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# The local boundary regularity of the solution of the $\bar{\partial}$-equation

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**Abstract.** Let $D$ be a domain of holomorphy on a complex closed submanifold of $C^n$ and let $E$ be an open subset of $\partial D$, which is smooth and strictly pseudoconvex at every point. We prove that, given $\varepsilon < 1/2$, every $\bar{\partial}$-closed differential form $f$ on $D$ with coefficients of class $\mathscr{C}^k(D \cup E)$ admits a solution $u$ of the equation $\bar{\partial}u = f$, such that the coefficients of $u$ are of class $\mathscr{C}^{k+\varepsilon}(D \cup E)$. We give also applications to the extension and division of holomorphic functions in pseudoconvex domains.

**1. Introduction.** In 1970 Grauert and Lieb [7] and Henkin [9] proved that if $f$ is $\bar{\partial}$-closed differential form of type $(0, q)$ with bounded coefficients in a strictly pseudoconvex bounded domain $D$ in $C^n$, then there exists a solution $u$ of the equation $\bar{\partial}u = f$ which also has bounded coefficients in $D$. Henkin and Romanov [11] showed that the equation $\bar{\partial}u = f$ for $f$ bounded admits a solution with coefficients which are 1/2-Hölder continuous in $\bar{D}$. The corresponding result for forms with coefficients of class $\mathscr{C}^k(\bar{D})$ (the solution having coefficients of class $\mathscr{C}^{k+1/2}(\bar{D})$) was proved by Siu [20] for $(0, 1)$-forms, and by Lieb and Range [17] for arbitrary $(0, q)$-forms, $q \geq 1$. Similar theorems, for strictly pseudoconvex, but not necessarily bounded domains, were obtained by Tomassini [21].

In this note we prove a theorem which shows that these results are, in a sense, of local character. Let $D$ be a domain in a closed complex submanifold $M$ of $C^n$ and let $E$ be an open subset of $\partial D$. We say that $\partial D$ is *smooth and strictly pseudoconvex at all points of $E$* iff the following holds:

(1.1) For every $z \in E$ there exist a neighbourhood $U$ of $z$ and a function $\sigma$ which is smooth and strictly plurisubharmonic in $U$, such that $d\sigma(s) \neq 0$ for every $s \in U$, and $D \cap U = \{s \in U \mid \sigma(s) < 0\}$.

THEOREM 1.1. *Let $D$ be a domain of holomorphy in a closed complex submanifold $M$ of $C^n$ and let $E$ be an open subset of $\partial D$, such that $\partial D$ is smooth and strictly pseudoconvex at all points of $E$. Suppose that $f$ is a $\bar{\partial}$-closed*

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*differential form of type* $(0, q)$ *with coefficients in* $\mathscr{C}^k(D \cup E)$ *and smooth in* $D$, $k \in N \cup \{\infty\}$, $q = 1, \ldots, n$. *Then for every* $\varepsilon$ *with* $0 \leqslant \varepsilon < 1/2$ *there exists a differential form* $u$ *of type* $(0, q-1)$ *with coefficients of class* $\mathscr{C}^{k+\varepsilon}(D \cup E)$ *and smooth in* $D$, *such that* $\bar{\partial} u = f$.

[Here, as usually, $\mathscr{C}^k(D \cup E)$, $k \in N$ (resp. $\mathscr{C}^t(D \cup E)$, $t > 0$, $t \neq 1, 2, \ldots$) denotes the space of all functions $k$-times differentiable in $D$, such that their derivatives up to order $k$ extend continuously to $D \cup E$ (resp. the space of all functions $f$ of class $\mathscr{C}^{[t]}(D \cup E)$), such that, given $z \in D \cup E$ and a coordinate neighbourhood $U$ of $z$ in $M$, the derivatives of $f$ of order $[t]$ (computed in the local coordinates in $U$) satisfy the Hölder condition with exponent $t-[t]$ in $(D \cup E) \cap U$.]

The idea of the proof is the same as in [21]. We construct an exhausting sequence of strictly pseudoconvex and bounded domains $\{D_i\}_{i=1}^{\infty}$ in $M$ such that $\bigcup_{i=1}^{\infty} \bar{D}_i = D \cup E$. On every domain $D_i$, the equation $\bar{\partial} u = f$ can be solved with prescribed boundary regularity of solution, because of [17]. It is then proved that the solutions $u_i$ can be chosen in such a way that $u_{i+1}$ is, in a sense, close to $u_i$ on a large partion of $D_i$.

