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# Reduction of invariant differential operators and expansions of conical distributions for $\mathbf{SO}_0(n, 1)$

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**Abstract.** We regard the unit $d$-dimensional sphere $S^d$ as the homogeneous space $\mathbf{SO}(d+1)/\mathbf{SO}(d)$ and propose a construction of radial parts for systems of differential operators transforming like a vector under the action of $\mathbf{SO}(d)$. This is applied to give a short self-contained proof of all basic facts concerning conical distributions and intertwining operators for the spherical principal series representations of $\mathbf{SO}_0(d+1, 1)$.

**Introduction.** Let $G$ be a connected semisimple Lie group of real rank one, $G = KAN$ an Iwasawa decomposition of $G$ and $S = K/M$ the maximal boundary space of the symmetric space $G/K$. Set $G_0 = \mathbf{SU}(2, 1)$ unless $G$ is locally isomorphic to $\mathbf{SO}(n, 1)$ in which case set $G_0 = \mathbf{SU}(1, 1)$ and denote by $S_0$ the maximal boundary space of $G_0/K_0$. Helgason's $\mathbf{SU}(2, 1)$ reduction implies that there are compatible imbeddings $G_0 \to G$ and $S_0 \to S$, i.e., denoting by $S^*$ the image of $S_0$ in $S$, the action of $G_0$ on $S^*$ by means of the former imbedding is equivalent to the action of $G_0$ on $S_0$.

By virtue of Kostant double transitivity the natural map $M \times S^* \to S$ is surjective; hence every function on $S$ of a given $M$-type (i.e., such that $M$ acts by a multiple of a given irreducible representation on the space spanned by $M$-translates of the function) is determined by its restriction to $S^*$, the same being true in an appropriate sense for distributions. Thus one might try to use $M$-covariance to reduce some questions concerning $G$ to questions concerning much simpler group $G_0$. In an intended series of papers, of which the present one is the first, we plan to use this method of reduction to study problems in the representation theory connected with conical distributions. This is motivated partly by Lepowsky's algebraic transference for conical vectors [16] and partly by the desire to put on a firm footing our earlier and somewhat ad hoc approach to determining conical vectors for $\mathbf{SO}_0(n, 1)$ [20]. Two earlier closely related constructions are in [10], namely the computation of the Poisson kernel for rank one symmetric spaces by means of $\mathbf{SU}(2, 1)$-reduction (cf. also [19]) and the construction of conical functions for not necessarily rank one semisimple Lie groups by extending conical functions for $\mathbf{SU}(1, 1)$. However, in those both cases the reduction has been used for

$M$-invariant smooth functions only, whereas we apply it here mainly for invariant distributions.

In the previous paper [20] we proposed a differential equation, or rather an $M$-invariant system of equations, whose $M$-invariant distribution solutions were precisely conical vectors. These were then found for $\mathbf{SO}_0(n, 1)$ by an explicit calculation, which by use of suitable coordinates reduced the system to a single equation on the circle $S^1$. Here by using the coordinate-free functorial description of $M$-invariant distributions based on [8] we show this reduction is a particular case of a construction of radial parts of invariant systems of differential operators, or, as we prefer to view things, of invariant operators acting on vector valued functions. The radial parts turn out to be scalar differential operators (in general singular) on $S^*$. This approach is taken from the forthcoming paper of the author [21] and is presented here without proofs in Section 2. Applying this construction to gradient-like systems obtained from the infinitesimal spherical representation of $\mathbf{SO}_0(n, 1)$ (we consider here those corresponding to $M$-irreducible subspaces of $\mathfrak{n}$ appearing in the restricted root spaces decomposition) we show these radial parts to be precisely the operators of the infinitesimal spherical representation of the group $G_0 = \mathbf{SO}_0(2, 1) \approx \mathbf{SU}(1, 1)$. This opens the way to a short proof of all the main facts concerning conical vectors for $\mathbf{SO}_0(n, 1)$, which were established previously in [9], [10], [13]. By using ultraspherical expansions of generalized functions on $S^*$ representing conical vectors we obtain at the same time the diagonalization of intertwining operators for the spherical principal series representations ([14], [23], [25]) together with some Sobolev type estimates for intertwining operators. This is done in Section 3.

Some more details concerning the content of the paper can be read off at the beginning of each section. In an immediate sequel to the paper we shall discuss the analytic counterpart of Lepowsky transference for conical vectors by using the present formalism.

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**1. Notations and structural preliminaries.** For an integer $d \geqslant 1$, we shall denote by $S^d$ the unit sphere in $\boldsymbol{R}^{d+1}$ and by $G = G^{(d)}$ the group $\mathbf{SO}_0(d+1, 1)$, i.e., the identity component of the group of all automorphisms of $\boldsymbol{R}^{d+2}$ which preserve the bilinear form

$$[x|y] = \sum_{i=0}^{d} x_i y_i - x_{d+1} y_{d+1}.$$

We shall write elements of $G$ in the matrix form (superscript $t$ denoting transposition)

$$g = \begin{bmatrix} A & u \\ {}^t w & z \end{bmatrix} \tag{1.1}$$

with $A \in M_{d+1}(\mathbf{R})$, $u, w \in \mathbf{R}^{d+1}$ and $z \in \mathbf{R}$ (satisfying appropriate conditions assuring invariance of the form $[\cdot|\cdot]$). It is well known that $G$ is isomorphic to the group of orientation-preserving conformal (with respect to the standard riemannian metric) diffeomorphisms of $S^d$ – Möbius transformations when $S^d$ is regarded as $\mathbf{R}^d \cup \{\infty\}$. We fix matters by defining an action of $G^{(d)}$ on $S^d$ by transfering to $S^d$ the projective action on the upper sheet of the cone $\{x \in \mathbf{R}^{d+2} - \{0\} | [x| x] = 0\}$. Explicitely, given $b \in S^d$ we denote $\tilde{b} = (b, 1) \in \mathbf{R}^{d+2}$ the corresponding point of the cone and for $g \in G^{(d)}$ we set

$$g \cdot b := |[e_{d+1}| g\tilde{b}]|^{-1} g\tilde{b} \tag{1.2}$$

(here $e_i$, $i = 0, \ldots, d+1$, denotes vectors of the standard base for $\mathbf{R}^{d+2}$ and $\mathbf{R}^{d+1}$ is considered as the hyperplane $x_{d+1} = 0$ in $\mathbf{R}^{d+2}$).

The formula make sense also if $b \in B = \{x \in \mathbf{R}^{d+1} | \|x\| < 1\}$, the open unit ball in $\mathbf{R}^{d+1}$ and the thus arising action of $G^{(d)}$ on $B$ gives rise to a realization of $G^{(d)}$ as the connected component of the isometry group of the $(d+1)$-dimensional real hyperbolic space (cf. [12]).

