi 17 (1971), p. 437-446.
- [10] — *Higher order torsions of spaces with Cartan connection*, Cahiers Topologie Géom. Différentielle, 12 (1971), p. 137-146.
- [11] — *On the contact of spaces with connection*, to appear in J. Differential Geometry.
- [12] M. Kucharzewski, *Elementy teorii obiektów geometrycznych*, Katowice 1969.
- [13] — and M. Kuczma, *Basic concepts of the theory of geometric objects*, Rozprawy Mat. 43, Warszawa 1964.
- [14] N. Que, *Du prolongement des espaces fibrés et des structures infinitésimales*, Ann. Inst. Fourier (Grenoble) 17 (1967), p. 157-223.
- [15] A. Szybiak, *Covariant differentiation of geometric objects*, Rozprawy Mat. 56, Warszawa 1967.
- [16] — *Canonical forms on a grupoid related to regular prolongations of Lie groups*, to appear in Colloq. Math.

INSTITUTE OF MATHEMATICS OF THE ČSAV
BRNO, CZECHOSLOVAKIA

Reçu par la Rédaction le 29. 10. 1971
