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An equivalent property to Ivory's theorem from the standpoint of conformal mapping

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Abstract. This paper proves a theorem which states an equivalent property to Ivory's theorem from the standpoint of conformal mapping and deduces a theorem in the theory of the conic sections from this property.

1. Introduction. Ivory's theorem (see [2], [4], [5], [6], [8], [14], [3], p. 32) says:

For a family of confocal conics, let P_1, P_2, P_3, P_4 be the four vertices of a curvilinear rectangle formed by any four members of this family arbitrarily chosen. Then $\overline{P_1P_3} = \overline{P_2P_4}$ holds.

In [2], [4], [5], [6], [8] and [3], p. 32, this theorem was proved by the use of the mapping functions $\cos z$ and z^2 from the standpoint of conformal mapping and, moreover, it was proved that this property characterizes the confocal conic sections.

Let $f = f(z)$ be a non-constant entire function of a complex variable z and let D be a non-empty simply connected domain, where f is univalent. We denote the set of all domains D satisfying the above conditions by $\mathscr{S}$.

Let D be an arbitrarily fixed domain belonging to $\mathscr{S}$. Let $A_1A_2A_3A_4$ be an arbitrary rectangle contained entirely in D whose sides are parallel to the real and imaginary axes on the z-plane and let M_1, M_2, M_3, M_4 be the midpoints of the four sides $A_1A_2, A_2A_3, A_3A_4, A_4A_1$, respectively. Furthermore, let M be the point of intersection of the line segments M_1M_3, M_2M_4.

We consider the conformal mapping of D under the mapping function $w = f(z)$. Let $P_j = f(A_j)$, $V_j = f(M_j)$ $(j = 1, 2, 3, 4)$ and $V = f(M)$ on the w-plane. Since $w = f(z)$ is regular and univalent in D, we obtain the four curvilinear rectangles $P_1V_1VV_4$, $V_1P_2V_2V$, $VV_2P_3V_3$, $V_4VV_3P_4$ on the w-plane as the images of the four plane rectangles $A_1M_1MM_4$, $M_1A_2M_2M$, $MM_2A_3M_3$, $M_4MM_3A_4$ on the z-plane under the mapping function $w = f(z)$. We denote the areas of the four curvilinear rectangles $P_1V_1VV_4$, $V_1P_2V_2V$, $VV_2P_3V_3$, $V_4VV_3P_4$ by S_1, S_2, S_3, S_4, respectively.

Now, we consider the following two properties:

PROPERTY I (Ivory's property). $\overline{P_1P_3} = \overline{P_2P_4}$ *for an arbitrary rectangle* $A_1A_2A_3A_4$ *satisfying the conditions mentioned above in each* D *belonging to* $\mathscr{S}$.

PROPERTY II. $S_1+S_3 = S_2+S_4$ *for an arbitrary rectangle* $A_1A_2A_3A_4$ *satisfying the conditions mentioned above in each* D *belonging to* $\mathscr{S}$.

We formulate the result stated at the beginning of this section as follows:

THEOREM A. *Property* I *holds if and only if* $f(z) = a\sin \alpha z + b\cos \alpha z + c$ *or* $f(z) = a\sinh \alpha z + b\cosh \alpha z + c$ *or* $f(z) = az^2+bz+c$, *where* a, b, c *are arbitrary complex constants and* α *is an arbitrary real constant with* $|a|+|b| > 0$ *and* $\alpha \neq 0$. *In other words, Property* I *characterizes the confocal conic sections, including degenerate cases, from the standpoint of conformal mapping* (see the remark below).

Remark. After some computations, we see that the following two hold:

(i) The horizontal and vertical lines $\mathrm{Im}(z) = \text{const}$ and $\mathrm{Re}(z) = \text{const}$ on the z-plane are transformed by the function $w = f(z) = a\sin \alpha z + b\cos \alpha z + c$ $(a^2+b^2 \neq 0)$ or $f(z) = a\sinh \alpha z + b\cosh \alpha z + c$ $(a^2-b^2 \neq 0)$ (where a, b, c are complex constants and α is a real constant with $\alpha \neq 0$) into a family of confocal ellipses and hyperbolas on the w-plane.