Let $K = K^{(d)}$ be the isotropy subgroup of $G^{(d)}$ at $0 \in B$; clearly, $K$ consists of matrices

$$\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}, \qquad R \in \mathbf{SO}(d+1)$$

and acts transitively on $S^d$. Denoting by $M = M^{(d)}$ the isotropy subgroup of $K$ at $e_0 \in S^d$, we see that $M \approx \mathbf{SO}(d)$ and $S^d = K/M$. Let now $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$ denote the matrix Lie algebras of $G$, $K$ and $M$, respectively, and let us further define

$$H := \begin{bmatrix} 0 & e_0 \\ {}^t e_0 & 0 \end{bmatrix}, \tag{1.3}$$

$e_0 \in \mathbf{R}^{d+1}$ (cf. (1.1)) and

$$Z(x) = \begin{bmatrix} 0 & {}^t x & 0 \\ -x & 0 & x \\ 0 & {}^t x & 0 \end{bmatrix}, \qquad \bar{Z}(x) = \begin{bmatrix} 0 & {}^t x & 0 \\ x & 0 & x \\ 0 & {}^t x & 0 \end{bmatrix} \tag{1.4}$$

for any $x \in \mathbf{R}^d$. Then

$$[H, Z(x)] = Z(x), \qquad [H, \bar{Z}(x)] = -\bar{Z}(x), \qquad x \in \mathbf{R}^d.$$

Setting $\mathfrak{a} := \boldsymbol{R}H$, $\mathfrak{n} := \{Z(x) | x \in \boldsymbol{R}^d\}$, $\bar{\mathfrak{n}} := \{\bar{Z}(x) | x \in \boldsymbol{R}^d\}$, we have the decomposition of Iwasawa

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

and the restricted root spaces decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}.$$

We note that $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are abelian Lie algebras isomorphic to $\boldsymbol{R}^d$ and that in terms of the parametrization (1.4) the action of $M$ on $\mathfrak{n}$ ($\bar{\mathfrak{n}}$, resp.) via adjoint representation is just the natural representation of $\mathbf{SO}(d)$ acting on $\boldsymbol{R}^d$. Moreover, the restricted root spaces decomposition is a decomposition into $M$-irreducible subspaces.

Let $A$, $N$, $\bar{N}$ denote the analytic subgroups of $G$ with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$, resp., and set

$$m_* := \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & 1 & \\ & & & \\ 0 & & & 1 \end{bmatrix} \in K.$$

Then we have (global) Iwasawa decomposition $G = KAN$ and the Bruhat decomposition $G = MAN \cup Nm_* MAN$ (disjoint union). Observe that $MAN =: P$ is the isotropy subgroup of $G^{(d)}$ at the point $e_0 \in S^d$. We shall write $g = k(g) \exp t(g) Hn$ with $k(g) \in K$, $\exp t(g) H \in A$, $n \in N$, and observe that

$$e^{t(g)} = |[e_{d+1} | g(e_0 + e_{d+1})]|, \tag{1.5}$$

$$g \cdot ke_0 = k(gk) e_0. \tag{1.6}$$

Assume now $d \geq 2$ and consider the embedding $G^{(1)} \ni g \to g^* \in G^{(d)}$ obtained by defining $g^*$ to be identity on the subspace of $\boldsymbol{R}^{d+2}$ spanned by $e_2, \ldots, e_d$ and $g$ on the span of $e_0, e_1, e_{d+1}$. Further, let $X \to X^*$ denote the corresponding imbedding of the Lie algebra $\mathfrak{g}^{(1)}$ into $\mathfrak{g}^{(d)} := \mathfrak{g}$ and let $\iota$: $S^1 \to S^d$ be given as $S^1 \ni s \to \iota(s) = (s, 0, \ldots, 0) \in S^d$. Then these imbeddings are compatible in the sense that

$$g^* \cdot \iota(s) = \iota(gs), \tag{1.7}$$

for all $g \in G^{(1)}$, $s \in S^1$ (cf. [12]) and furthermore the mapping

$$M \times S^1 \ni (m, s) \to m \cdot \iota(s) \in S^d$$

is surjective. Note the obvious relation of the latter map to the system of spherical coordinates on $S^d$.

**2. Invariant distributions and radial parts of invariant systems of differential operators.** We state now some results taken from [21], concerning a method of reduction for $M$-invariant systems of differential operators. They are motivated chiefly by the application to conical vectors, to be discussed in the following section, but since such systems occur rather frequently in the recent literature (cf. [7] and the references there), we do not restrict ourselves to the particular system determining conical vectors. Our approach is based on an explicit representation of $M$-invariants in distribution spaces on $S^d$ in terms of their (appropriately defined) restrictions to the maximal torus $S^1 \subset S^d$ and consists in associating to each $M$-invariant system of differential operators on $S^d$ a single differential operator, possibly with singularities, acting between spaces of these restrictions. This resembles the coordinate-free description of the radial part of a single differential operator (cf. [11]) and, in fact, is a natural extension of that construction. For more details we refer to the forthcoming paper of the author [21].

We use standard notation $\mathcal{E}(S^d)$, $\mathcal{E}'(S^d)$, resp., for the space of smooth (complex valued) functions on $S^d$ taken with its customary (Schwartz) topology and resp. for the space of distributions (usually endowed with the weak topology). Also we shall denote by $\mathcal{E}(S^d; \boldsymbol{C}^d)$, resp. $\mathcal{E}'(S^d; \boldsymbol{C}^d)$, the space of $\boldsymbol{C}^d$-valued smooth functions on $S^d$ with the natural topology, resp. of $\boldsymbol{C}^d$-distributions, i.e., continuous linear functionals on $\mathcal{E}(S^d; \boldsymbol{C}^d)$. In the sequel one uses substantially the fact that $\mathcal{E}(S^d)$ and $\mathcal{E}(S^d; \boldsymbol{C}^d)$ are Fréchet spaces. $M$ is acting naturally on $\mathcal{E}(S^d)$ and also on $\mathcal{E}(S^d; \boldsymbol{C}^d)$, the latter action given by

$$(2.1) \qquad m \cdot F(b) = mF(m^{-1} b), \qquad b \in S^d,$$

where on the right-hand side the natural matrix action of $M \approx \mathbf{SO}(d)$ on $\boldsymbol{C}^d$ is denoted by juxtaposition. In the respective distribution spaces we shall consider dual actions of $M$. Note that we can imbed $\mathcal{E}(S^d) \to \mathcal{E}'(S^d)$ using the canonical rotationally invariant measure $db$ on $S^d$ (which we shall always assume normalized by $\mathrm{vol}(S^d) = 1$) and similarly we imbed $\mathcal{E}(S^d; \boldsymbol{C}^d) \to \mathcal{E}'(S^d; \boldsymbol{C}^d)$ by means of the bilinear pairing

$$(2.2) \qquad \mathcal{E}(S^d; \boldsymbol{C}^d) \times \mathcal{E}(S^d; \boldsymbol{C}^d) \ni (F, H) \to \langle F, H \rangle := \int_{S^d} \big(F(b) | H(b)\big) db \in \boldsymbol{C},$$

where $(\cdot|\cdot)$ denotes both the standard inner product on $\boldsymbol{R}^d$ and its complex bilinear extension to $\boldsymbol{C}^d$.

These imbeddings are compatible with the action of $M$.

Let $\mathcal{E}(S^d)^M$, $\mathcal{E}(S^d; \boldsymbol{C}^d)^M$ be the subspaces of $M$-invariant elements in $\mathcal{E}(S^d)$ or $\mathcal{E}(S^d; \boldsymbol{C}^d)$, respectively, and denote the subspaces of $M$-invariant distributions accordingly by $\mathcal{E}'(S^d)^M$, $\mathcal{E}'(S^d; \boldsymbol{C}^d)^M$. Also consider the action of the group $\boldsymbol{Z}_2$ of integers mod 2 on the torus $S^1 \subset \boldsymbol{R}^2$, which for the nontrivial element $w \in \boldsymbol{Z}_2$ is just the reflection in the $x_0 = 0$ axis. Usually we shall regard $S^1$ as contained in $S^d$ via the imbedding $\iota$ from Section 1 and then this action is just the action of the Weyl group of $(K, M)$, $W = M_1'/M_1$ on $S^1$, where $M_1'$, $M_1$ are the normalizer, respectively, centralizer of $S^1$ in $M$. Regarding the induced action of $\boldsymbol{Z}_2$ on the space of $\mathscr{E}(S^1)$ of smooth functions on $S^1$, we shall write $\mathscr{E}_+ = \mathscr{E}_+(S^1)$ for the subspace of $\boldsymbol{Z}_2$-invariant (i.e., even) functions and $\mathscr{E}_- = \mathscr{E}_-(S^1)$ for the subspace of odd functions which we consider with their respective subspace topology.

