ANNALES
POLONICI MATHEMATICI
XV (1964)

Investigation of the existence and uniqueness of differentiable solutions of a functional equation

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In the present paper we are concerned with the functional equation

$$\varphi(x) = H\big(x, \varphi[f(x)]\big) , \tag{1}$$

where $\varphi(x)$ is the unknown function, and $f(x)$ and $H(x, z)$ are known real-valued functions of real variables.

Equation (1) is the most general functional equation of the first rank and the first order (¹) which does not contain superpositions of the unknown function. It may be regarded as a generalization of the linear functional equation

$$\varphi[f(x)] - k(x)\varphi(x) = F(x) \tag{2}$$

(where $\varphi(x)$ denotes the unknown function, the remaining functions are known). Equation (2) finds applications in various problems of mathematics (e.g. functional characterisations of certain functions, the theory of continuous iterations, etc; comp. [12]) as well as in particular problems of related sciences (e.g. in hydrodynamics [5]).

Equation (2) and its particular cases have often been dealt with and many authors have been concerned with the problems of the existence and uniqueness of its solutions fulfilling some additional conditions, like boundedness, monotonity, convexity, continuity, etc. In particular, theorems regarding the existence or uniqueness of continuous solutions of equation (2) have been proved by A. Bielecki and J. Kisyński [3] for $k(x) \equiv 1$, M. Kuczma [9] for $k(x) \equiv -1$, and by J. Kordylewski and M. Kuczma [8] for equation (2) in the general form.

The most important results concerning equations (2) and (1) and their applications are collected in [12]. Here we shall quote only the results regarding the continuous solutions of equation (1).

(¹) A definition of the rank (German *Stufe*) of a functional equation may be found in [1], p. 20, [12]. That of the order in [6], [12].

As has been proved in [7], equation (1) possesses infinitely many solutions which are continuous in an open interval (a, b) under the supposition that the function $f(x)$ is defined, continuous and strictly increasing in the interval $\langle a, b\rangle$, where a and b are two consecutive fixed points of the function $f(x)$ (i.e. $f(a) = a$, $f(b) = b$ and $f(x) \neq x$ in (a, b)), and under some further suppositions regarding the function $H(x, z)$.

The situation becomes more complicated when one requires the solution to be continuous in an interval closed on one side: $(a, b\rangle$ or $\langle a, b)$, the hypotheses regarding the function $f(x)$ being analogous to those in the previous case. M. Kuczma [10], [11] established the conditions of the existence of exactly one, or at most one, or infinitely many solutions of equation (1), continuous in the considered interval. These conditions involve the function $H(x, z)$.

From other results concerning equation (1) we mention here a theorem on the existence and uniqueness of bounded solutions, proved by M. Bajraktarević [2].

In the present paper we discuss the problem of the existence and uniqueness of solutions of equation (1) that are of class C^r, $1 \leqslant r < \infty$, in the interval $(a, b\rangle$ or $\langle a, b)$, where a and b are two consecutive fixed points of the function $f(x)$. The assumptions will be similar to those occurring in [10] and [11] in the case of an analogous problem for continuous solutions of equation (1).

The problem of regular solutions of equation (1), as it was formulated above, has not been considered in the general case. However, A. H. Read [15] treated equation (1) on the complex plane and found its analytic solutions. A. Pelczar [14] proved the existence of a solution of equation (1) in the class of functions satisfying the Lipschitz condition with a common constant. A theorem regarding the solutions of equation (2) with $k(x) = 1$ which belong to class C^1 is to be found in [3].

The existence and uniqueness of differentiable solutions has been proved by M. Kuczma [13] only for linear equation (2). For equation (1), however, only the problem of the solutions of class C^r in the open interval (a, b) has been solved ([4]; comp. Lemma 8 below).

In § 1 we formulate the hypotheses, prove some lemmas and quote the results contained in papers [8], [11] and [4], as we shall need them in our further considerations. In § 2 we prove Theorem 1 regarding the existence of a solution of equation (1) which is of class C^r in the interval $(a, b\rangle$. § 3 contains Theorem 2 concerning the existence and uniqueness of the C^r solution of equation (1) in $(a, b\rangle$, and two corollaries from Theorems 1 and 2. The case where there is an infinity of C^r solutions of equation (1) in $(a, b\rangle$ is discussed in § 4 (Theorem 5). Finally, the theorems corresponding to Theorems 1–5 and concerning solutions of class C^r in the interval $\langle a, b)$ as well as some final conclusions and remarks make the contents of § 5.

§ 1. We shall consider the problem of the existence and uniqueness of differentiable solutions of equation (1) under the following hypotheses regarding the functions $f(x)$ and $H(x, z)$ (r denotes here an arbitrary, but fixed positive integer):

(I^r) *The function $f(x)$ is defined and of class C^r in an interval $\langle a, b\rangle$ such that $f(a) = a$, $f(b) = b$. Furthermore, the following inequalities are fulfilled:*

$$f(x) > x \quad \textit{for} \quad x \in (a, b)\,,$$
$$f'(x) > 0 \quad \textit{for} \quad x \in (a, b\rangle\,.$$

(II^r) *The function $H(x, z)$ is defined and of class C^r in an open region Ω, normal with respect to the x-axis* ([2]), *and*

$$\frac{\partial H(x, z)}{\partial z} \neq 0 \quad \textit{for} \quad (x, z) \in \Omega\,.$$

(III) *$\Omega_x \neq \emptyset$, $\Gamma_x = \Omega_x$ for $x \in \langle a, b\rangle$, where Ω_x denotes the x-section of the region Ω of the form:*

$$\Omega_x \overset{\mathrm{df}}{=} \{z\colon\ (f^{-1}(x), z) \in \Omega\}$$

($f^{-1}(x)$ is the function inverse to the function $f(x)$), $\emptyset$ is the empty set, and Γ_x denotes the set of the values of the function $H(x, z)$ for $z \in \Omega_{f(x)}$, i.e.

$$\Gamma_x \overset{\mathrm{df}}{=} \Big\{y\colon\ \sum_z \big(z \in \Omega_{f(x)},\ y = H(x, z)\big)\Big\}\,.$$

(IV^r) *There exist a number d such that*

$$d = H(b, d) \tag{3}$$

and $(b, d) \in \Omega$ and numbers $d_1, d_2, \ldots, d_r$ such that

$$d_k = H_k(b, d, d_1, \ldots, d_k)\,, \qquad k = 1, \ldots, r\,, \tag{4}$$

where the functions $H_k(x, z, z_1, \ldots, z_k)$ are defined by the recurrent relations:

$$H_1(x, z, z_1) \overset{\mathrm{df}}{=} H'_x(x, z) + H'_z(x, z) f'(x) z_1\,,$$

$$H_{k+1}(x, z, z_1, \ldots, z_{k+1}) \overset{\mathrm{df}}{=} \frac{\partial H_k}{\partial x} + f'(x)\left(\frac{\partial H_k}{\partial z} z_1 + \ldots + \frac{\partial H_k}{\partial z_k} z_{k+1}\right), \tag{5}$$
$$k = 1, \ldots, r-1\,.$$

([2]) I.e. every x-section of the region Ω is either an open interval, or an empty set.

In the sequel d will always denote a fixed root of equation (3) such that $(b, d) \in \Omega$.

Hypotheses (I^r), (II^r) and (III) guarantee the existence of an infinite number of solutions of equation (1) which are of class C^r in the interval (a, b) (comp. Lemma 8). In particular, assumption (III) is essential for the existence of this solution in the whole interval (a, b) (comp. [7]). Assumption (IV^r) establishes a necessary condition of the existence of a solution of equation (1), of class C^r in the interval $(a, b\rangle$ (it will be proved later, in Lemma 4).

Now we introduce further notation.

We denote by $G(x, y)$ the function inverse to the function $H(x, z)$ with respect to the variable z, i.e.

$$z = G(x, y) \equiv y = H(x, z)\,.$$

This function exists according to assumption (II^r) and is of class C^r in the region

$$\Omega' \stackrel{\mathrm{df}}{=} \{(x, y):\ x \in \langle a, b\rangle\,,\ y \in \Gamma_x\}\,.$$

Further we put

$$G_1(x, y, y_1) \stackrel{\mathrm{df}}{=} [f'(x)]^{-1}\big(G'_x(x, y) + G'_y(x, y)y_1\big)\,,$$

$$G_{k+1}(x, y, y_1, \ldots, y_{k+1}) \stackrel{\mathrm{df}}{=} [f'(x)]^{-1}\left(\frac{\partial G_k}{\partial x} + \frac{\partial G_k}{\partial y}y_1 + \ldots + \frac{\partial G_k}{\partial y_k}y_{k+1}\right), \tag{6}$$

$$k = 1, \ldots, r-1\,.$$

The expressions defined by formulae (5) and (6) have some properties which are expressed in Lemmas 1–3.