(ii) The horizontal and vertical lines $\mathrm{Im}(z) = \text{const}$ and $\mathrm{Re}(z) = \text{const}$ on the z-plane are transformed by the function $w = f(z) = az^2+bz+c$ (where a, b, c are complex constants with $a \neq 0$) into a family of confocal parabolas on the w-plane.

The purpose of this note is to prove the following theorem and to deduce a theorem in the theory of the conic sections from Property II.

THEOREM 1. *Property* I *and Property* II *are equivalent.*

2. Lemmas. We shall apply the following four lemmas.

LEMMA 1. *Let* $f = f(s, t)$ *be a real-valued function of two real variables* s, t *and let* f *be of class* C^2 *in a domain containing a rectangle* R *with vertices* $R_1 = (a_1, b_1)$, $R_2 = (a_2, b_1)$, $R_3 = (a_2, b_2)$, $R_4 = (a_1, b_2)$, *where* $a_1 \leqslant a_2$ *and* $b_1 \leqslant b_2$. *Then,*

$$\iint_R (\partial^2 f/\partial t\partial s)\, ds\, dt = f(R_1)-f(R_2)+f(R_3)-f(R_4).$$

Proof. See [1], p. 248–249.

Before we state Lemma 2, we state Theorem B.

THEOREM B (Nevanlinna–Pólya Theorem). (See [9], [13].) *If* $f = f(z)$, $g = g(z)$, $h = h(z)$, $k = k(z)$ *are regular functions of* z *in a non-empty* *domain* D *and satisfy* $|f(z)|^2+|g(z)|^2 = |h(z)|^2+|k(z)|^2$ *in* D, *then there exists a unitary matrix* $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ *such that*

$$h(z) = \alpha f(z)+\beta g(z), \qquad k(z) = \gamma f(z)+\delta g(z)$$

hold in D. *Here* $\alpha, \beta, \gamma, \delta$ *are complex constants.*

We may now state Lemma 2. (See [7].)

LEMMA 2. *Suppose that* D *is a domain including* 0. *If* $f = f(z)$, $g = g(z)$, $h = h(z)$, $k = k(z)$ *are regular functions of* z *in* D *and satisfy* $|f'(z)|^2+|g'(z)|^2 = |h'(z)|^2+|k'(z)|^2$ *in* D *and if* $f(0) = g(0) = h(0) = k(0) = 0$, *then* $|f(z)|^2+|g(z)|^2 = |h(z)|^2+|k(z)|^2$ *holds in* D.

Proof. By hypothesis and by Theorem B there exists a unitary matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ such that

$$h'(z) = \alpha f'(z)+\beta g'(z), \tag{1}$$

$$k'(z) = \gamma f'(z)+\delta g'(z) \tag{2}$$

hold in D. Here $\alpha, \beta, \gamma, \delta$ are complex constants.

By (1), (2) and by $f(0) = g(0) = h(0) = k(0) = 0$ we have

$$h(z) = \alpha f(z)+\beta g(z), \tag{3}$$

$$k(z) = \gamma f(z)+\delta g(z). \tag{4}$$

Since $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is a unitary matrix, by (3), (4) we have $|f(z)|^2+|g(z)|^2 = |h(z)|^2+|k(z)|^2$ in D. Thus the lemma is proved.

Before we state Lemma 4, we prove Lemma 3.

LEMMA 3. *Suppose that* D *is a non-empty domain and that* EF *is a line segment contained entirely in* D *with midpoint* M.

(i) *If* $w = f(z) = z^2$ *is univalent in* D *and if we denote* $f(E)$, $f(F)$, $f(M)$ *by* E', F', M', *respectively, then the tangent line to the arc* $f(EF)$ *at* M' *is parallel to the chord* $E'F'$ *joining its extremities on the* w-*plane.*

(ii) *If* $w = f(z) = \cos z$ *is univalent in* D *and if we denote* $f(E)$, $f(F)$, $f(M)$ *by* E', F', M', *respectively, then*

(a) *under the additional hypothesis that* EF *is parallel to the real axis on the* z-*plane the same conclusion as that of part* (i) *holds,*

(b) *under the additional hypothesis that* EF *is parallel to the imaginary axis on the* z-*plane the same conclusion as that of part* (i) *holds.*

Proof. We shall apply the following Theorem C (see [10], p. 15).