LEMMA 2.1. (i) *The map* $\theta^*$: $\mathscr{E}_+(S^1) \to \mathscr{E}(S^d)^M$ *defined by*

$$\theta^* f(m\iota(s)) := f(s), \quad s \in S^1, \ m \in M, \tag{2.3}$$

*is a topological isomorphism.*

(ii) *Let* $\theta_1^*$: $\mathscr{E}_-(S^1) \to \mathscr{E}(S^d; \boldsymbol{C}^d)$ *be defined by*

$$\theta_1^* f(m\iota(s)) = f(s) m e_1, \quad s \in S^1, \ m \in M. \tag{2.4}$$

*Then, for* $d > 2$, $\theta_1^*$ *is a topological isomorphism onto* $\mathscr{E}(S^d; \boldsymbol{C}^d)^M$ *and for* $d = 2$, $\theta_1^*$ *is a topological isomorphism onto* $\mathscr{E}(S^d; \boldsymbol{C}^d)^{\tilde{M}}$, *where* $\tilde{M} = \mathbf{O}(2)$ *is considered as the isotropy subgroup of* $\mathbf{O}(3)$ *at the point* $e_0 \in \boldsymbol{R}^3$ *and its action on* $\mathscr{E}(S^d; \boldsymbol{C}^d)$ *is still defined by* (2.1).

Part (i) is a special case of Dadok's result characterizing functions on symmetric spaces invariant with respect to the isotropy group [2]. For functions on spheres invariant with respect to other subgroups of the orthogonal subgroup see [5] and for an extension of (ii) to other irreducible representations of $M$ see [21].

The corresponding parametrization of invariant distributions is obtained as follows. The invariant integral on $S^d$ can be written in the form (essentially expressing the integration in the spherical coordinates)

$$\int_{S^d} f(b)\, db = \int_{S^1} \delta(s) \int_M f(m\iota(s))\, dm\, ds, \tag{2.5}$$

where $dm$ is the normalized Haar measure on $M$, $ds$ the normalized Lebesgue measure on $S^1$ and $\delta$ is the density function (recall the coordinates on $S^1$ are $s_0 = (s|e_0)$ and $s_1 = (s|e_1)$)

$$\delta(s) = c_d |(s|e_1)|^{d-1}, \tag{2.6}$$

where

$$c_d = \pi^{1/2}\,\Gamma\left(\frac{d+1}{2}\right) \Big/ \Gamma\left(\frac{d}{2}\right)$$

is the normalization factor, $\Gamma(\cdot)$ denoting the Euler gamma function. Now introduce the spaces $\mathscr{M}_+(S^1) = \mathscr{M}_+^d(S^1)$ and $\mathscr{M}_-(S^1) = \mathscr{M}_-^d(S^1)$ consisting respectively, of all functions of the form $f = \delta f_0$, $f_0 \in \mathscr{E}_+(S^1)$, resp., $f_0 \in \mathscr{E}_-(S^1)$, endowed with the topology carried over from $\mathcal{E}_+(S^1)$, $\mathcal{E}_-(S^1)$ resp., by means of the multiplication with $\delta$. By $\langle\cdot,\cdot\rangle$ we shall denote the nonsingular pairing between $\mathcal{E}_+(S^1)$ and $\mathcal{H}_+(S^1)$, resp., between $\mathcal{E}_-(S^1)$ and $\mathcal{H}_-(S^1)$ given in each case by the integration over $S^1$, i.e.,

$$\langle f, h\rangle := \int_{S^1} fh\,ds. \tag{2.7}$$

One sees easily that the dual map to $\theta^*$, denoted $\theta_*$, maps continuously $\mathcal{E}(S^d)$ onto $\mathcal{H}^d_+(S^1)$ and similarly the dual $\theta_{1*}$ to $\theta_1^*$ maps continuously $\mathcal{E}(S^d; \mathbf{C}^d)$ onto $\mathcal{H}^d_-(S^1)$. Let now $\mathcal{H}'_+(S^1)$, $\mathcal{H}'_-(S^1)$ be the spaces of continuous linear functionals on $\mathcal{H}_+(S^1)$, resp., $\mathcal{H}_-(S^1)$. Denoting $dm_\alpha(s) := |(s|e_1)|^{2\alpha}ds$, $\alpha = (d-1)/2$, we consider the subspaces $L^1_+(S^1, dm_\alpha)$ and $L^1_-(S^1, dm_\alpha)$ in $L^1(S^1, dm_\alpha)$ consisting of even, odd, resp., functions and observe that via (2.7) they can be considered as subspaces of $\mathcal{H}'_+(S^1)$, resp., $\mathcal{H}'_-(S^1)$.

PROPOSITION 2.2. (i) *The dual to $\theta_*$ maps isomorphically $\mathcal{H}'_+(S^1)$ onto $\mathcal{E}'(S^d)^M$.*

(ii) *The dual to $\theta_{1*}$ maps isomorphically $\mathcal{H}'_-(S^1)$ onto $\mathcal{E}'(S^d; \mathbf{C}^d)^M$ for $d > 2$ and onto $\mathcal{E}'(S^d; \mathbf{C}^d)^{\tilde{M}}$ if $d = 2$.*

*Since these maps extend $\theta^*$, resp. $\theta_1^*$, defined previously on smooth even, odd, resp., functions on $S^1$, we shall continue to use $\theta^*$, resp. $\theta_1^*$, to denote these extensions as well.*

Remark. One can view the maps $\theta^*$, $\theta_1^*$ as "pull-back"-maps and $\theta_*$, $\theta_{1*}$ as "push-forward" maps with respect to the orbit map $\theta$: $S^d \to M\backslash S^d \approx S^1/\mathbf{Z}_2$. See [8], Chapter VI, for the terminology and the construction in great generality.

Due to the functorial character of those constructions the treatment of invariant systems of differential operators is particularly simple. Let us consider a system $D_i$, $i = 1, \ldots, d$, of (scalar) differential operators $D_i$: $\mathcal{E}(S^d) \to \mathcal{E}(S^d)$ transforming like a vector under the action of $M$ ($\approx \mathbf{SO}(d)$). That is, defining for a differential operator $D$ the "shifted by $m\in M$" operator $D^m$ by

$$D^m = \tau(m)\circ D\circ\tau(m^{-1}),$$

where $\tau(m)$ is the translation $\tau(m)f(b) = f(m^{-1}b)$ on $\mathcal{E}(S^d)$, we assume that

$$D_i^m = \sum_{k=1}^{d} m_{ki} D_k, \tag{2.8}$$

where the matrix $m_0 = [m_{ki}]\in\mathbf{SO}(d)$ is given by

$$m = \begin{bmatrix} 1 & & \\ & m_0 & \\ & & 1 \end{bmatrix} \in M.$$

Then the differential operator $D\colon \mathcal{E}(S^d) \to \mathcal{E}(S^d; C^d)$ defined by

$$Df := (D_1 f, \ldots, D_d f)$$

is easily verified to be $M$-invariant and so is its formal adjoint $D^*\colon \mathcal{E}(S^d; C^d) \to \mathcal{E}(S^d)$ defined by

$$\langle D^* F, f\rangle := \langle F, Df\rangle, \qquad F \in \mathcal{E}(S^d; C^d),\ f \in \mathcal{E}(S^d).$$