LEMMA 1. *Let hypotheses* (I^r) *and* (II^r) *be fulfilled. Then the expressions* $H_k(x, z, z_1, \ldots, z_k)$, $k = 1, \ldots, r$, *are functions of the variables* $(x, z, z_1, \ldots, z_k)$ *defined and of class* C^{r-k} *for* $(x, z) \in \Omega$ *and arbitrary* z_i $(i = 1, \ldots, k)$. *Moreover, we have*

$$H_k(x, z, z_1, \ldots, z_k) = P_k(x, z, z_1) + Q_k(x, z, z_k) + R_k(x, z, z_1, \ldots, z_{k-1})\,, \tag{7}$$

$$k = 2, \ldots, r\,,$$

where

$$P_k(x, z, z_1) = \sum_{i=0}^{k}\binom{k}{i}\frac{\partial^k H(x, z)}{\partial x^{k-i}\partial z^i}[f'(x)]^i z_1^i\,, \tag{8}$$

$$Q_k(x, z, z_k) = \frac{\partial H(x, z)}{\partial z}[f'(x)]^k z_k\,, \tag{9}$$

(10) $R_k(x, z, z_1, \ldots, z_{k-1})$ *is a polynomial of the variables* $z_1, \ldots, z_{k-1}$, *whose coefficients are functions of the variables* (x, z), *of class* C^{r-k} *with respect to* x *and of class* C^{r-k+1} *with respect to* z.

Proof. At first we shall prove (7). For $k=2$ equality (7) follows from definition (5), and since in this case $R_2(x, z, z_1) = f''(x)H_z'(x, z)z_1$, (10) evidently is fulfilled. If (7) holds for $k=p$, $2 \leqslant p < r$, then we get by (5) according to (8) and (9) ($k=p$)

$$H_{p+1}(x, z, z_1, \ldots, z_{p+1}) = P_{p+1}(x, z, z_1) + Q_{p+1}(x, z, z_{p+1}) + \\ + \left[\frac{\partial R_p}{\partial x} + f'(x)\left(\frac{\partial R_p}{\partial z} z_1 + \ldots + \frac{\partial R_p}{\partial z_{p-1}} z_p\right) + f'(x)\frac{\partial P_p}{\partial z_1} z_2 + \right. \\ \left. + \sum_{i=0}^{p} i \binom{p}{i} \frac{\partial^p H(x, z)}{\partial x^{p-i} \partial z^i} [f'(x)]^{i-1} f''(x) z_1^i + \frac{\partial Q_p}{\partial x} + f'(x)\frac{\partial Q_p}{\partial z} z_1\right].$$

Let us denote by $R_{p+1}(x, z, z_1, \ldots, z_p)$ the expression in the parentheses on the right-hand side of the above relation. In order to prove that relation (7) holds for $k=p+1$ it is enough to show that the expression R_{p+1} thus defined fulfills (10). But R_{p+1} really is a polynomial of the variables $(z_1, \ldots, z_p)$, because R_p, Q_p and P_p are polynomials of the variables $(z_1, \ldots, z_{p-1})$, z_p and z_1, respectively. The coefficients of this polynomial are functions of the variables (x, z), of class C^{r-p-1} with respect to x and C^{r-p} with respect to z, because the coefficients of the polynomial R_p are with respect to x, and z of class C^{r-p} and C^{r-p+1}, respectively, and the remaing components that do not contain R_{p+1} are at least of class C^{r-p} with respect to x and z. Thus relation (7) holds for $k \leqslant r$. The rest of the assertions of the lemma follows immediately from (7)-(10). This completes the proof.

For the expressions defined by formulae (6) we have the following

LEMMA 2. *Let hypotheses* (I^r) *and* (II^r) *be fulfilled. Then the expressions* $G_k(x, y, y_1, \ldots, y_k)$ *are functions of the variables* $(x, y, y_1, \ldots, y_k)$, *defined and of class* C^{r-k} *for* $(x, y) \in \Omega'$ *and arbitrary* y_i, $i = 1, \ldots, k$. *Furthermore, we have*

$$\frac{\partial G_k}{\partial y_k} = G_y'(x, y)[f'(x)]^{-k}, \quad k = 1, \ldots, r. \tag{11}$$

If, moreover, the relations

$$y = H(x, z), \quad y_i = H_i(x, z, z_1, \ldots, z_i), \quad i = 1, \ldots, r, \tag{12}$$

hold, then we have also

$$G_k(x, y, y_1, \ldots, y_k) = z_k, \quad k = 1, \ldots, r. \tag{13}$$

Proof. The first part of the assertion of the lemma is a consequence of the fact that the function $G(x, y)$ is of class C^r in Ω' and that the function $f(x)$ is of class C^r in $\langle a, b\rangle$. A more precise proof would run similarly as the proof of Lemma 1.

Now we are going to prove relation (13). For $k = 1$ it results from (5) and (6) and from the relations between the first derivatives of the functions $H(x, z)$ and $G(x, y)$ (where $y = H(x, z)$):

$$G'_y(x, y) = [H'_z(x, z)]^{-1}, \qquad G'_x(x, y) = -H'_x(x, z)[H'_z(x, z)]^{-1}.$$

If (13) holds for $k = 1, \ldots, p$, $p < r$, then we have

$$G_p\big(x, H(x, z), H_1(x, z, z_1), \ldots, H_p(x, z, z_1, \ldots, z_p)\big) = z_p .$$

We differentiate both the sides of the above relation with respect to $x, z, z_1, \ldots, z_p$, consecutively. After suitable calculations we get on account of (5) and (12) (applied for $k = 1, \ldots, p+1$)

$$\frac{\partial G_p}{\partial x} + \frac{\partial G_p}{\partial y} y_1 + \ldots + \frac{\partial G_p}{\partial y_p} y_{p+1} = f'(x) z_{p+1} .$$

Hence, dividing both the sides by $f'(x)$ and making use of (6), we obtain (13) for $k = p+1$.

Almost evident relation (11) also can be easily proved by induction.

The next lemma expresses the relation between a solution of equation (1) and the functions H_k and G_k.

LEMMA 3. *Let hypotheses* (I^r), (II^r) *and* (III) *be fulfilled, and let* $\varphi(x)$ *be a* C^r *solution of equation* (1) *in an interval* $I \subset \langle a, b\rangle$ *such that* $\varphi(x) \in \Omega_x$ *for* $x \in I$. *Then the functions* $H_k\big(x, \varphi[f(x)], \varphi'[f(x)], \ldots, \varphi^{(k)}[f(x)]\big)$ *and* $G\big(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(k)}(x)\big)$ *are of class* C^{r-k} *in the interval* I *and the derivatives* $\varphi^{(k)}(x)$ *of the function* $\varphi(x)$ *fulfill the equations*

$$\varphi^{(k)}(x) = H_k\big(x, \varphi[f(x)], \ldots, \varphi^{(k)}[f(x)]\big), \qquad k = 1, \ldots, r, \tag{14}$$

and

$$\varphi^{(k)}[f(x)] = G_k\big(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(k)}(x)\big), \qquad k = 1, \ldots, r. \tag{15}$$

Proof. The first part of the assertion of the lemma follows from Lemmas 1 and 2 on account of the assumptions regarding the function $\varphi(x)$ and of hypothesis (III). Formulae (14) result from the relations

$$H_{k+1}\big(x, \varphi[f(x)], \ldots, \varphi^{(k+1)}[f(x)]\big) = \frac{d}{dx} H_k\big(x, \varphi[f(x)], \ldots, \varphi^{(k)}[f(x)]\big)$$

that hold for $k = 1, \ldots, r-1$. Finally, if (14) is fulfilled for $k = 1, \ldots, p$, then, according to (13), also (15) is fulfilled for $k = p$. Since p may be any one of the numbers $1, 2, \ldots, r$, the lemma has been completely proved.

The next lemma, which has already been announced, concerns the role of hypothesis (IV^r).

LEMMA 4. *Let hypotheses* (I^r) *and* (II^r) *be fulfilled.*

1° *Hypothesis* (IV^r) *is a necessary condition of the existence of a solution of equation* (1) *that belongs to the class* C^r *in* $(a, b\rangle$.

2° *If numbers* $d, d_1, \ldots, d_r$ *fulfill relations* (3) *and* (4), *then they fulfill also the relations*

$$d = G(b, d), \quad d_k = G_k(b, d, d_1, \ldots, d_k), \quad k = 1, \ldots, r. \tag{16}$$

3° *If*

$$|H_z'(b, d)|[f'(b)]^r > 1, \tag{17}$$

then there exists one and only one system of numbers $d_1, \ldots, d_r$ *fulfilling relations* (4).

Proof. 1° If a function $\varphi(x)$ is of class C^r in the interval $(a, b\rangle$ and satisfies equation (1) for $x \in (a, b\rangle$, then its derivatives $\varphi^{(k)}(x)$, $k = 1, \ldots, r$, satisfy equations (14) in this interval (cf. Lemma 3). Setting in (14) $x = b$ we obtain in view of the equality $f(b) = b$

$$\varphi^{(k)}(b) = H_k\big(b, \varphi(b), \varphi'(b), \ldots, \varphi^{(k)}(b)\big) \text{ (}^3\text{)}, \quad k = 1, \ldots, r.$$

This means that the numbers $d = \varphi(b)$, $d_1 = \varphi'(b), \ldots, d_r = \varphi^{(r)}(b)$ fulfill equations (3) and (4), which proves the first assertion of the lemma.