THEOREM C. *Suppose that* $w = f(z)$ *is defined in a closed disk* K *with centre at* $z = z_0$ *and is differentiable at* $z = z_0$. *Suppose further that* $f'(z_0) \neq 0$.

If the point z moves along the ray R: $\arg(z-z_0) = \varphi$ ($=$ const) *emanating from the point* $z = z_0$ *on the z-plane, then the arc* $f(R\cap K)$ *possesses a directed tangent line at the point* $w = f(z_0)$ *which makes an angle* $\varphi+\arg\left(f'(z_0)\right)$ *with the real axis on the w-plane.*

Proof of (i). Let E, F, M represent the complex numbers $x-y$, $x+y$, x, respectively. If the point z moves along the ray $\arg(z-x) = \arg(y)$ emanating from the point $z = x$, then by Theorem C we have $\psi = \arg(y) + \arg\left(f'(x)\right)$, where ψ is an angle made by the directed tangent line to the arc $f(MF)$ at M' with the real axis on the w-plane. Hence we have $\psi = \arg(y)+\arg(2x) = \arg(y)+\arg(4x) = \arg(4xy) = \arg\left((x+y)^2 - (x-y)^2\right) = \arg\left(f(F)-f(E)\right) = \arg(\overrightarrow{E'F'})$ which leads to the desired result on the w-plane.

Proof of (ii) (a). Let E, F, M represent the complex numbers $x-t$, $x+t$, x, respectively, where we may assume that $t > 0$.

If the point z moves along the ray $\arg(z-x) = \arg(t) = 0$ emanating from the point $z = x$, then by Theorem C we have $\psi = \arg(t)+\arg\left(f'(x)\right) = \arg\left(f'(x)\right)$, where ψ is an angle made by the directed tangent line to the arc $f(MF)$ at M' with the real axis on the w-plane. Hence we have $\psi = \arg(-\sin x) = \arg(-2\sin x \sin t)$ ($2\sin t > 0$ by the inequality $0 < t < \pi$ which follows from the positiveness of t and the univalency of f in D) $= \arg\left(\cos(x+t)-\cos(x-t)\right) = \arg\left(f(F)-f(E)\right) = \arg(\overrightarrow{E'F'})$ which leads to the desired result on the w-plane.

Proof of (ii) (b). Since it is similar to that of (ii) (a), we omit it.

We may now state Lemma 4.

LEMMA 4. *Let* $P_1P_2P_3P_4$ *be a curvilinear rectangle formed by any four members of a family of confocal conics and let* V_k *be the point on the arc* P_kP_{k+1} ($k = 1, 2, 3, 4$), *where the tangent line is parallel to the chord* P_kP_{k+1} *joining its extremities. Then there exists one and only one member of this family passing through* V_1, V_3. *Similarly, there exists one and only one member of this family passing through* V_2, V_4.

Proof. We discuss two cases.

Case 1. Consider a family of confocal ellipses and hyperbolas.

We may assume that the family lies on the w-plane. Furthermore, we may assume that the two common foci are at 1 and -1. Consider the mapping function $w = f(z) = \cos z$. Then there exist a non-empty simply connected domain D and four points A_1, A_2, A_3, A_4 on the z-plane satisfying the following three conditions:

(i) f is regular and univalent in D.

(ii) The four points A_1, A_2, A_3, A_4 form the four vertices of a rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes on the z-plane.

(iii) The four points $f(A_1), f(A_2), f(A_3), f(A_4)$ coincide with P_1, P_2, P_3, P_4, respectively, on the w-plane.

The above facts follow from the following mapping property of $f(z) = \cos z$.