Therefore $D\big(\mathcal{E}(S^d)^M\big) \subset \mathcal{E}(S^d; C^d)^M$, so there exists a unique mapping of $\mathcal{E}_+(S^1)$ to $\mathcal{E}_-(S')$ such that the diagram below commutes

$$\begin{array}{ccc} \mathcal{E}(S^d) & \xrightarrow{D} & \mathcal{E}(S^d; C^d) \\ \theta^* \uparrow & & \theta_1^* \uparrow \\ \mathcal{E}_+(S^1) & \longrightarrow & \mathcal{E}_-(S^1) \end{array}$$

and there is a similar commutative diagram for $D^*$, too. We shall denote the mapping in the lower line by $\Delta(D)$, resp. by $\Delta(D^*)$, and call it the *radial part* of $D$, resp. $D^*$, so that we have

$$\text{(2.9)} \qquad \begin{aligned} D\theta^* f &= \theta_1^* \Delta(D) f, \qquad f \in \mathcal{E}_+(S^1), \\ D^* \theta_1^* f &= \theta^* \Delta(D^*) f, \qquad f \in \mathcal{E}_-(S^1). \end{aligned}$$

Now note that the set $\iota(S^1)_r$ of regular points in the maximal torus $\iota(S^1) \subset S^d$ (that is, the torus $\iota(S^1)$ with the two poles $\pm e_0$ deleted) satisfies the transversality condition of Proposition 2.2 from [11], and hence there exists a unique differential operator $\Delta(D_1)$, the radial part of $D_1$, defined on $\iota(S^1)_r$ and satisfying (the bar denoting the restriction to $\iota(S^1)_r$)

$$\Delta(D_1)\bar{f} = (D_1 f)^-$$

for each $f \in \mathcal{E}(S^d)^M$. Actually this holds for much broader class of functions which are locally invariant in an appropriate sense, but we shall not need this fact.

To simplify notation in the following we shall identify the torus $S^1$ with its imbedded image $\iota(S^1)$ in $S^d$, so we shall write $S^1_r$ instead of $\iota(S^1)_r$ and identify smooth functions on $\iota(S^1)$ with elements of $\mathcal{E}(S^1)$. Now we may state the following proposition.

PROPOSITION 2.3. *For every* $f \in \mathcal{E}_+(S^1)$

$$\big(\Delta(D) f\big)\big|_{S^1_r} = \Delta(D_1)\big(f|_{S^1_r}\big)$$

*and similarly, for every* $f \in \mathcal{E}_-(S^1)$,

$$\big(\Delta(D^*) f\big)\big|_{S^1_r} = \delta^{-1} \circ \Delta(D_1) \circ \delta\big(f|_{S^1_r}\big).$$

*Here* $\delta$ *is considered as an operator of multiplication by* $\delta$, *which is obviously smooth on* $S^1_r$ *and* $\circ$ *denotes superposition of operators.*

This is a particular case of a more general result on radial parts of covariant systems of differential operators proved in [21]. Consider now the validity of the diagram as above for spaces of invariant distributions. We argue the case of $D$, the other being completely analogous. By virtue of $M$-invariance of $D$ there is an operator which we call $\Delta(D)^+$, for reasons to become clear very soon, such that $\theta_* D^* = \Delta(D)^+ \theta_{1*}$. Its continuity follows by applying the open map theorem to $\theta_*$. Formulas (2.9) and the elementary properties of $\theta_1^*$ and $\theta_*$ imply that for each $f \in \mathcal{E}_+(S^1)$ and each $h \in \mathcal{M}_-(S^1)$

$$\langle \Delta(D) f, h \rangle = \langle f, \Delta(D)^+ h \rangle,$$

that is, $\Delta(D)^+$ is dual to $\Delta(D)$ with respect to the pairings given by (2.7). Taking the mapping of $\mathcal{M}'_+(S^1)$ into $\mathcal{M}'_-(S^1)$ dual to $\Delta(D)^+$ we see that it extends $\Delta(D)$: $\mathcal{E}_+(S^1) \to \mathcal{E}_-(S^1)$ and, so denoting it still by $\Delta(D)$, we obtain commutativity of the diagram:

$$\begin{array}{ccc} \mathcal{E}'(S^d) & \xrightarrow{D} & \mathcal{E}'(S^d; C^d) \\ \theta^* \uparrow & & \theta_1^* \uparrow \\ \mathcal{M}'_+(S^1) & \xrightarrow{\Delta(D)} & \mathcal{M}'_-(S^1). \end{array}$$

For the sequel we shall need the following (standard) notation. If $X \in \mathfrak{g}$, then the induced vector field on $S^d$ is denoted by $X^+$:

$$X^+ f(b) := -\frac{d}{dt}\Big|_{t=0} f(\exp tX \cdot b).$$

EXAMPLE 2.4. Let $\mathfrak{l} \subset \mathfrak{k}$ be the orthocomplement (with respect to the negative of the Killing form) of $\mathfrak{m}$. Then $\mathfrak{l}$ is $\mathrm{Ad}(M)$-invariant and the representation of $M$ in $\mathfrak{l}$ is equivalent to the natural matrix action of $\mathbf{SO}(d)$ on $\boldsymbol{R}^d$. Let $(X_i)$ be an orthonormal basis for $\mathfrak{l}$ and assume that $X_1$ is tangent to the torus $S^1$ at $e_0$ $(= eM) \in S^d$. Then the operators $X_i^+$ satisfy (2.8), and hence the preceding applies to the operator $D_\mathfrak{l} = (X_1^+, \ldots, X_d^+)$. Letting $\omega = -D_\mathfrak{l}^* D_\mathfrak{l}$, we see that, for each $f \in \mathcal{E}(S^d)^M$, $\omega(f) = L(f)$ where $L$ is the Laplace–Beltrami operator on $S^d$. Since by functoriality we have $\Delta(D_\mathfrak{l}^* D_\mathfrak{l}) = \Delta(D_\mathfrak{l}^*)\Delta(D_\mathfrak{l})$, we may immediately deduce from Proposition 2.3 the familiar expression for the radial part of the Laplace–Beltrami operator.

Our main application of the preceding formalism is the case of differential operators arising from the infinitesimal action of $G$ corresponding to the spherical principal series representation of $G$ in the so-called *compact picture*, i.e., acting on functions on $S^d$. Recall the Poisson kernel for the hyperbolic space $B = G^{(d)}/K^{(d)}$ given by $P(gK, kM) = |[ge_{d+1}|\ k(e_0 + e_{d+1})]|^{-d}$ and set

$$\sigma(g, b) := |[ge_{d+1}|\ \tilde{b}]|, \quad g \in G^{(d)},\ b \in S^d, \tag{2.10}$$

where as in Section 1 we write $\tilde{b} = (b, 1) \in \boldsymbol{R}^{d+2}$. Then for any $\lambda \in \boldsymbol{C}$ one has representation $\tau_\lambda = \tau_\lambda^{(d)}$ of $G^{(d)}$ acting on $\mathcal{E}(S^d)$ by

$$\tau_\lambda(g) f(b) = \sigma(g, b)^{i\lambda - d/2} f(g^{-1} \cdot b), \quad g \in G^{(d)},\ b \in S^d.$$