2° Relation (16) follows from (3), (4) and (13).

3° According to (4) and (7)–(10) the numbers $d_1, \ldots, d_r$ are roots of a system of r linear equations (with r unknowns). The determinant D of this system has the value

$$D = \prod_{i=1}^{r} \big(1 - H_z'(b, d)[f'(b)]^i\big).$$

But on account of hypothesis (I^r) $f'(b) \leqslant 1$. Hence it follows by (17) that

$$|H_z'(b, d)|[f'(b)]^i > 1 \quad \text{for} \quad i = 1, \ldots, r.$$

Consequently $D \neq 0$ and system (4) has exactly one solution.

Thus the lemma has been completely proved.

Remark. From the proof of Lemma 4 we see that in general (unless something like relation (17) is supposed) it can happen that there exist infinitely many systems of numbers $d_1, \ldots, d_r$ fulfilling (4) or that such numbers do not exist at all. Therefore we had to make hypothesis (IV^r).

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(3) The symbols $\varphi^{(k)}(b)$ denote here the left-side derivatives of the function $\varphi(x)$ at the point b.

We assume also the following convention:

$U(\eta, \delta)$ will always denote the neighbourhood of the point (b, d) of the form

$$U(\eta, \delta) = \langle b-\eta, b\rangle \times \langle d-\delta, d+\delta\rangle \,, \tag{18}$$

where η and δ are some positive numbers.

Now we shall prove the following

LEMMA 5. *Suppose that hypotheses* (I^r), (II^r) *and* (III) *are fulfilled and that there exist a neighbourhod $U(\eta, \delta)$ and constants Θ and L such that*

$$|H'_z(x, z)|[f'(x)]^r < \Theta \quad \text{for} \quad (x, z) \in U(\eta, \delta) \,, \tag{19}$$

$$\left|\frac{\partial^r H(x, \bar{z})}{\partial x^k \partial z^{r-k}} - \frac{\partial^r H(x, \bar{\bar{z}})}{\partial x^k \partial z^{r-k}}\right| \leqslant L|\bar{z}-\bar{\bar{z}}| \tag{20}$$

for $(x, \bar{z}), (x, \bar{\bar{z}}) \in U(\eta, \delta)$, $k = 0, 1, \ldots, r$.

Then the function $H_r(x, z, z_1, \ldots, z_r)$ given by formulae (5) *satisfies a Lipschitz condition with respect to the variables $z, z_1, \ldots, z_r$ in the set*

$$Z \overset{\mathrm{df}}{=} U(\eta, \delta) \times \langle a_1, \beta_1\rangle \times \ldots \times \langle a_r, \beta_r\rangle \tag{21}$$

where $-\infty < a_k < \beta_k < +\infty$ $(k = 1, \ldots, r)$ are arbitrary real numbers.

More precisely: *There exist constants $L_0, L_1, \ldots, L_r$ independent of x such that for $(x, \bar{z}, \bar{z}_1, \ldots, \bar{z}_r)$, $(x, \bar{\bar{z}}, \bar{\bar{z}}_1, \ldots, \bar{\bar{z}}_r) \in Z$ we have*

$$|H_r(x, \bar{z}, \bar{z}_1, \ldots, \bar{z}_r) - H_r(x, \bar{\bar{z}}, \bar{\bar{z}}_1, \ldots, \bar{\bar{z}}_r)| \leqslant L_0|\bar{z}-\bar{\bar{z}}| + \sum_{k=1}^{r} L_k|\bar{z}_k - \bar{\bar{z}}_k| \tag{22}$$

and moreover (cf. (19))

$$L_r = \Theta \,. \tag{23}$$

Proof. Let us fix arbitrary numbers $a_k < \beta_k$ and let us consider the function $H_r(x, z, z_1, \ldots, z_r)$ in the set Z defined by (21). In virtue of Lemma 2 this function is defined and continuous in the set Z and can be expressed in the form (7). Thus the function H_r fulfills a Lipschitz condition with respect to the variables $z_1, \ldots, z_r$ in the set Z, viz. it is a polynomial of these variables with coefficients which are continuous functions of the variables (x, z) in the closed set $U(\eta, \delta)$. It remains to prove that the function H_r satisfies also a Lipschitz condition with respect to the variable z. But this is a consequence of formulae (7)–(10). In fact, on account of (8) and (20) the function $P_r(x, z, z_1)$ fulfills a Lipschitz condition with respect to z, and the functions $Q_r(x, z, z_r)$ and $R_r(x, z, z_1 \ldots, z_r)$ are at least of class C^1 with respect to z and at least continuous with respect to the remaining variables (cf. (9) and (10)).

Finally, (23) results from (7), (9) and (19), which completes the proof of the lemma.

In Lemmas 6–8 we quote some theorems, which were proved in [8], [11] and [4] (under assumptions weaker than (I^r) and (II^r)). The following Lemma 6 concerns equation (2) and its proof is to be found in [8]:

LEMMA 6. *Suppose that*

(1) *the function* $f(x)$ *fulfills hypothesis* (I^r);

(2) *the functions* $k(x)$ *and* $F(x)$ *are continuous in the interval* $(a, b\rangle$, *and* $k(x) \neq 0$ *in* (a, b);

(3) *there exist numbers* $\eta > 0$ *and* $0 < \vartheta < 1$ *such that the inequality* $|k(x)| < \vartheta$ *holds for* $x \in \langle b-\eta, b\rangle$.

Then every function $\varphi(x)$ *satisfying equation* (2) *and continuous in the interval* (a, b) *is also continuous in the interval* $(a, b\rangle$.

LEMMA 7. *Let hypotheses* (I^r), (II^r) *and* (III) *be fulfilled. If* $|H'_z(b, d)| < 1$, *then equation* (1) *possesses exactly one solution* $\varphi(x)$ *that is continuous in the interval* $(a, b\rangle$ *and fulfills the condition* $\varphi(b) = d$.

The proof of this lemma is to be found in [11].

LEMMA 8. *If hypotheses* (I^r), (II^r) *and* (III) *are fulfilled, then for every point* $x_0 \in (a, b)$ *and every function* $\bar{\varphi}(x)$ *which is of class* C^r *in the interval* $\langle x_0, f(x_0)\rangle$ *and fulfills the conditions*

$$\bar{\varphi}(x) \in \Omega_x \ \ for \ \ x \in \langle x_0, f(x_0)\rangle\,, \qquad \bar{\varphi}(x_0) = H\big(x_0, \bar{\varphi}[f(x_0)]\big)\,,$$

$$\bar{\varphi}^{(k)}(x_0) = H_k\big(x_0, \bar{\varphi}[f(x_0)], \ldots, \bar{\varphi}^{(k)}[f(x_0)]\big)\,, \qquad k = 1, \ldots, r\,;$$

there exists exactly one function $\varphi(x)$ *defined and of class* C^r *in the interval* (a, b), *satisfying equation* (1) *and such that*

$$\varphi(x) = \bar{\varphi}(x) \qquad for \qquad x \in \langle x_0, f(x_0)\rangle\,.$$

This lemma has been proved in [4].

§ 2. In the present section we shall formulate a theorem on the existence of solutions of equation (1) which are of class C^r in the interval $(a, b\rangle$. For the sake of clarity and in order to avoid burdensome calculations we shall present the proof only in the case $r = 1$. For $r > 1$ the theorem can be proved by the same method, i.e. by an application of Schauder's fixed point theorem.

THEOREM 1. *Suppose that*

a) *hypotheses* (I^r), (II^r), (III) *and* (IV^r) *are fulfilled*;

b) *there exists a positive number* $\bar{\eta}$ *such that*

$$f'(x) \leqslant 1 \qquad for \qquad x \in \langle b-\bar{\eta}, b\rangle; \tag{24}$$

c) *we have*

$$|H'_z(b, d)|[f'(b)]^r < 1 . \tag{25}$$

Then for every system of numbers $d_1, \ldots, d_r$ fulfilling (4) *there exist at least one solution $\varphi(x)$ of equation* (1) *that is of class C^r in the interval $(a, b\rangle$ and fulfills the conditions $\varphi(b) = d$, $\varphi'(b) = d_1, \ldots, \varphi^{(r)}(b) = d_r$.*

Proof. Let $r = 1$. We assume hypotheses (I^1), (II^1), (III) and (IV^1), and we fix numbers d and d_1. Inequality (25) takes now the form

$$|H'_z(b, d)|f'(b) < 1 ,$$

from which it follows that there exist a number $\Theta < 1$ and a neighbourhood $U(\eta_0, \delta_0)$ of the point (b, d) (cf. (18)) such that

$$|H'_z(x, z)|f'(x) < \Theta < 1 \quad \text{for} \quad (x, z) \in U(\eta_0, \delta_0) . \tag{26}$$

Let K be an arbitrarily chosen positive number, in the sequel regarded as fixed. We put

$$K_0 \overset{\mathrm{df}}{=} \frac{(1-\Theta)K}{1+|d_1|} . \tag{27}$$

We may assume that the neighbourhood $U(\eta_0, \delta_0)$ has been chosen in such a manner that the inequalities

$$\begin{gathered} |H'_x(x, z) - H'_x(b, d)| \leqslant K_0 , \\ |H'_z(x, z)f'(x) - H'_z(b, d)f'(b)| \leqslant K_0 \end{gathered} \tag{28}$$

hold for $(x, z) \in U(\eta_0, \delta_0)$. Such a neighbourhood can be found for any number $K_0 > 0$, since the first derivatives of the function $H(x, z)$ are continuous at the point (b, d) and the function $f'(x)$ is continuous at the point b. We put also

$$M \overset{\mathrm{df}}{=} K + |d_1| . \tag{29}$$

Now we introduce the interval

$$I \overset{\mathrm{df}}{=} \langle b-\eta, b\rangle ,$$

whose length η is chosen to fulfill

$$\eta \leqslant \min(\bar{\eta}, \eta_0, 1, \delta_0/M) \tag{30}$$

and so that the set

$$\Lambda \overset{\mathrm{df}}{=} \{(x, z)\colon f(x) \in I ,\ |z-d| \leqslant M|x-b|\} \tag{31}$$

be contained in the set Ω.