The horizontal and vertical lines $\mathrm{Im}(z) = \mathrm{const}$ and $\mathrm{Re}(z) = \mathrm{const}$ on the z-plane are transformed by the function $w = f(z) = \cos z$ into the family of confocal ellipses and hyperbolas with common foci at 1 and -1 on the w-plane.

By Lemma 3 we have

$$(5) \qquad f(M_j) = V_j \quad (j = 1, 2, 3, 4),$$

where M_1, M_2, M_3, M_4 denote the midpoints of $A_1A_2, A_2A_3, A_3A_4, A_4A_1$, respectively.

We denote the straight line joining M_1, M_3 by M_1M_3. Obviously, $f(M_1 M_3)$ is a member of the family of confocal ellipses and hyperbolas. Since by (5) $f(M_1) = V_1$ and $f(M_3) = V_3$, this member passes through V_1 and V_3. Similarly, since by (5) $f(M_2) = V_2$ and $f(M_4) = V_4$, there exists one member of the family of confocal ellipses and hyperbolas passing through V_2 and V_4. The proof of the uniqueness part of the lemma is clear.

Case 2. Consider a family of confocal parabolas.

By using the mapping function $f(z) = z^2$ and a similar method to that in Case 1 we can prove the lemma in this case.

3. Proof of Theorem 1. We shall use the same notation as above.

Proof that Property I implies Property II. Let D be an arbitrarily fixed domain belonging to $\mathscr{S}$. As above let $A_1A_2A_3A_4$ be an arbitrary rectangle contained entirely in D whose sides are parallel to the real and imaginary axes on the z-plane, let M_1, M_2, M_3, M_4 be the midpoints of the four sides $A_1A_2, A_2A_3, A_3A_4, A_4A_1$, respectively, and let M be the point of intersection of the segments M_1M_3, M_2M_4. If we represent the point M by the complex number x, then we can represent the four vertices of the rectangle $A_1A_2A_3A_4$ by the complex numbers $x+y, x-\bar{y}, x-y, x+\bar{y}$. By hypothesis we have for arbitrary points $x+y, x-\bar{y}, x-y, x+\bar{y}$ belonging to D

$$(6) \qquad |f(x+y)-f(x-y)| = |f(x+\bar{y})-f(x-\bar{y})|.$$

Squaring both sides of (6) yields

$$(7) \qquad |f(x+y)-f(x-y)|^2 = |f(x+\bar{y})-f(x-\bar{y})|^2.$$

Taking the Laplacians $\Delta = \partial^2/\partial s^2 + \partial^2/\partial t^2$ of both sides of (7) with respect to $x = s+it$ (s, t real), we have

$$4\,|f'(x+y)-f'(x-y)|^2 = 4\,|f'(x+\bar{y})-f'(x-\bar{y})|^2,$$

or

$$(8)\qquad |f'(x+y)-f'(x-y)|^2 = |f'(x+\overline{y})-f'(x-\overline{y})|^2,$$

since, by [12], p. 94, $\Delta|g(z)|^2 = 4|g'(z)|^2$, where $g = g(z)$ is a regular function of z.

By (7) we have

$$(9)\qquad |f(x+y)-f(x-y)|^2 = |\overline{f(x+\overline{y})}-\overline{f(x-\overline{y})}|^2.$$

We next take the Laplacians $\Delta = \partial^2/\partial u^2+\partial^2/\partial v^2$ of both sides of (9), taking into account the fact that $\overline{f(x+\overline{y})}, \overline{f(x-\overline{y})}$ are regular functions of y, where $x+y, x-\overline{y}, x-y, x+\overline{y} \in D$, with respect to $y = u+iv$ (u, v real) and obtain

$$4|f'(x+y)+f'(x-y)|^2 = 4|\overline{f'(x+\overline{y})}+\overline{f'(x-\overline{y})}|^2,$$

or

$$(10)\qquad |f'(x+y)+f'(x-y)|^2 = |f'(x+\overline{y})+f'(x-\overline{y})|^2.$$

Adding (8), (10) side by side and using the Parallelogram Law $|a+b|^2+$ $+|a-b|^2 = 2|a|^2+2|b|^2$ (a, b complex) yields for arbitrary points $x+y$, $x-\overline{y}, x-y, x+\overline{y}$ belonging to D

$$(11)\qquad |f'(x+y)|^2+|f'(x-y)|^2 = |f'(x+\overline{y})|^2+|f'(x-\overline{y})|^2.$$

We denote by $2h$ the length of the sides of $A_1A_2A_3A_4$ which are parallel to the real axis and by $2k$ the length of the sides of $A_1A_2A_3A_4$ which are parallel to the imaginary axis.