The infinitesimal representation of the Lie algebra $\mathfrak{g}^{(d)}$, which we denote by the same symbol, is given by

$$\tau_\lambda(X)f(b) = \big(X^+ - (i\lambda - d/2)\,\sigma(X, b)\big)f(b), \tag{2.11}$$

where for $X \in \mathfrak{g}^{(d)}$ we have

$$\sigma(X, b) = \frac{d}{dt}\Big|_{t=0} \sigma(\exp tX, b) = [Xe_{d+1}|\ \tilde{b}],$$

and satisfies (* denoting the adjoint with respect to the measure $db$ on $S^d$)

$$\tau_\lambda(X)^* = -\tau_{-\lambda}(X), \qquad X \in \mathfrak{g}.$$

Consider bases $Z_i := Z(e_i)$, $i = 1, \ldots, d$, and $\bar{Z}_i := \bar{Z}(e_i)$, $i = 1, \ldots, d$, for $\mathfrak{n}$, $\bar{\mathfrak{n}}$, resp., defined according to (1.4). For $d = 1$ write $Z = Z(1)$ and $\bar{Z} = \bar{Z}(1)$ for the corresponding generators of $\mathfrak{n}^{(1)}$, $\bar{\mathfrak{n}}^{(1)}$, resp., and $h$ for the generator of $\mathfrak{a}^{(1)}$. Note $H = h^*$, and note also that in this latter case $\mathfrak{g}^{(1)} = \bar{\mathfrak{n}}^{(1)} + \mathfrak{a}^{(1)} + \mathfrak{n}^{(1)}$. It has been observed before that the action of $M$ on each of $\mathfrak{n}$, $\bar{\mathfrak{n}}$ in terms of the above-given bases is just the matrix action of $\mathbf{SO}(d)$ on $\mathbf{R}^d$; hence by virtue of the $M$-equivariance of the mapping

$$\mathfrak{g} \times \mathscr{E}(S^d) \ni (X, f) \to \tau_\lambda(X) f \in \mathscr{E}(S^d)$$

the sets $\{\tau_\lambda(Z_i)\}$ and $\{\tau_\lambda(\bar{Z}_i)\}$ satisfy (2.8). We write

$$D_\lambda := \big(\tau_\lambda(Z_1), \ldots, \tau_\lambda(Z_d)\big), \qquad \bar{D}_\lambda := \big(\tau_\lambda(\bar{Z}_1), \ldots, \tau_\lambda(\bar{Z}_d)\big).$$

The following result systematizes an argument from [20].

PROPOSITION 2.5. *Given $\lambda \in \mathbf{C}$, set $\mu = \lambda + i\alpha$, $\alpha = (d-1)/2$. Then*

$$\Delta(D_\lambda) = \tau_\mu^{(1)}(Z), \qquad \Delta(\bar{D}_\lambda) = \tau_\mu^{(1)}(\bar{Z}), \qquad \Delta\big(\tau_\lambda(H)\big) = \tau_\mu^{(1)}(h). \tag{2.12}$$

*In the last equation the radial part is that of a scalar differential operator.*

Remark. To be completely precise we should have indicated that there are restrictions to $\mathscr{E}'_+(S^1)$ on the right-hand side in each of the above equalities, but we shall not do that in order to keep notation simple.

Proof. Using the notation from Section 1, if $X^* \in \mathfrak{g}^{(d)}$ is the image of $X \in \mathfrak{g}^{(1)}$ under the imbedding $\mathfrak{g}^{(1)} \to \mathfrak{g}^{(d)}$, then the vector fields $(X^*)^+$ on $S^d$ and $X^+$ on $S^1$ are $\iota$-related by virtue of (1.7). Letting $\sigma^{(1)}$ to denote the cocycle (2.10) for the action of $G^{(1)}$ on $S^1$, a simple matrix computation shows that

$$\sigma\big(X^*, \iota(s)\big) = \sigma^{(1)}(X, s).$$

Since $Z_1 = Z^*$, $\bar{Z}_1 = \bar{Z}^*$, $H = h^*$, Proposition 2.3 and (2.11) give the result.

For the later use we shall record here the coordinate form of the operators appearing in Proposition 2.5. If $\varphi$ denotes the angular coordinate on the torus $S^1$, then

$$\tau_\mu^{(1)}(Z) = (1-\cos\varphi)\frac{d}{d\varphi} - (i\mu - \tfrac{1}{2})\sin\varphi,$$

$$\tau_\mu^{(1)}(\bar{Z}) = -(1+\cos\varphi)\frac{d}{d\varphi} - (i\mu - \tfrac{1}{2})\sin\varphi, \tag{2.13}$$

$$\tau_\mu^{(1)}(h) = \sin\varphi\frac{d}{d\varphi} - (i\mu - \tfrac{1}{2})\cos\varphi.$$

Another useful application of the representation of $M$-invariant distributions in the form

$$\Psi = \theta^* T, \qquad T \in \mathcal{M}'_+(S^1) \tag{2.14}$$

is to analyse the operators $\mathcal{E}(S^d) \to \mathcal{E}(S^d)$ commuting with the action of $K$ on $\mathcal{E}(S^d)$.

In fact, every such map assuming appropriate continuity can be written as the convolution operator $f \to f * \Psi$ with an uniquely determined $\Psi \in \mathcal{E}'(S^d)^M$. Here $*$ denotes the convolution on $S^d$ induced by the usual convolution on $K$ (cf. [10], p. 84) which is given explicitely for $M$-invariant $\Psi$ by the formula

$$f * \Psi(ke_0) = \Psi(\tau(k^{-1})f),$$

with $\tau$ denoting the action of $K$ on $S^d$ by translation. Any such convolution operator, regarded as an (in general unbounded) operator in $L^2(S^d)$ with domain $\mathcal{E}(S^d)$ is diagonalized by the decomposition

$$L^2(S^d) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n(S^d) \tag{2.15}$$

in (surface) spherical harmonics (which is the decomposition of the representation $\tau$ into irreducibles).

The actual diagonalization is achieved by using expansion of $\Psi$ into spherical harmonics or related decomposition of $T$. At first we observe that the usual definition of ultraspherical expansions of functions defined on $[0, \pi]$ can be rephrased so as to apply to $\mathcal{M}'_+(S^1)$.

In fact, let $P_n^{(\alpha)}$ denote the ultraspherical polynomial of type $\alpha = (d-1)/2$ and degree $n$,

$$P_n^{(\alpha)}(x) = q(\alpha, n)(1-x^2)^{-\alpha+1/2}\left(\frac{d}{dx}\right)^n (1-x^2)^{n+\alpha-1/2},$$

where the normalization constant $q(\alpha, n)$ is given as

$$q(\alpha, n) = \frac{(-2)^n}{n!}\frac{\Gamma(n+\alpha)\,\Gamma(n+2\alpha)}{\Gamma(\alpha)\,\Gamma(2n+2\alpha)}.$$

We recall that $S^d \ni b \to P_n^{(\alpha)}(1)^{-1} P_n^{(\alpha)}((b|e_0)) =: \omega_n(b)$ are the zonal spherical harmonics of degree $n$ on $S^d$ normalized by $\omega_n(e_0) = 1$ and that

$$\int_{S^d} \omega_n(b)^2\, db = P_n^{(\alpha)}(1)^{-2} \int_{S^1} P_n^{(\alpha)}((s|e_0))^2\, \delta(s)\, ds = \dim \mathscr{H}^n(S^d)^{-1}.$$

We set

$$\gamma_n^\alpha := \dim \mathscr{H}^n(S^d) = (2n+d-1)\frac{(n+d-2)!}{n!\,(d-1)!}.$$

Given $T \in \mathscr{M}'_+(S^1)$, we define its *ultraspherical Fourier coefficients* by

(2.16) $$c_n(T) := \langle T, \delta R_n^\alpha \rangle,$$

where

$$R_n^\alpha(s) = P_n^{(\alpha)}(1)^{-1} P_n^{(\alpha)}((s|e_0)), \quad s \in S^1,$$

and we call the series $\sum_{n=0}^{\infty} \gamma_n^\alpha c_n(T) R_n^\alpha$ the *ultraspherical expansion* of $T$, to be denoted by

(2.17) $$T = \sum_{n=0}^{\infty} \gamma_n^\alpha c_n(T) R_n^\alpha.$$

Regarding even functions on $S^1$ as functions of the angular variable $\varphi \in [0, \pi]$, we recover the usual definition of the ultrasperical expansions (see e.g. [22]).