The inclusion $\Lambda \subset \Omega$ can be realized, since by (IV^1) $(b, d) \in \Omega$. Let us also notice that for $x \in I$ inequality (24) holds and for $(x, z) \in U(\eta, \delta_0) \subset U(\eta_0, \delta_0)$ inequalities (26) and (28) are fulfilled.

Now we shall define an auxiliary set $\mathfrak{G}$ of functions defined in the interval I. Let ε be an arbitrary positive number and let us write

$$\varepsilon' \overset{\mathrm{df}}{=} \frac{(1-\Theta)\varepsilon}{1+M} . \tag{32}$$

The first derivatives of the function $H(x, z)$ are uniformly continuous in the closed set Λ (cf. (31)) and the function $f'(x)$ is uniformly continuous in the closed interval I. Consequently there exist positive numbers η', δ' such that the inequalities

$$\begin{aligned} |H'_x(\bar{x}, \bar{z}) - H'_x(\bar{\bar{x}}, \bar{\bar{z}})| &< \varepsilon' , \\ |H'_z(\bar{x}, \bar{z})f'(\bar{x}) - H'_z(\bar{\bar{x}}, \bar{\bar{z}})f'(\bar{\bar{x}})| &< \varepsilon' \end{aligned} \tag{33}$$

hold for

$$|\bar{x} - \bar{\bar{x}}| < \eta' . \quad |\bar{z} - \bar{\bar{z}}| < \delta' . \tag{34}$$

We put

$$\bar{\delta} \overset{\mathrm{df}}{=} \min\left(\frac{\delta'}{M}, \eta'\right) , \tag{35}$$

the number $\bar{\delta}$ depends only on the choice of the number ε.

Definition. We denote by $\mathfrak{G}$ the set of the functions $a(x)$ which are defined in the interval I and such that for an arbitrary number $\varepsilon > 0$ and for every $\bar{x}, \bar{\bar{x}} \in I$ and fulfilling the condition

$$|\bar{x} - \bar{\bar{x}}| < \bar{\delta} \tag{36}$$

(where the number $\bar{\delta}$ is defined by (35)) the inequality

$$|a(\bar{x}) - a(\bar{\bar{x}})| < \varepsilon \tag{37}$$

holds.

The set $\mathfrak{G}$ is not empty and the functions $a(x) \in \mathfrak{G}$ are equicontinuous in the interval I.

A fundamental part in the proof of our theorem will be played by the function space $\mathcal{F}$.

Definition. We denote by $\mathcal{F}$ the set of all the functions which are defined and of class C^1 in the interval I. To a function $u \in \mathcal{F}$ a norm is ascribed by the formula

$$\|u\| = \max\left(\sup_I |u(x)| , \sup_I |u'(x)|\right) . \tag{38}$$

The space $\mathcal{F}$ is a vector space over the field of real numbers and the convergence in the sense of norm (38) means the uniform convergence of the functions and their first derivatives in the interval I. Hence it follows that $\mathcal{F}$ is a Banach space.

In turn we define a subset $\mathcal{S}$ of the space $\mathcal{F}$.

DEFINITION. The set $\mathcal{S}$ is formed by those functions φ from the set $\mathcal{F}$ which fulfill the following conditions:

(i) $\varphi(b) = d, \quad \varphi'(b) = d_1$,

(ii) $\varphi(x) \in \Omega_x$ for $x \in I$,

(iii) $|\varphi'(x) - d_1| \leqslant K$ for $x \in I$,

(iv) $\varphi'(x) \in \mathcal{C}$.

Let us notice that if $\varphi_1 \in \mathcal{S}$ and $\varphi_2 \in \mathcal{S}$, then

$$\sup_I |\varphi_1(x) - \varphi_2(x)| \leqslant \sup_I |\varphi_1'(x) - \varphi_2'(x)| . \tag{39}$$

In fact, the functions $\varphi_1(x)$ and $\varphi_2(x)$ are of class C^1 in I and fulfill conditions (i). By the mean value theorem we obtain

$$|\varphi_1(x) - \varphi_2(x)| \leqslant \eta |\varphi_1'(\xi) - \varphi_2'(\xi)| ,$$

where $x < \xi < b$, whence

$$\sup_{x \in I} |\varphi_1(x) - \varphi_2(x)| \leqslant \eta \sup_{\xi \in I} |\varphi_1'(\xi) - \varphi_2'(\xi)| .$$

Hence (39) results in view of (30). Thus we have accordingly to the definition of the norm (38)

$$\|\varphi_1 - \varphi_2\| = \sup_{x \in I} |\varphi_1'(x) - \varphi_2'(x)| \quad \text{for} \quad \varphi_1 \in \mathcal{S} , \ \varphi_2 \in \mathcal{S} . \tag{40}$$

Let us also note the following simple inequalities:

$$|\varphi'(x)| \leqslant M \quad \text{for} \quad \varphi \in \mathcal{S} , \tag{41}$$

$$|\varphi(x) - d| \leqslant M |x - b| \quad \text{for} \quad \varphi \in \mathcal{S} . \tag{42}$$

Now we shall prove that

($*$) *The set $\mathcal{S}$ is a compact, closed and convex subset of the space $\mathcal{F}$.*

The set $\mathcal{S}$ is compact. Let $\{\varphi_n\}$ be an infinite sequence of functions of the set $\mathcal{S}$. Consider the sequence $\{\varphi_n'(x)\}$. It follows from (41), (iv) and the definition of the set $\mathcal{C}$ that the derivatives of the functions $\varphi \in \mathcal{S}$ form a set of functions which are equibounded and equicontinuous in the interval I. Thus one can choose from the sequence $\{\varphi_n'(x)\}$ a subsequence $\{\varphi_{k_n}'(x)\}$ uniformly convergent in the interval I. By (40) the subsequence $\{\varphi_{k_n}\}$ of the sequence $\{\varphi_n\}$ converges in the sense of the norm (38), which was to be proved.

The set $\mathcal{S}$ is closed. Let $\varphi_n \in \mathcal{S}$ and $\lim\limits_{n\to\infty} \varphi_n = \varphi$ (in the sense of the norm (38)). We shall show that $\varphi \in \mathcal{S}$. Evidently $\varphi \in \mathcal{F}$, since the sequences $\{\varphi_n(x)\}$ and $\{\varphi_n'(x)\}$ are uniformly convergent in the interval I to the functions $\varphi(x)$ and $\varphi'(x)$, respectively. The function $\varphi(x)$ fulfills conditions (i), (iii) and (iv), which can be easily verified by passing to the limit. It remains to prove that $\varphi(x) \in \Omega_x$. Let us fix an $x \in I$ and let $z_n \stackrel{\text{df}}{=} \varphi_n(x)$. It follows from (42) and definition (31) that $(x, z_n) \in \Lambda$. The set Λ is closed and thus $(x, z) \in \Lambda$, where $z \stackrel{\text{df}}{=} \lim\limits_{n\to\infty} z_n = \varphi(x)$. Further it follows from definition (31) that if $x \in I$ and $(x, z) \in \Lambda$, then $(f^{-1}(x), z) \in \Lambda$. Consequently $(f^{-1}(x), \varphi(x)) \in \Lambda \subset \Omega$, i.e. $\varphi(x) \in \Omega_x$, which was to be proved.

The set $\mathcal{S}$ is convex. We omit a simple verification of this fact.