Putting $y = u+iv$ (u, v real) in (11) and integrating both sides of the resulting equality with respect to u, v over the rectangle $R = \{(u, v)|$ $0 \leqslant u \leqslant h,\ 0 \leqslant v \leqslant k\}$ yields

$$(12)\qquad \iint_R |f'(x+u+iv)|^2 du\,dv + \iint_R |f'(x-u-iv)|^2 du\,dv$$
$$= \iint_R |f'(x+u-iv)|^2 du\,dv + \iint_R |f'(x-u+iv)|^2 du\,dv.$$

Using linear transformations in (12), applying the Transformation Theorem for Double Integrals (see [11], p. 374) and rearranging the resulting equality if necessary, we have

$$(13)\qquad \iint_{A_1M_1MM_4} |f'(z)|^2 dp\,dq + \iint_{MM_2A_3M_3} |f'(z)|^2 dp\,dq$$
$$= \iint_{M_1A_2M_2M} |f'(z)|^2 dp\,dq + \iint_{M_4MM_3A_4} |f'(z)|^2 dp\,dq,$$

where $z = p+iq$ (p, q real).

If G is a domain enclosed by a simple closed contour C and if $g = g(z)$ ($\neq$ const) is regular and univalent in $E = C \cup G$, then the area of the image of E under the mapping function $g = g(z)$ is given by

$$\iint_E |g'(z)|^2 dp\, dq,$$

where $z = p + iq$ (p, q real) (see [10], p. 183, [12], p. 96).

Hence, by (13) we have $S_1 + S_3 = S_2 + S_4$ and obtain Property II.

Proof that Property II implies Property I.

Let $F = F(z)$, using the complex notation $z = p + iq$ (p, q real), be a real-valued function of a complex variable z satisfying in D

$$\partial^2 F / \partial q \partial p = |f'(z)|^2. \tag{14}$$

We represent the four vertices of $A_1A_2A_3A_4$ by the complex numbers $x+y, x-\bar{y}, x-y, x+\bar{y}$, x denoting the centre of $A_1A_2A_3A_4$. By hypothesis (13) holds. By (13), (14) and by Lemma 1 we have for all combinations of signs of u and v, grouping all terms involved on the left-hand side,

$$\begin{aligned} F(x+y) + F(x-y) + F(x-\bar{y}) + F(x+\bar{y}) - 2\big(F(x+u) + \\ + F(x-u) + F(x+iv) + F(x-iv) - 2F(x)\big) = 0, \end{aligned} \tag{15}$$

where $y = u + iv$ (u, v real) and $x+y, x-\bar{y}, x-y, x+\bar{y} \in D$.

Operating on both sides of (15) with $\partial^2/\partial v \partial u$ and using (14) yields

$$|f'(x+y)|^2 + |f'(x-y)|^2 = |f'(x+\bar{y})|^2 + |f'(x-\bar{y})|^2. \tag{16}$$

We take the Laplacians $\Delta = \partial^2/\partial s^2 + \partial^2/\partial t^2$ of both sides of (16) with respect to $x = s + it$ (s, t real) and obtain

$$|f''(x+y)|^2 + |f''(x-y)|^2 = |f''(x+\bar{y})|^2 + |f''(x-\bar{y})|^2, \tag{17}$$

since $\Delta |g(z)|^2 = 4|g'(z)|^2$, where $g = g(z)$ is a regular function of z.