From the standard Sobolev-type estimates one sees that (2.17) is an expansion of a smooth even function on $S^1$ if and only if $c_n(f) = o(n^k)$ for all natural $k$ and that it is an expansion of $T \in \mathscr{M}'_+(S^1)$ if and only if $c_n(T)$ have at most polynomial growth.

On the other hand, for any $\Psi \in \mathscr{E}'(S^d)$ there is an expansion in spherical harmonics, $\Psi = \sum_{n=0}^{\infty} \Psi_n$, where $\Psi_n \in \mathscr{H}^n(S^d)$ and the series converges in the weak topology. The projection $\Psi \to \Psi_n \in \mathscr{H}^n(S^d)$ is given by

$$\Psi_n(ke_0) = \gamma_n^\alpha \langle \Psi, \tau(k)\, \omega_n \rangle.$$

LEMMA 2.6. *Given $\Psi = \theta^* T \in \mathscr{E}'(S^d)^M$, let $c_n(T)$ be defined by* (2.16). *Then*

(i) *the decomposition of $\Psi$ in spherical harmonics is*

$$\Psi = \sum_{n=0}^{\infty} \dim \mathscr{H}^n(S^d)\, c_n(T)\, \omega_n,$$

(ii) *the convolution operator $K_\Psi\colon f \to f * \Psi$, $f \in \mathscr{E}(S^d)$ decomposes, with respect to* (2.15), *as*

$$K_\Psi = \sum_{n=0}^{\infty} c_n(T)\, id_{\mathscr{H}^n(S^d)}.$$

Convenient references for the basic facts from analysis on sphere $S^d$ are [1] and [25].

Remark. Among other things the lemma enables the study of the continuity properties of the operator $K_\Psi$ to be reduced to the study of the convergence of the ultraspherical expansion of $T \in \mathcal{M}'_+(S^1)$ representing $\Psi$ via (2.17).

We shall use it this way in the next section to obtain Sobolev continuity of convolution operators defined by conical distributions.

**3. Expansions of conical vectors and intertwining operators.** The reduction technique of the preceding section will now be applied to derive most of the known properties of conical vectors for $\mathbf{SO}_0(d, 1)$ with $d > 2$ in a uniform way. We show how the results of [10] and [13] establishing the dimension of the space of conical vectors (cf. also [9], [4] and [20]) and their relation with the representation theory of the groups $\mathbf{SO}_0(n, 1)$ (in addition to the papers quoted, see [23], [25], [14]) follow easily by analysing the difference equation satisfied by the coefficients in the ultraspherical expansion of the generalized function $T \in \mathcal{M}'_+(S^1)$ corresponding to the given conical vector via (2.14). Solving the difference equation by elementary methods produces explicit expressions for the coefficients of basis vectors. As a byproduct of these formulas we obtain the diagonalized form of the intertwining operators for the spherical principal series representations which is basic for understanding such properties like reducibility, existence of the complementary series, etc. We shall not go into this, as it is adequately covered in the literature, and only record for subsequent use a simple estimate for intertwining operators and a characterization of the complementary series in terms of the Sobolev spaces on $S^d$, results which do not seem to be available in print although they are certainly known. (Almost identical estimates are given in [18] for the case of $\mathbf{SO}(p, q)$ with $p > 2$, $q > 2$. Also the characterization of complementary series was mentioned by J. Faraut in a discussion with the author.)

We note that the approach in this section could be traced back to a rather old idea of the author (*On conical distributions for* $\mathbf{SU}(2, 1)$, unpublished research report dated 1976) of determining conical distributions for $\mathbf{SU}(1, 1)$ by means of recurrence relations for their Fourier coefficients.

The following introduces the main object of the study in the sequel (cf. [10], [16]).

DEFINITION. $\Psi \in \mathcal{E}'(S^d)$ is called $\lambda$-*conical vector* if

$$\tau_\lambda(mn)\Psi = \Psi, \qquad \forall\, mn \in \mathrm{MN}. \tag{3.1}$$

The subspace of $\mathcal{E}'(S^d)$ consisting of all $\lambda$-conical vectors is called the *conical space* of the representation $\tau_\lambda$ and will be denoted by $\mathrm{Con}(\tau_\lambda)$. Since A normalizes MN, the conical space of $\tau_\lambda$ is invariant under $\tau_\lambda(A)$; a $\lambda$-conical vector which is also a common eigenvector for $\tau_\lambda(A)$ will be called an *A-homogeneous conical vector* (conical restricted weight vector in the standard terminology, cf. [16]).

Condition (3.1) is clearly equivalent to being an M-invariant solution of the equation (cf. [20])

$$D_\lambda \Psi = 0; \tag{3.2}$$

hence from the preceding results we obtain the following proposition.

PROPOSITION 3.1. *$\theta^*$ maps isomorphically the kernel of the operator $\tau_{\lambda+i\alpha}(Z)$: $\mathcal{M}'_+(S^1) \to \mathcal{M}'_-(S^1)$ onto $\mathrm{Con}(\tau_\lambda)$. Moreover, smooth conical vectors (i.e., elements of $\mathrm{Con}(\tau_\lambda) \cap \mathcal{E}(S^d)$) correspond this way to smooth elements of the kernel (i.e., those represented by elements of $\mathcal{E}_+(S^1) \subset \mathcal{M}'_+(S^1)$).*

*Similarly, $\Psi = \theta^* T$ satisfies*

$$\tau_\lambda(\exp tH)\,\Psi = e^{\varkappa t}\,\Psi$$

*for some $\varkappa \in \boldsymbol{C}$ and all $t \in \boldsymbol{R}$ if and only if*

$$\tau^{(1)}_{\lambda+i\alpha}(h)\,T = \varkappa T. \tag{3.3}$$

We shall determine the solutions in $\mathcal{M}'_+(S^1)$ of

$$\tau^{(1)}_{\lambda+i\alpha}(Z)\,T = 0 \tag{3.4}$$

by determining the coefficients in their ultraspherical expansions. The computations are based on the following relations for the ultraspherical polynomials $P_n^{(\alpha)}$, which can be easily derived using [22], pp. 80–83. Let $\boldsymbol{Z}_+ := \{0, 1, \ldots\}$ (nonnegative integers).

LEMMA 3.2. *For any $\gamma \in \boldsymbol{C}$ and $n \in \boldsymbol{Z}_+$*

(i) $$\left((1-x)\frac{d}{dx}+\gamma\right)P_n^{(\alpha)} = \frac{\alpha}{n+\alpha}\left((\gamma-n)\,P_n^{(\alpha+1)}+2(n+\alpha)\,P_{n-1}^{(\alpha+1)}-(n+\gamma+2\alpha)\,P_{n-2}^{(\alpha+1)}\right),$$

(ii) $$\left((1+x)\frac{d}{dx}-\gamma\right)P_n^{(\alpha)} = \frac{\alpha}{n+\alpha}\left((n-\gamma)\,P_n^{(\alpha+1)}+2(n+\alpha)\,P_{n-1}^{(\alpha+1)}+(n+\gamma+2\alpha)\,P_{n-2}^{(\alpha+1)}\right),$$

(iii) $$\left((1-x^2)\frac{d}{dx}+\gamma x\right)P_n^{(\alpha)} = \frac{n+1}{2(n+\alpha)}(\gamma-n)\,P_{n+1}^{(\alpha)}+\frac{n+2\alpha-1}{2(n+\alpha)}(n+\gamma+2\alpha)\,P_{n-1}^{(\alpha)}, \tag{3.5}$$

*where as usual we set $P_{-1}^{(\alpha)} = P_{-2}^{(\alpha)} = 0$.*

Now, substituting for $T \in \mathscr{M}'_+(S^1)$ expression (2.17),

$$T = \sum_{n=0}^{\infty} \gamma_n^\alpha c_n R_n^\alpha,$$

where

$$R_n^\alpha(s) = P_n^{(\alpha)}(1)^{-1} P_n^{(\alpha)}((s|e_0))$$

and using the coordinate expressions (2.13) we obtain:

$$\tau^{(1)}_{\lambda+i\alpha}(Z)\, T = 0$$

if and only if for each $n \in \boldsymbol{Z}_+$

$$(i\lambda - \varrho - n)\, c_n + (2\varrho + 2n + 1)\, c_{n+1} - (i\lambda + \varrho + n + 1)\, c_{n+2} = 0, \tag{3.6}$$

where we use for clarity the customary notation $\varrho = d/2$.