Now we consider the transformation

$$\Phi[\varphi] \stackrel{\text{df}}{=} H\big(x, \varphi[f(x)]\big) \tag{43}$$

for $\varphi \in \mathcal{S}$. We shall prove that

(**) *Φ maps the set $\mathcal{S}$ into itself, i.e. $\varphi \in \mathcal{S}$ implies $\Phi[\varphi] \in \mathcal{S}$.*

(***) *Φ is a continuous transformation, i.e. $\|\varphi_n - \varphi\| \to 0$ implies $\|\Phi[\varphi_n] - \Phi[\varphi]\| \to 0$.*

In order to prove (**) we take a $\varphi \in \mathcal{S}$ and we write $\psi \stackrel{\text{df}}{=} \Phi[\varphi]$. We have

$$\psi(x) = H\big(x, \varphi[f(x)]\big)\,, \tag{44}$$

$$\psi'(x) = H'_x\big(x, \varphi[f(x)]\big) + H'_z\big(x, \varphi[f(x)]\big)f'(x)\varphi'[f(x)]\,. \tag{45}$$

The derivatives on the right-hand side of (45) exist and are continuous for $x \in I$ on account of hypotheses (I^1) and (II^1) and of the fact that $\varphi(x)$ is of class C^1. Thus $\psi \in \mathcal{F}$. We are going to verify that the function $\psi(x)$ fulfills conditions (i)-(iv).

Condition (i). Since $\varphi(x)$ fulfills (i), we get from (44) and (45)

$$\psi(b) = H(b, d)\,, \qquad \psi'(b) = H'_x(b, d) + H'_z(b, d)f'(b)d_1\,.$$

By (3) and (4) we obtain hence $\psi(b) = d$, $\psi'(b) = d_1$, which was to be proved.

Condition (ii). If $x \in I$, then $\varphi[f(x)] \in \Omega_{f(x)}$. In virtue of hypothesis (III) and relation (44) we obtain hence $\psi(x) \in \Omega_x$, which was to be proved.

Condition (iii). According to (4) and (45) we have

$$\begin{aligned}|\psi'(x) - d_1| \leqslant \big|H'_x\big(x, \varphi[f(x)]\big) - H'_x(b, d)\big| &+ \\ + |d_1|\,\big|H'_z\big(x, \varphi[f(x)]\big)f'(x) - H'_z(b, d)f'(b)\big| &+ \\ + \big|H'_z\big(x, \varphi[f(x)]\big)\big|\,f'(x)|\varphi'[f(x)] - d_1|&\,.\end{aligned}$$

By (42) we have $|\varphi(x)-d| \leqslant M\eta$ for $x \in I$. If an $x \in I$, then also $f(x) \in I$ and we obtain by (30) $|\varphi[f(x)]-d| \leqslant \delta_0$. Thus for $x \in I$ and $\varphi \in S$ the point $(x, \varphi[f(x)])$ belongs to $U(\eta_0, \delta_0)$ and we may make use of inequalities (26) and (28). Taking into account (iii) for the function $\varphi(x)$ and definition (27) we obtain

$$|\psi'(x)-d_1| \leqslant K_0+|d_1|K_0+\Theta K = K ,$$

which was to be proved.

Condition (iv). Let ε be an arbitrary positive number. Suppose that $\bar{x}, \bar{\bar{x}} \in I$ and that inequalities (36) are fulfilled. We write shortly $\bar{z} = \varphi[f(\bar{x})]$, $\bar{\bar{z}} = \varphi[f(\bar{\bar{x}})]$. Thus we have

$$\begin{aligned}|\psi'(\bar{x})-\psi'(\bar{\bar{x}})| \leqslant |H'_x(\bar{x}, \bar{z})-H'_x(\bar{\bar{x}}, \bar{\bar{z}})| &+ \\ + |\varphi'[f(\bar{\bar{x}})]|\,|H'_z(\bar{x}, \bar{z})f'(\bar{x})-H'_z(\bar{\bar{x}}, \bar{\bar{z}})f'(\bar{\bar{x}})| &+ \\ + f'(\bar{x})\,|H'_z(\bar{x}, \bar{z})|\,|\varphi'[f(\bar{x})]-\varphi'[f(\bar{\bar{x}})]| &.\end{aligned}$$

On account of inequality (24) we have

$$|f(\bar{x})-f(\bar{\bar{x}})| \leqslant |\bar{x}-\bar{\bar{x}}| \quad \text{for} \quad \bar{x}, \bar{\bar{x}} \in I ,$$

and the function $\varphi[f(x)]$ fulfills the inequalities

$$|\varphi[f(\bar{x})]-\varphi[f(\bar{\bar{x}})]| \leqslant |\varphi'(\zeta)|\,|f(\bar{x})-f(\bar{\bar{x}})| \leqslant M|\bar{x}-\bar{\bar{x}}| , \tag{46}$$

where $\zeta \overset{\mathrm{df}}{=} f(\bar{x})+q[f(\bar{\bar{x}})-f(\bar{x})] \in I$, $0 < q < 1$. From the above inequalities we conclude that: 1° the function $\varphi'[f(x)]$ belongs to $\mathfrak{C}$, i.e. for $\alpha(x) = \varphi'[f(x)]$ inequality (37) holds, 2° we may make use of inequality (33), since we have by (46) and (36)

$$|\bar{z}-\bar{\bar{z}}| < M\bar{\delta} \leqslant \delta' \quad \text{and} \quad |\bar{x}-\bar{\bar{x}}| < \eta' ,$$

i.e. inequalities (34) are fulfilled. Thus we have

$$|\psi'(\bar{x})-\psi'(\bar{\bar{x}})| \leqslant \varepsilon'+M\varepsilon'+\Theta\varepsilon = \varepsilon$$

(comp. (32)). Consequently the function $\psi'(x) \in \mathfrak{C}$, which was to be proved.

Assertion (**) has been completely proved. We shall only outline the proof of assertion (***).

We write $\psi \overset{\mathrm{df}}{=} \Phi[\varphi]$ and $\psi_n \overset{\mathrm{df}}{=} \Phi[\varphi_n]$. In order to prove that the convergence $\|\varphi_n-\varphi\| \to 0$ implies the convergence $\|\psi_n-\psi\| \to 0$ it is enough to show that $\psi'_n(x) \rightrightarrows \psi'(x)$ in the interval I. (This follows from relation (40) in view of the fact that φ and φ_n belong to S.)

The uniform convergence of the sequence $\{\psi'_n(x)\}$ to the function $\psi'(x)$ in the interval I results from the uniform continuity of the first derivatives of the function $H(x, z)$ in the set Λ and from the uniform convergence $\varphi_n(x) \rightrightarrows \varphi(x)$, $\varphi'_n(x) \rightrightarrows \varphi'(x)$, which is equivalent to the convergence $\|\varphi_n-\varphi\| \to 0$.

Thus we see that transformation (43) is continuous (comp. (***)) in a convex, closed and compact subset S of the Banach space $\mathcal{F}$ (comp. (*)) and maps the set S into itself (comp. (**)). On account of Schauder's theorem there exists at least one invariant point of such a transformation in the set S, i.e. there exists a function $\varphi(x)$ which is of class C^1 in the interval I, fulfills conditions $\varphi(b) = d$, $\varphi'(b) = d_1$, $\varphi(x) \in \Omega_x$ for $x \in I$ and $\Phi[\varphi] = \varphi$. This last equality means that the function $\varphi(x)$ is a solution of equation (1).

As follows from Lemma 8, this solution can be extended onto the whole interval $(a, b\rangle$, the class C^1 being preserved. This completes the proof of Theorem 1 for $r = 1$.

Remark. In the present theorem we have first proved the existence of a local regular solution of equation (1) in the closed interval I. In order to be able to extend this solution onto the whole interval $(a, b\rangle$ we had to make use of Lemma 8 and consequently to make suitable assumptions.

It seems that some of the hypotheses of Theorem 1 could be weakened if we required only local solutions, in a neighbourhood of the point $x = b$. The same remark applies also to Theorem 2 below.

§ 3. In this section we shall prove a theorem on the existence of exactly one C^r solution of equation (1) in the interval $(a, b\rangle$ (Theorem 2). We shall make hypotheses analogous to those of Theorem 1, in particular inequality (25) turns out to be essential. Besides we shall be forced to make an additional assumption, which did not occur in Theorem 1. On the other hand, inequality (24) does not appear to admit the hypotheses of Theorem 2. Two other theorems of this section are corollaries from Lemma 7 and Theorems 1 and 2.

THEOREM 2. *Suppose that hypotheses* (I^r), (II^r), (III) *and* (IV^r) *are fulfilled, inequality* (25) *holds, and the partial derivatives of the r-th order of the function $H(x, z)$ fulfill a Lipschitz condition with respect to the variable z in a neighbourhood $U(\eta, \delta)$ of the point (b, d), i.e. inequalities* (20) *are fulfilled. Then there exists exactly one function $\varphi(x)$ that is of class C^r in $(a, b\rangle$, satisfies equation* (1) *in $(a, b\rangle$ and fulfills the conditions $\varphi(b) = d$, $\varphi'(b) = d_1, \ldots, \varphi^{(r)}(b) = d_r$.*

Proof. Inequalities (20) are fulfilled for $(x, z) \in U(\eta, \delta)$. We may assume that also the inequality

$$(47) \qquad |H_z'(x, z)|[f'(x)]^r < \Theta < 1$$

holds for $(x, z) \in U(\eta, \delta)$. (Such a number Θ and a neighbourhood $U(\eta, \delta)$ exist on account of inequality (25) and the continuity of the functions $H_z'(x, z)$ and $f'(x)$.)