When x is arbitrarily fixed in D, $f'(x+y) - f'(x)$, $f'(x-y) - f'(x)$, $\overline{f'(x+\bar{y})} - \overline{f'(x)}$, $\overline{f'(x-\bar{y})} - \overline{f'(x)}$ are regular functions of y, where $x+y$, $x-\bar{y}, x-y, x+\bar{y} \in D$, with $\big(f'(x+y) - f'(x)\big)_{y=0} = \big(f'(x-y) - f'(x)\big)_{y=0} = \big(\overline{f'(x+\bar{y})} - \overline{f'(x)}\big)_{y=0} = \big(\overline{f'(x-\bar{y})} - \overline{f'(x)}\big)_{y=0} = 0$. Moreover, by (17) and by observing the formula $|\gamma| = |\bar{\gamma}|$ (γ complex) we have

$$\begin{aligned} &\big|(\partial/\partial y)\big(f'(x+y) - f'(x)\big)\big|^2 + \big|(\partial/\partial y)\big(f'(x-y) - f'(x)\big)\big|^2 \\ &\quad = \big|(\partial/\partial y)\big(\overline{f'(x+\bar{y})} - \overline{f'(x)}\big)\big|^2 + \big|(\partial/\partial y)\big(\overline{f'(x-\bar{y})} - \overline{f'(x)}\big)\big|^2. \end{aligned}$$

Hence, by Lemma 2 we have

$$\begin{aligned} |f'(x+y) - f'(x)|^2 + |f'(x-y) - f'(x)|^2 = |\overline{f'(x+\bar{y})} - \overline{f'(x)}|^2 + \\ + |\overline{f'(x-\bar{y})} - \overline{f'(x)}|^2, \end{aligned}$$

or

$$(18) \quad |f'(x+y)-f'(x)|^2+|f'(x-y)-f'(x)|^2 = |f'(x+\bar{y})-f'(x)|^2+ + |f'(x-\bar{y})-f'(x)|^2.$$

Subtracting (18) from (16) side by side and using the identity $|a-b|^2 = |a|^2+|b|^2-2\,\mathrm{Re}(a\bar{b})$ (a, b complex), we see that

$$(19) \quad \mathrm{Re}\big(f'(x+y)\overline{f'(x)}\big)+\mathrm{Re}\big(f'(x-y)\overline{f'(x)}\big) = \mathrm{Re}\big(f'(x+\bar{y})\overline{f'(x)}\big)+ + \mathrm{Re}\big(f'(x-\bar{y})\overline{f'(x)}\big).$$

By the linearity of Re and using the formula $\mathrm{Re}(\gamma) = \mathrm{Re}(\bar{\gamma})$ (γ complex) (19) yields

$$(20) \quad \mathrm{Re}\Big(\big(f'(x+y)+f'(x-y)\big)\overline{f'(x)}-\big(\overline{f'(x+\bar{y})}+\overline{f'(x-\bar{y})}\big)f'(x)\Big) = 0,$$

where $x+y,\ x-\bar{y},\ x-y,\ x+\bar{y} \in D$.

Since $\big(f'(x+y)+f'(x-y)\big)\overline{f'(x)}-\big(\overline{f'(x+\bar{y})}+\overline{f'(x-\bar{y})}\big)f'(x)$ on the left-hand side of (20) is a regular function of y, by a famous theorem in analytic function theory we have

$$(21) \quad \big(f'(x+y)+f'(x-y)\big)\overline{f'(x)}-\big(\overline{f'(x+\bar{y})}+\overline{f'(x-\bar{y})}\big)f'(x) = A(x),$$

where $x+y,\ x-\bar{y},\ x-y,\ x+\bar{y} \in D$ and $A(x)$ is a function of x only.

Putting $y = 0$ in (21) yields $A(x) = 0$ at every point of D. Hence, by (21) we have

$$(22) \quad \big(f'(x+y)+f'(x-y)\big)\overline{f'(x)} = \big(\overline{f'(x+\bar{y})}+\overline{f'(x-\bar{y})}\big)f'(x).$$

Taking the absolute values of both sides of (22), using the formula $|\bar{\gamma}| = |\gamma|$ (γ complex) and taking into account the fact that $f'(x) \neq 0$ in D which follows from the univalency of f in D yields

$$(23) \quad |f'(x+y)+f'(x-y)| = |\overline{f'(x+\bar{y})}+\overline{f'(x-\bar{y})}|.$$