We claim that for each value of $\lambda$ the space of solutions of (3.6) is of dimension 2. This is clear in the regular case, i.e., if $i\lambda + \varrho + n + 1 \neq 0$ and $i\lambda - \varrho - n \neq 0$ for all $n \in \boldsymbol{Z}_+$, since then a solution is a linear function of the initial values $c_0$ and $c_1$. But if $i\lambda + \varrho = -k$ for some integer $k \geqslant 1$, then every solution is constant for all $n \leqslant k$, and hence is determined uniquely by specifying initial conditions at $k$, $k+1$. Also if $i\lambda - \varrho = k$ for some $k \in \boldsymbol{Z}_+$, then every solution is constant for $n > k$ and is likewise determined by its initial values $c_0$ and $c_1$. We note that, since the sum of the coefficients of equation (3.6) is 0 for each $n$, a constant sequence is always a solution. Therefore in each of the singular cases there is a solution vanishing over its interval of constancy.

By virtue of the foregoing we identify the space of solutions of (3.6) with $\boldsymbol{C}^2$ – the space of initial values $(c_0, c_1)$ if $i\lambda + \varrho$ is not a negative integer or $(c_k, c_{k+1})$ if $i\lambda + \varrho = -k$. Since $\tau^{(1)}_{\lambda+i\alpha}(h)$ permutes the solutions of (3.6) between themselves, it induces a linear map of $\boldsymbol{C}^2$ via this identification. Using Lemma 3.2 (iii), it is not difficult to see that the map is given by the matrix (with respect to the standard basis)

$$H(\lambda, k) = \begin{pmatrix} k, & -i\lambda - \varrho - k \\ -i\lambda + \varrho + k, & -2\varrho - k \end{pmatrix},$$

where $k = 0$ if $i\lambda + \varrho$ is not a negative integer and $k = -(i\lambda + \varrho)$ otherwise. The eigenvalues are $\varkappa_1 = -i\lambda - \varrho$, $\varkappa_2 = i\lambda - \varrho$ (independently of $k$) and the matrix is diagonalizable unless $\lambda = 0$. The solutions of (3.6) corresponding to the eigenvectors of $H(\lambda, k)$ are: constant sequences for $\varkappa_1$ and multiples of $(c_n(\lambda))$ for $\varkappa_2$, where

$$c_n(\lambda) = \prod_{j=0}^{n-1} \frac{(j - i\lambda + \varrho)}{(j + i\lambda + \varrho)}, \qquad n \in \boldsymbol{Z}_+ \tag{3.7a}$$

(the product of the empty family is 1) provided $i\lambda+\varrho \notin -\mathbf{Z}_+$ and

(3.7b) $$c_n(\lambda) = \begin{cases} 0, & n \leqslant k, \\ \prod_{j=k+1}^{n-1} \frac{(j-i\lambda+\varrho)}{(j+i\lambda+\varrho)}, & n > k, \end{cases}$$

if $i\lambda+\varrho = -k$, $k \in \mathbf{Z}_+$. However, for $\lambda = 0$ the solution given by (3.7) is also the constant solution and another independent solution can be obtained as the one, e.g. corresponding to the initial data $c_0 = 0$, $c_1 = 1$.

We observe that $c_n(\lambda)$ grow polynomially as $n \to \infty$ (a precise estimate is given in the course of the proof of the Proposition 3.6 below), unless $i\lambda-\varrho$ is a nonnegative integer in which case only a finite number of them is different from zero. Therefore the series in (3.5) converges to an element of $\mathcal{H}'_+(S^1)$ and furthermore the sum represents a smooth function if and only if all but a finite number of $c_n$'s vanish and this happens, as we have just seen, precisely when $i\lambda-\varrho$ is a nonnegative integer. Solutions of (3.4) with ultraspherical expansions determined above will be denoted as follows.

(3.8) $$S_0 = \sum_{n=0}^{\infty} \gamma_n^\alpha R_n^\alpha$$

corresponds to the constant solution of (3.5). Note that $\theta^* S_0$ is the Dirac delta at $e_0 \in S^d$. Further, let

(3.9) $$T_\lambda = \sum_{n=0}^{\infty} \gamma_n^\alpha c_n(\lambda) R_n^\alpha,$$

where for $\lambda \neq 0$ the coefficients $c_n(\lambda)$ are those of (3.7a) or (3.7b) and for $\lambda = 0$ are determined from (3.6) and the initial condition $c_0 = 1$, $c_1 = 0$.

Now, all the preceding can be summed up in the following theorem.

THEOREM 3.3 (Helgason [10], cf. also [13], [9]).

(1) *For every* $\lambda \in \mathbf{C}$ $\dim \mathrm{Con}(\tau_\lambda) = 2$. *In particular,* $\Phi = \theta^* S_0$ *and* $\Psi_\lambda \equiv \theta^* T_\lambda$ *form a basis for* $\mathrm{Con}(\tau_\lambda)$.

(2) *The representation of $A$ on* $\mathrm{Con}(\tau_\lambda)$, *obtained by restricting* $\tau_\lambda$, *is semisimple if and only if* $\lambda \neq 0$. *The basis* $\{\Phi, \Psi_\lambda\}$ *of* $\mathrm{Con}(\tau_\lambda)$ *consists of $A$-homogeneous vectors satisfying*

(3.10) $$\tau_\lambda(\exp tH)\,\Phi = e^{-(i\lambda+d/2)t}\,\Phi,$$

(3.11) $$\tau_\lambda(\exp tH)\,\Psi_\lambda = e^{(i\lambda-d/2)t}\,\Psi_\lambda.$$

(3) *There is a smooth vector in* $\mathrm{Con}(\tau_\lambda)$ *if and only if* $i\lambda-d/2 \in \mathbf{Z}_+$, *in which case it is a scalar multiple of* $\Psi_\lambda$.

One of the main reasons for interest in conical vectors is their relation to intertwining operators. By the latter we mean an operator (linear, continuous) $T\colon \mathcal{E}(S^d) \to \mathcal{E}(S^d)$ such that

$$T \circ \tau_\lambda(g) = \tau_\nu(g) \circ T \tag{3.12}$$

for some $\lambda$, $\nu \in \boldsymbol{C}$ and all $g \in G$. Since such an operator commutes with the natural action of $K$ on $\mathcal{E}(S^d)$, it is necessarily a convolution operator $f \to f * \Psi$ with an unique $\Psi \in \mathcal{E}'(S^d)^M$. Then from the above one readily obtains the following proposition.