Let us fix a positive number K. It follows from lemma 1 that the function $H_r(x, d, d_1, \ldots, d_r)$ is continuous at the point b. Then there exists a number $\eta_1 > 0$ such that the inequality

$$(48) \qquad |H_r(x, d, d_1, \ldots, d_r) - H_r(b, d, d_1, \ldots, d_r)| < \tfrac{1}{2}(1-\Theta)K$$

holds for $0 \leqslant b-x \leqslant \eta_1$, where the number Θ is that occurring in (47).

Let us put

$$(49) \qquad M_k = \sum_{i=k+1}^{r} |d_i| + K, \quad k = 1, \ldots, r-1; \; M_r = K,$$

and

$$(50) \qquad \alpha_k \stackrel{\text{df}}{=} d_k - M_k, \quad \beta_k \stackrel{\text{df}}{=} d_k + M_k, \quad k = 1, \ldots, r.$$

On account of Lemma 5 the function $H_r(x, z, z_1, \ldots, z_r)$ fulfills a Lipschitz condition with respect to the variables $z, z_1, \ldots, z_r$ in the set Z defined by (21), where α_k and β_k are given by formulae (50). In other words, relations (22) and (23) hold and now we have moreover $\Theta < 1$.

Now we choose a positive number σ in such a manner that the following inequalities be fulfilled:

$$(51) \qquad \sigma \leqslant \min(\eta, \eta_1, 1),$$

$$(52) \qquad \sum_{k=1}^{r} |d_k| \frac{\sigma^k}{k!} + K\frac{\sigma^r}{r!} < \delta,$$

$$(53) \qquad \sum_{k=0}^{r-1} \sum_{i=1}^{r-k} L_k |d_{k+i}| \frac{\sigma^i}{i!} + K \sum_{k=1}^{r} L_{r-k} \frac{\sigma^k}{k!} < \frac{(1-\Theta)K}{2},$$

and

$$(54) \qquad \sum_{k=0}^{r-1} L_k \frac{\sigma^{r-k}}{(r-k)!} + \Theta < \vartheta,$$

where

$$(55) \qquad \Theta < \vartheta < 1.$$

We put

$$I \stackrel{\text{df}}{=} \langle b-\sigma, b\rangle$$

and we define the *function space* $\mathfrak{K}$ as the set of the functions $\varphi(x)$ that are of class C^r in the interval I and fulfill the conditions

$$(56) \qquad \varphi(b) = d, \quad \varphi^{(k)}(b) = d_k, \quad k = 1, \ldots, r,$$

$$(57) \qquad |\varphi^{(r)}(x) - d_r| \leqslant K \quad \text{in } I,$$

$$(58) \qquad \varphi(x) \in \Omega_x \quad \text{for} \quad x \in I.$$

Next we define a metric ϱ in the set $\mathfrak{R}$ putting

$$\varrho(\varphi_1, \varphi_2) = \sup_{x \in I} |\varphi_1^{(r)}(x) - \varphi_2^{(r)}(x)| \tag{59}$$

for $\varphi_1, \varphi_2 \in \mathfrak{R}$. One can easily verify that the metric postulates are fulfilled. In particular, conditions (56) guarantee that if $\varrho(\varphi_1, \varphi_2) = 0$, then $\varphi_1 \equiv \varphi_2$. Moreover, we have for $\varphi_1 \in \mathfrak{R}$ and $\varphi_2 \in \mathfrak{R}$ on account of the mean-value theorem

$$\sup_{x \in I} |\varphi_1^{(k)}(x) - \varphi_2^{(k)}(x)| \leqslant \frac{\sigma^{r-k}}{(r-k)!} \sup_{x \in I} |\varphi_1^{(r)}(x) - \varphi_2^{(r)}(x)|$$

for $k = 0, 1, \ldots, r-1$, i.e. (comp. (59))

$$\sup_{x \in I} |\varphi_1^{(k)}(x) - \varphi_2^{(k)}(x)| \leqslant \frac{\sigma^{r-k}}{(r-k)!} \varrho(\varphi_1, \varphi_2), \quad k = 0, 1, \ldots, r. \tag{60}$$

The above relation shows that the convergence in the sense of metric (59) is equivalent to the uniform convergence of functions and their derivatives up to the order r (inclusively) in the interval I. Consequently the space $\mathfrak{R}$ with metric (59) is complete.

We shall show that transformation (43), regarded now as a transformation in the set $\mathfrak{R}$, maps $\mathfrak{R}$ into itself and that

$$\varrho(\Phi[\varphi_1], \Phi[\varphi_2]) \leqslant \vartheta \varrho(\varphi_1, \varphi_2), \tag{61}$$

where ϑ is the number occurring in (54). Thus we have according to (55) $\vartheta < 1$, i.e. (43) is a contraction mapping. Hence, on account of Banach's fixed-point theorem it follows that Φ has the unique invariant point, i.e. that there exists exactly one function $\varphi(x)$ that is of class C^r in the interval I, satisfies conditions (56) and (58) and

$$\varphi(x) = H\big(x, \varphi[f(x)]\big) \quad \text{in } I.$$

Thus we shall have the required solution in the interval I. But it follows from Lemma 8 that this solution can be extended onto the whole interval $(a, b\rangle$.

Thus the theorem will be completely proved if we verify that transformation (43) has the above mentioned properties.

Let $\varphi \in \mathfrak{R}$ and $\psi \overset{\mathrm{df}}{=} \Phi[\varphi]$. Since the functions $f(x)$ and $H(x, z)$ are of class C^r, so is also the function $\psi(x)$. Further we have from (14) for $x \in I$

$$\psi^{(k)}(x) = H_k\big(x, \varphi[f(x)], \ldots, \varphi^{(k)}[f(x)]\big), \quad k = 1, \ldots, r, \tag{62}$$

whence it follows that

$$\psi^{(k)}(b) = H_k\big(b, \varphi(b), \ldots, \varphi^{(k)}(b)\big), \quad k = 1, \ldots, r.$$

But the function φ fulfills conditions (56) and so we may apply (4) to the above equality. Accordingly, taking also (3) into consideration, we see that the function ψ fulfills conditions (56).

It follows easily from hypothesis (III) that the function ψ fulfills condition (58). We must prove that it fulfills also (57).

Since $\varphi \in \mathfrak{K}$, we have for $x \in I$

$$|\varphi^{(k)}(x)-d_k| \leqslant |\varphi^{(k+1)}(b)|\sigma + \ldots + \big|\varphi^{(r)}\big(b+\tau_k(x-b)\big)\big|\frac{\sigma^{r-k}}{(r-k)!},$$
$$0<\tau_k<1, \quad k=0,1,\ldots,r-1.$$

Inequality (57) implies the relation

$$|\varphi^{(r)}(x)| \leqslant |d_r|+K \quad \text{for} \quad x \in I,$$

whence

$$|\varphi^{(k)}(x)-d_k| \leqslant \sum_{i=1}^{r-k}|d_{k+i}|\frac{\sigma^i}{i!}+K\frac{\sigma^{r-k}}{(r-k)!} \tag{63}$$

for $x \in I$ and $k=0,1,\ldots,r-1$. Hence we get by (51) ($\sigma \leqslant 1$)

$$|\varphi^{(k)}(x)-d_k| \leqslant \sum_{i=1}^{r-k}|d_{k+i}|+K, \quad k=1,\ldots,r-1.$$

Thus we have by (49) and (57) (for $k=r$)

$$|\varphi^{(k)}(x)-d_k| \leqslant M_k \quad \text{for} \quad x \in I,\ k=1,\ldots,r. \tag{64}$$

On the other hand, we obtain from (63) for $k=0$

$$|\varphi(x)-d| \leqslant \sum_{i=1}^{r}|d_i|\frac{\sigma^i}{i!}+K\frac{\sigma^r}{r!}$$

and hence by (52)

$$|\varphi(x)-d| \leqslant \delta \quad \text{for} \quad x \in I. \tag{65}$$

Now we have according to (62) and (4) (for $k=r$)

$$\begin{aligned}|\psi^{(r)}(x)-d_r| &= \big|H_r\big(x,\varphi[f(x)],\varphi'[f(x)],\ldots,\varphi^{(r)}[f(x)]\big)-d_r\big| \\ &\leqslant \big|H_r\big(x,\varphi[f(x)],\varphi'[f(x)],\ldots,\varphi^{(r)}[f(x)]\big)-H_r(x,d,d_1,\ldots,d_r)\big| + \\ &\quad + |H_r(x,d,d_1,\ldots,d_r)-H_r(b,d,d_1,\ldots,d_r)|.\end{aligned}$$

According to (51), (64) and (65), for $x \in I$ the points $\big(x,\varphi[f(x)],\ldots,$ $\ldots,\varphi^{(r)}[f(x)]\big)$ and $(x,d,d_1,\ldots,d_r)$ belong to the set Z (cf. (21) and (50)). Thus, since by (51) $\sigma \leqslant \eta_1$, we may make use of (22) and (48). We obtain

$$|\psi^{(r)}(x)-d_r| \leqslant L_0|\varphi[f(x)]-d| + \sum_{k=1}^{r-1}L_k|\varphi^{(k)}[f(x)]-d_k| +$$
$$+\Theta|\varphi^{(r)}[f(x)]-d_r| + \tfrac{1}{2}(1-\Theta)K.$$