When x is arbitrarily fixed in D, $f'(x+y)+f'(x-y)$, $\overline{f'(x+\bar{y})}+\overline{f'(x-\bar{y})}$ are regular functions of y, where $x+y,\ x-\bar{y},\ x-y,\ x+\bar{y} \in D$. Hence, by the Maximum Modulus Theorem we have

$$(24) \quad f'(x+y)+f'(x-y) = C\big(\overline{f'(x+\bar{y})}+\overline{f'(x-\bar{y})}\big),$$

where C is a complex constant of modulus 1 and may or may not depend on x.

Integrating both sides of (24) with respect to y yields

$$(25) \quad f(x+y)-f(x-y) = C\big(\overline{f(x+\bar{y})}-\overline{f(x-\bar{y})}\big)+K,$$

where K is a complex constant and may or may not depend on x.

Putting $y = 0$ in (25), we have $K = 0$ and so

$$f(x+y)-f(x-y) = C\left(\overline{f(x+\bar{y})}-\overline{f(x-\bar{y})}\right),$$

where $|C| = 1$. Hence we have

$$|f(x+y)-f(x-y)| = |\overline{f(x+\bar{y})}-\overline{f(x-\bar{y})}|,$$

or

$$|f(x+y)-f(x-y)| = |f(x+\bar{y})-f(x-\bar{y})|,$$

where $x+y$, $x-\bar{y}$, $x-y$, $x+\bar{y} \in D$. Consequently, we obtain Property I.

We may now prove the following

THEOREM 2. *Property* II *holds if and only if* $f(z) = a\sin az + b\cos az + c$ *or* $f(z) = a\sinh az + b\cosh az + c$ *or* $f(z) = az^2 + bz + c$, *where* a, b, c *are arbitrary complex constants and* a *is an arbitrary real constant with* $|a| + |b| > 0$ *and* $a \neq 0$. *In other words, Property* II *characterizes the confocal conic sections, including degenerate cases, from the standpoint of conformal mapping.*

Proof. The proof is clear from Theorem 1 and Theorem A.

4. A theorem in the theory of the conic sections deduced from Property II.

THEOREM 3. *Let* $P_1P_2P_3P_4$ *be a curvilinear rectangle formed by any four members of a family of confocal conics and let* V_k *be the point on the conical arc* P_kP_{k+1} $(k = 1, 2, 3, 4)$, *where the tangent line is parallel to the chord* P_kP_{k+1} *joining its extremities. Furthermore, let the point of intersection of the two members of this family stated in Lemma* 4, *one passing through* V_1, V_3 *and the other passing through* V_2, V_4, *be* V *and let the areas of the four curvilinear rectangles* $P_1V_1VV_4$, $V_1P_2V_2V$, $VV_2P_3V_3$, $V_4VV_3P_4$ *be* S_1, S_2, S_3, S_4, *respectively. Then we have*

$$S_1+S_3 = S_2+S_4.$$

Proof. We discuss two cases.

Case 1. Consider a family of confocal ellipses and hyperbolas.

We may assume that the family lies on the w-plane. Furthermore, we may assume that the two common foci are at 1 and -1. Consider the mapping function $w = f(z) = \cos z$. We shall use the same notation as in Lemma 4. Since by Theorem A Property I holds under the mapping function $w = f(z) = \cos z$, by Theorem 1 Property II holds under the mapping function $w = f(z) = \cos z$. By (5) we obtain the four curvilinear rectangles $P_1V_1VV_4$, $V_1P_2V_2V$, $VV_2P_3V_3$, $V_4VV_3P_4$ on the w-plane as the images of the four plane rectangles $A_1M_1MM_4$, $M_1A_2M_2M$, $MM_2A_3M_3$, $M_4MM_3A_4$ on the z-plane under the mapping function $w = f(z) = \cos z$. Hence the proof is complete in this case.

Case 2. Consider a family of confocal parabolas.

If we consider the mapping function $w = f(z) = z^2$, then we can similarly prove the desired result in this case.

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