PROPOSITION 3.4. *Given $\lambda \neq 0$, there exists a non-zero operator intertwining $\tau_\lambda$ with $\tau_\nu$ if either $\nu = \lambda$ in which case the operator is a scalar multiple of the identity or $\nu = -\lambda$ and the operator is a scalar multiple of the convolution operator*

$$A(\lambda)\colon f \to f * \Psi_{-\lambda}, \tag{3.13}$$

*with $\Psi_{-\lambda}$ as in Theorem 3.3. In the latter case*

$$A(\lambda)|_{\mathcal{H}^n(S^d)} = \prod_{j=0}^{n-1} \frac{(j+i\lambda+d/2)}{(j-i\lambda+d/2)} I \tag{3.14a}$$

*if $\lambda \neq -i(k+d/2)$, $k \in \boldsymbol{Z}_+$ (and with the usual convention that the product of the empty family is 1) and*

$$A(\lambda)|_{\mathcal{H}^n(S^d)} = \begin{cases} 0, & n \leqslant k, \\ \displaystyle\prod_{j=k+1}^{n-1} \frac{(j+i\lambda+d/2)}{(j-i\lambda+d/2)} I, & n > k, \end{cases} \tag{3.14b}$$

*if $\lambda = -i(k+d/2)$ for some $k \in \boldsymbol{Z}_+$.*

*If $\lambda = 0$ then the only non-zero intertwining operators are the scalar multiples of the identity.*

Proof. If suffices to show that a convolution operator $f \to f * \Psi$ intertwines $\tau_\lambda$ with $\tau_\nu$ if and only if $\Psi$ is a $(-\lambda)$-conical vector satisfying the homogenity condition

$$\tau_{-\lambda}(\exp tH)\, \Psi = e^{(i\lambda - d/2)t}\, \Psi. \tag{3.15}$$

In fact, using the $M$-invariance of $\Psi$, we have that the convolution $f * \Psi$ is given by

$$f * \Psi(ke_0) = \Psi(\tau(k^{-1}) f);$$

hence, denoting $s = \exp tHn \in AN$, we get

$$\begin{aligned} \Psi(\tau_\lambda(s) f) &= (\tau_\lambda(s) f) * \Psi(e_0) = \tau_\lambda(s)(f * \Psi)(e_0) \\ &= e^{(i\lambda - d/2)t} f * \Psi(e_0) = e^{(i\lambda - d/2)t}\, \Psi(f). \end{aligned}$$

Conversely, writing the Iwasawa decomposition $g = k(g)\exp t(g) Hn(g)$ as $g = k(g)s(g)$ with $k(g) \in K$, $s(g) \in AN$, we have

$$(3.16)\qquad \begin{aligned}\Psi\big(\tau(k)^{-1}\tau_\lambda(g)f\big) &= \Psi\big(\tau_\lambda(g^{-1}k)^{-1}f\big)\\ &= \Psi\big(\tau_\lambda(s(g^{-1}k))^{-1}\tau(k(g^{-1}k))^{-1}f\big)\\ &= e^{(i\nu-d/2)t(g^{-1}k)}\Psi\big(\tau(k(g^{-1}k))^{-1}f\big)\end{aligned}$$

in virtue of (3.15). Upon comparing with the definition of $\tau_\lambda$ (cf. (1.5), (1.6)) the desired intertwining property is clear. The diagonalized form (3.14) follows from (3.7) by applying Lemma 2.6. □

Remark. Formula (3.13) expressing intertwining operators for the spherical principal series by means of conical distributions was given (for arbitrary semi-simple $G$) in [10] (cf. also [9] and [15]). The diagonalized form (3.14) was written down in [9] for $\mathbf{SO}(n, 1)$ groups and in [14] for all classical rank one groups, however, for operators giving rise to the complementary series for $\mathbf{SO}(n, 1)$ (cf. below) was found already in [23] and [25].

Corollary 3.5. *The intertwining operator $A(\lambda)$ is hermitean if and only if $\lambda$ is purely imaginary and is positive definite if and only if $-d/2 < i\lambda < d/2$. In addition, for $i\lambda = k+d/2$, $k \in \mathbf{Z}_+$ and $i\lambda = -d/2$, $A(\lambda)$ is positive semi-definite.*

The corollary follows immediately from (3.14a) and (3.14b), resp. For those values of $\lambda$ where the ($\tau_\lambda$-invariant) sesqui-linear form on $\mathcal{E}(S^d)$

$$(3.17)\qquad (f, h)_\lambda := \int_{S^d} A(\lambda) f\bar{h}\, db$$

is positive (semi-)definite by the standard procedure of Hilbert completion (possibly after factoring out the null space of the form) one obtains unitary representations of $G$ (cf. e.g. [17]). The unitary representations so obtained for $\lambda \in ]-id/2, id/2[$ are said to form the complementary series of the spherical representations of $G$.

Following [6], we define the Sobolev $s$-norm of $\Psi \in \mathcal{E}'(S^d)$ (for $s \in \mathbf{R}$) to be

$$(3.18)\qquad \|\Psi\|_s^2 = \|(1-L)^{s/2}\Psi\|^2 = \sum_{n=0}^{\infty}(1-\lambda_n)^s\|\Psi_n\|^2,$$

where $\lambda_n = -n(n+d-1)$ is the eigenvalue of the Laplace–Bettrami operator $L$ corresponding to $\mathscr{H}^n(S^d)$ and for $\Psi \in \mathcal{E}'(S^d)$ by $\Psi_n$ we have denoted its projection onto $\mathscr{H}^n(S^d)$ and the norm $\|\cdot\|$ without subscript is the $L^2$-norm. The Sobolev spaces $H^s = H^s(S^d)$ are then

$$H^s = \{\Psi \in \mathcal{E}'(S^d) \mid \|\Psi\|_s^2 < \infty\}.$$

We now state

PROPOSITION 3.6. *The intertwining operator* $A(\lambda)$ *defined by* (3.13) *extends to a bounded operator*

$$A(\lambda)\colon H^s \to H^{s+2Im\lambda}$$

*for any real s. Further, if* $\lambda = i\sigma$ *with* $\sigma \in \boldsymbol{R}$ *and* $|\sigma| < d/2$, *then the norm* $f \to (A(\lambda)f, f)^{1/2}$ *of the complementary series representation is equivalent to the Sobolev norm* $\|\cdot\|_{-\sigma}$.

Proof. Let $c_n(\lambda)$ denote the eigenvalue of the operator $A(\lambda)$ corresponding to the space $\mathscr{H}^n(S^d)$ (as we know, they are equal to the coefficients in the ultraspherical expansion of the generalized function $T_{-\lambda} \in \mathscr{M}'_+(S^1)$, for which $\Psi_{-\lambda} = \theta^* T_{-\lambda}$). It suffices to prove

$$A(1+n)^{-2Im\lambda} < |c_n(\lambda)| < A_1(1+n)^{-2Im\lambda} \tag{3.19}$$

for some positive constants $A$, $A_1$ and sufficiently large $n \in \boldsymbol{Z}_+$. In fact, since

$$\|A(\lambda)f\|_{s+t}^2 = \sum_{n=0}^{\infty} (1-\lambda_n)^{s+t} |c_n(\lambda)|^2 \|f_n\|^2 < \sup_n \{(1-\lambda_n)^t |c_n(\lambda)|^2\} \|f\|_s^2,$$

the right-hand side of the inequality in (3.19) implies the boundedness of $A(\lambda)$ if $t = 2Im\lambda$. The equivalence of the norms follows using similar estimate on both sides.

Now, (3.14a) can be written as

$$c_n(\lambda) = \frac{\Gamma(-i\lambda+d/2)}{\Gamma(i\lambda+d/2)} \frac{\Gamma(i\lambda+d/2+n)}{\Gamma(-i\lambda+d/2+n)}$$

and from the known asymptotic behaviour of a quotient of two gamma functions (cf. [3], 1.18(4)) we get

$$|c_n(\lambda)| \sim \left|\frac{\Gamma(-i\lambda+d/2)}{\Gamma(i\lambda+d/2)}\right| (d/2+n)^{-2Im\lambda}$$

what clearly is equivalent to (3.19). The case where $\lambda = -i(k+d/2)$, $k \in \boldsymbol{Z}_+$, requires only a trivial modification and is therefore omitted. □

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