Taking into account (63) and (57) we obtain hence

$$|\psi^{(r)}(x)-d_r| \leqslant L_0\sum_{k=1}^{r}|d_k|\frac{\sigma^k}{k!}+K\frac{\sigma^r}{r!}+\Theta K+$$

$$+\frac{(1-\Theta)K}{2}+\sum_{k=1}^{r-1}L_k\sum_{i=1}^{r-k}|d_{k+i}|\frac{\sigma^i}{i!}+\sum_{k=1}^{r-1}L_kK\frac{\sigma^{r-k}}{(r-k)!}$$

$$=\frac{(1+\Theta)K}{2}+\sum_{k=0}^{r-1}\sum_{i=1}^{r-k}|d_{k+i}|L_k\frac{\sigma^i}{i!}+K\sum_{k=1}^{r}L_{r-k}\frac{\sigma^k}{k!}.$$

This together with (53) yields

$$|\psi^{(r)}(x)-d_r| \leqslant \tfrac{1}{2}(1+\Theta)K+\tfrac{1}{2}(1-\Theta)K = K\,,$$

i.e. the function ψ fulfills condition (57).

Now let us write $\psi_1 \overset{\mathrm{df}}{=} \Phi[\varphi_1]$, $\psi_2 \overset{\mathrm{df}}{=} \Phi[\varphi_2]$. We have by (22)

$$|\psi_1^{(r)}(x)-\psi_2^{(r)}(x)|$$

$$=\left|H_r\big(x,\varphi_1(s),\varphi_1'(s),\dots,\varphi_1^{(r)}(s)\big)-H_r\big(x,\varphi_2(s),\varphi_2'(s),\dots,\varphi_2^{(r)}(s)\big)\right|$$

$$\leqslant L_0|\varphi_1(s)-\varphi_2(s)|+\sum_{k=1}^{r-1}L_k|\varphi_1^{(k)}(s)-\varphi_2^{(k)}(s)|+\Theta\,|\varphi_1^{(r)}(s)-\varphi_2^{(r)}(s)|$$

(where $s=f(x)$), since by (51), (64) and (65) the points $\big(x,\varphi(s),\dots,\varphi^{(r)}(s)\big)$ belong to Z for every $x\in I$ and every function φ from the set $\mathfrak{R}$. Further we obtain from the above inequality

$$\varrho(\psi_1,\psi_2) \leqslant \Theta\varrho(\varphi_1,\varphi_2)+L_0\sup_{x\in I}|\varphi_1(x)-\varphi_2(x)|+$$

$$+\sum_{k=1}^{r-1}L_k\sup_{x\in I}|\varphi_1^{(k)}(x)-\varphi_2^{(k)}(x)|\,.$$

We make use of (60):

$$\varrho(\psi_1,\psi_2) \leqslant \Theta\,\varrho(\varphi_1,\varphi_2)+\sum_{k=0}^{r-1}L_k\frac{\sigma^{r-k}}{(r-k)!}\,\varrho(\varphi_1,\varphi_2)\,,$$

and finally we get by (54)

$$\varrho(\psi_1,\psi_2) \leqslant \vartheta\varrho(\varphi_1,\varphi_2)\,.$$

Consequently inequality (61) holds and the proof of the theorem has been finished.

As a consequence of Theorems 1 and 2 we have the following

THEOREM 3. *Suppose that*

1° *inequality* (25) *holds for a certain exponent* $r=p\geqslant 1$;

2° hypotheses (I^q), (II^q), (III) *and* (IV^q) *are fulfilled and* $q > p$, *so in particular the functions* $f(x)$ *and* $H(x, z)$ *are of class* C^q;

3° inequality (24) *is fulfilled.*

Then for every fixed system $d_1, \ldots, d_p, \ldots, d_q$ *there exists exactly one solution of equation* (1) *that is of class* C^q *in the interval* $(a, b\rangle$ *and fulfills the conditions*

$$\varphi(b) = d\,, \quad \varphi^{(k)}(b) = d_k \tag{66}$$

for $k = 1, \ldots, q$.

Proof. From 1° and 3° it results that (25) holds also for $r = q$. Thus the assumptions of Theorem 1 are fulfilled (with $r = q$) and there exists at least one solution $\varphi(x)$ of equation (1) which is of class C^q in $(a, b\rangle$ and fulfills conditions (66) for $k = 1, \ldots, q$. This solution is also of class C^q in $(a, b\rangle$ (since $q > p$) and fulfills conditions (66) for $k = 1, \ldots, p$. Since the function $H(x, z)$ is of class C^q, its derivatives of the order p satisfy a Lipschitz condition in a suitable set. Thus it follows from Theorem 2, in view of 1°, that there exists exactly one solution of equation (1) which is of class C^p in $(a, b\rangle$ and fulfills (66) for $k = 1, \ldots, p$. Consequently it must be identical with $\varphi(x)$ for $x \in (a, b\rangle$, which completes the proof.

THEOREM 4. *Let hypotheses* (I^r), (II^r), (III) *and* (IV^r) *be fulfilled. If* (24) *holds and* $|H_z'(b, d)| < 1$, *then there exists exactly one solution* $\varphi(x)$ *of equation* (1) *which is of class* C^r *in the interval* $(a, b\rangle$ *and fulfills conditions* (66) *for* $k = 1, \ldots, r$.

Proof. We infer from Lemma 7 that there exists exactly one solution $\varphi(x)$ of equation (1) which is continuous in $(a, b\rangle$ and fulfills the condition $\varphi(b) = d$.

Similarly as in the proof of Theorem 3 we conclude that this solution is also the unique solution of class C^r in the interval $(a, b\rangle$ fulfilling (66) for $k = 1, \ldots, r$.

§ 4. The theorems obtained in the preceding sections concerned the case where inequality (26) holds. The question arises, how essential is this condition for the problem of the existence and uniqueness of differentiable solutions of equation (1). It turns out that (26) ensures the uniqueness. Namely we shall prove the following

THEOREM 5. *If hypotheses* (I^r), (II^r), (III) *and* (IV^r) *are fulfilled and inequality* (17) *holds, then equation* (1) *possesses infinitely many solutions that are of class* C^r *in the interval* $(a, b\rangle$.

More precisely: *There exist numbers* $\eta > 0$, $\varepsilon > 0$ *such that if* $\bar{\varphi}(x)$ *is a solution of equation* (1) *which is of class* C^r *in* (a, b) *and fulfills the condition*

$$|\bar{\varphi}(x) - d| < \varepsilon \quad \textit{for} \quad x \in \langle x_0, f(x_0)\rangle\,, \tag{67}$$

where x_0 is an arbitrary point of the interval $(b-\eta, b)$, then the function

$$\varphi(x) \stackrel{\mathrm{df}}{=} \begin{cases} \overline{\varphi}(x) & \text{for} \quad x \in (a, b)\,, \\ d & \text{for} \quad x = b \end{cases} \tag{68}$$

is a solution of equation (1), which is of class C^r in the interval $(a, b\rangle$ and fulfills the conditions $\varphi(b) = d$, $\varphi'(b) = d_1, \ldots, \varphi^{(r)}(b) = d_r$, where the numbers $d_1, \ldots, d_r$ are the unique roots of equations (4).

Proof. The proof will be by induction. Let $r = 1$, i.e. instead of inequality (17) we have

$$|H_z'(b, d)|f'(b) > 1\,,$$

and then, since, by assumption (I^1), $f'(b) \leqslant 1$, also

$$|H_z'(b, d)| > 1\,.$$

Hence it follows that

$$|G_y'(b, d)| < 1\,, \tag{69}$$

since $d = G(b, d)$ and $H_z'(b, d) = [G_y'(b, d)]^{-1}$. We have also

$$|G_y'(b, d)|[f'(b)]^{-1} < 1\,. \tag{70}$$

Let a function $\overline{\varphi}(x)$ satisfy equation (1) in (a, b) and be of class C^1 there. According to (69) and hypotheses (I^1)–(III) we may apply a theorem proved by M. Kuczma [10], from which it follows that there exist numbers $\eta > 0$, $\varepsilon > 0$ such that if the function $\overline{\varphi}(x)$ fulfills (67), then the function $\varphi(x)$ defined by (68) is a continuous solution of equation (1) in $(a, b\rangle$.

So we need only prove that function (68) is of class C^1 in $(a, b\rangle$. We have by (15) for $x \in (a, b)$

$$\varphi'[f(x)] = G_1\big(x, \varphi(x), \varphi'(x)\big)\,,$$

which, if we put (cf. (6))

$$u(x, y) \stackrel{\mathrm{df}}{=} G_x'(x, y)[f'(x)]^{-1}\,, \qquad v(x, y) \stackrel{\mathrm{df}}{=} G_y'(x, y)[f'(x)]^{-1}\,,$$

can be written in the form

$$\varphi'[f(x)] = u\big(x, \varphi(x)\big) + v\big(x, \varphi(x)\big)\varphi'(x)\,.$$

This means that the function $\varphi'(x)$ satisfies the linear equation

$$\chi[f(x)] - v\big(x, \varphi(x)\big)\chi(x) = u\big(x, \varphi(x)\big) \tag{71}$$

in the interval (a, b). The functions $u(x, y)$ and $v(x, y)$ are continuous in Ω' (Lemma 2), $v\big(x, \varphi(x)\big) \neq 0$ for $x \in (a, b\rangle$, and inequality (70) holds. Consequently there exist a neighbourhood $U(\eta_0, \delta_0)$ of the point (b, d)

and a number $\vartheta < 1$ such that

$$|v(x, y)| < \vartheta \quad \text{for} \quad (x, y) \in U(\eta_0, \delta_0) .$$

The number η_0 may be chosen so small that for $x \in \langle b-\eta_0, b\rangle$ the point $(x, \varphi(x))$ belong to $U(\eta_0, \delta_0)$, and consequently

$$\left|v\left(x, \varphi(x)\right)\right| < \vartheta \quad \text{for} \quad x \in \langle b-\eta_0, b\rangle .$$

The above relation holds, since the function $\varphi(x)$ is continuous at the point b and $\varphi(b) = d$.

Thus we may apply Lemma 6 to equation (71) and we see that every function satisfying equation (71) and continuous in (a, b) is also continuous at the point b. Hence it follows that there exists the limit

$$\lim_{x\to b-0} \varphi'(x) = g . \tag{72}$$

But by (71) we have $g = u(b, d)+v(b, d)g$, i.e. $g = G_1(b, d, g)$. Consequently $g = d_1$, since by Lemma 4 there exists the unique number d_1 fulfilling the equation $d_1 = H_1(b, d, d_1)$ and it is also the unique root of the equation $d_1 = G_1(b, d, d_1)$.

Thus the limit (72) is equal to d_1 and, as results from the mean-value theorem, it is also the value of the left-side derivative of the function $\varphi(x)$ at the point b. Consequently the function $\varphi(x)$ is of class C^1 in $(a, b\rangle$, i.e. for $r = 1$ the theorem has been proved.

If our theorem is true for $r = p \geqslant 1$ and the function $\varphi(x)$ is of class C^p in $(a, b\rangle$ and of class C^{p+1} in (a, b), then we have according to (15)

$$\varphi^{(p+1)}(x) = G_{p+1}\left(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(p+1)}(x)\right) .$$

This means that the derivative $\varphi^{(p+1)}(x)$ satisfies the equation

$$\chi[f(x)] - B\left(x, \varphi(x)\right)\chi(x) = A\left(x, \varphi(x)\right) , \tag{73}$$

where

$$A\left(x, \varphi(x)\right) \stackrel{\text{df}}{=} G_{p+1}\left(x, \varphi(x), \ldots, \varphi^{(p)}(x), 0\right) ,$$

$$B\left(x, \varphi(x)\right) \stackrel{\text{df}}{=} G'_y\left(x, \varphi(x)\right)[f'(x)]^{-p-1} .$$

Relation (73) results from (6) and (11).

On account of the inductive hypotheses and Lemma 3 the functions $A(x, \varphi(x))$ and $B(x, \varphi(x))$ are continuous in the interval $(a, b\rangle$ and the latter fulfills moreover the inequality (comp. (17) for $r = p+1$)

$$0 < \left|B\left(x, \varphi(x)\right)\right| < \vartheta_1 < 1 \quad \text{for} \quad x \in \langle b-\eta, b\rangle .$$

Similarly as in the proof in the case $r = 1$ we deduce from Lemmas 7 and 4 that there exists the limit

$$\lim_{x \to b-0} \varphi^{(p+1)}(x) = d_{p+1}$$

and that the function $\varphi(x)$ is of class C^{p+1} in the interval $(a, b\rangle$.

We have proved that every solution of equation (1) which is defined and of class C^r in the interval (a, b) and fulfills condition (68) may be additionally defined at the point b in such a manner that it will be of class C^r in the interval $(a, b\rangle$. The existence of an infinite number of such solutions follows from Lemma 8.

This completes the proof.

§ 5. As a supplement to the considerations of §§ 1–4 we shall formulate here Theorems 6–10, which are analogues of Theorems 1–5 for the interval $\langle a, b)$ closed on the left side.

We must replace hypothesis (IV^r) by the following hypothesis

(V^r) *There exists numbers $c, c_1, \ldots, c_r$ such that $(a, c) \in \Omega'$ and*

$$c = G(a, c), \quad c_k = G_k(a, c, c_1, \ldots, c_k), \quad k = 1, \ldots, r. \tag{74}$$

We shall denote by $V(\eta, \delta)$ the neighbourhood of the point (a, c) of the form

$$V(\eta, \delta) \overset{\mathrm{df}}{=} \langle a, a+\eta\rangle \times \langle c-\delta, c+\delta\rangle,$$

where η and δ are some positive numbers.

In the sequel (in Theorems 6–10) we shall assume that hypotheses (I^r), (II^r), (III) and (V^r) are fulfilled, which will be not repeated at every particular instant. We make also the following convention: in the sequel by a solution of equation (1) we shall understand a solution which is of class C^r in the interval $\langle a, b)$ and fulfills the conditions

$$\varphi(a) = c, \quad \varphi^{(k)}(a) = c_k, \quad k = 1, \ldots, r,$$

where the numbers $c, c_1, \ldots, c_r$ are an arbitrarily fixed system of roots of equations (74).

THEOREM 6. *If*

$$|H'_z(a, c)|[f'(a)]^r > 1 \tag{75}$$

and if there exists a positive number η such that

$$f'(x) \geqslant 1 \quad \text{for} \quad x \in \langle a, a+\eta\rangle, \tag{76}$$

then there exists at least one solution of equation (1).

THEOREM 7. *If* (75) *holds and if there exists a neighbourhood $V(\eta, \delta)$ such that the derivatives of the r-th order of the function $H(x, z)$ satisfy a Lipschitz condition for $(x, z) \in V(\eta, \delta)$ with respect to the variable z with a constant L independent of x, then there exists exactly one solution of equation* (1).

THEOREM 8. *If inequality* (75) *holds for an* $r_0 < r$ *and inequality* (76) *is fulfilled, then there exists exactly one solution of equation* (1).

THEOREM 9. *If* $|H'_z(a, c)| > 1$ *and inequality* (76) *is fulfilled, then there exists exactly one solution of equation* (1).

THEOREM 10. *If*

$$|H'_z(a, c)|[f'(a)]^r < 1 ,$$

then there exist infinitely many solutions of equation (1).

Remark 1. In order to prove e.g. Theorem 6 we write equation (1) in the form

$$\varphi[f(x)] = G\big(x, \varphi(x)\big)$$

and then, putting

$$w(x) \stackrel{\mathrm{df}}{=} f^{-1}(x) , \qquad \bar{G}(x, y) \stackrel{\mathrm{df}}{=} G\big(f^{-1}(x), y\big) ,$$

we transform it into

$$\varphi(x) = \bar{G}\big(x, \varphi[w(x)]\big) . \tag{77}$$

Now, inequalities (75) and (76) imply that

$$|\bar{G}'_y(a, c)|\,[w'(x)]^r < 1 , \qquad w'(x) \leqslant 1 \text{ in } \langle a, a+\eta\rangle ,$$

Thus one may apply to equation (77) the method of the proof of Theorem 1. Similarly Theorems 7–10 can be proved on the lines of proofs of Theorems 2–5.

Remark 2. In hypothesis (I^r) it would be enough to suppose that $f(x) \neq x$ in (a, b), instead of assuming $f(x) > x$. In fact, if $f(x) < x$, then $f^{-1}(x) > x$ in (a, b), and transforming equation (1) into (77) reduces the problem to the former one. But in this case Theorems 1–5 (with the point (b, d) replaced by (a, c) in all the hypotheses) will be valid for interval $\langle a, b)$, and theorems 6–10 (now with (b, d) in place of (a, c)) for the interval $(a, b\rangle$.

Remark 3. In Theorems 1–5 a may be equal $-\infty$ as well as in Theorems 6–10 b may be equal $+\infty$. This results from the fact that the theorem on solutions of equation (1) that are of class C^r in the interval (a, b) (our Lemma 8) remains valid if $a = -\infty$ or $b = +\infty$.

Remark 4. Theorems 1–5 and 6–10 remain true under somewhat weaker suppositions, viz. if in hypotheses (I^r) and (III) we replace the interval $\langle a, b\rangle$ by the interval $(a, b\rangle$ and $\langle a, b)$, respectively.

Remark 5. The results of the present paper do not say anything about the regular solutions of equation (1) if $|H'_z(b, d)|[f'(b)]^r = 1$. Unfortunately, our methods cannot be applied in this case. Thus this important problem remains open.

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Reçu par la Rédaction le 7. 3. 1962