In the last section we give applications of this theorem to the problem of extension of holomorphic functions from a submanifold to a domain of holomorphy and to the problem of division of holomorphic functions, vanishing on a submanifold of a domain of holomorphy.

We use the following notation.

Given an open subset $U$ of $M$, $\mathcal{O}(U)$ denotes the space of holomorphic functions in $U$.

If $K$ is a compact subset of $U$, the set $\hat{K}_{\mathcal{O}(U)} = \{z \in U |$ for every $f \in \mathcal{O}(U)$, $|f(z)| \leqslant \|f\|_K\}$ is the holomorphically convex hull of $K$ in $U$. ($\| \ \|_K$ is the usual sup norm on $K$.)

The complex hessian of the function $f \in \mathscr{C}^2(U)$ at a point $z \in U$, acting on a vector $w \in C^n$, will be denoted by $[(Hf)(z)](w)$; in local coordinates $(\xi_1, \ldots, \xi_n)$ around, $[(Hf)(z)](w) = \sum_{i,j=1}^{n} \partial^2 f / \partial z_i \, \partial \bar{z}_j (z) \, w_i \bar{w}_j$.

If $S$ and $S'$ are two smooth hypersurfaces in $M$ (i.e., $\dim_{\boldsymbol{R}} S = \dim_{\boldsymbol{R}} S' = 2n-1$, where $\dim_{\boldsymbol{C}} M = n$), we say that $S$ and $S'$ intersect transversally if, for every $z \in S \cap S'$, $T_z S + T_z S' = T_z M$. ($T_z S$ denotes the real tangent space to $S$ at a point $z$.)

A domain $D$ in $M$ is called *strictly pseudoconvex* if $D$ is relatively compact in $M$ and there exist a neighbourhood $V$ of $\partial D$ in $M$ and a function $\sigma$, smooth and strictly plurisubharmonic in $V$, such that $d\sigma(z) \neq 0$ at every point $z \in V$ and

$$D = (D \setminus V) \cup \{z \in V | \ \sigma(z) < 0\}. \tag{1.2}$$

$\sigma$ is then called a (strictly plurisubharmonic) defining function for $D$ in $V$.

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**2. Preliminaries.** We need a following lemma on the existence of a strictly plurisubharmonic defining function for a strictly pseudoconvex hypersurface $E$, defined in a whole neighbourhood of $E$ in $M$.

LEMMA 2.1. *Let $M$, $D$ and $E$ be as in the assumption of Theorem 1.1. Then there exist a neighbourhood $U$ of $E$ in $M$ and a function $\sigma$, smooth and strictly plurisubharmonic in $U$, such that $d\sigma \neq 0$ at every point of $U$ and $D \cap U = \{z \in U |\ \sigma(z) < 0\}$.*

Remark. The existence of such a function is well-known e.g. if $E = \partial D$, where $D$ is a strictly pseudoconvex and relatively compact domain in $M$ (and hence $E$ is compact). In our case, $E$ need not be a compact subset of $M$.

Proof. The proof is a simple modification of the construction from [8].

For every $z \in E$, there exist a neighbourhood $U_z$ of $z$ and a function $\sigma_z$ defined in $U_z$ with the properties listed in (1.1). Choose a family $\{U_{z_j}\}_{j=1}^{\infty}$ of sets of this type and family of functions $\{\varphi_j\}_{j=1}^{\infty}$ such that $\{U_{z_j}\}$ is locally finite, $\varphi_j \in \mathscr{C}^{\infty}(U_{z_j})$, $\operatorname{supp} \varphi_j$ is compact in $U_{z_j}$ for all $j$ and $E \subset \bigcup_{j=1}^{\infty} \operatorname{int}(\operatorname{supp} \varphi_j)$. Set $\varrho = \sum_{j=1}^{\infty} \varphi_j \sigma_{z_j}$ (where $\varphi_j \sigma_{z_j}$ is understood to be zero outside $U_{z_j}$). Choose a sequence $\{E_i\}_{i=1}^{\infty}$ of open relatively compact subsets of $E$ such that $\bar{E}_i \subset E_{i+1}$, $i = 1, 2, \ldots$, and $\bigcup_{i=1}^{\infty} E_i = E$. Exactly as in [8], Proposition IX.A 4, one can prove that for every $i \in N$ there exists $c_i > 0$ such that, for every function $A$ defined and smooth in a neighbourhood of $E_i$ in $M$, fulfilling the inequality $A \geqslant c_i$ on $\bar{E}_i$, the function $e^{A\varrho}\varrho$ is strictly plurisubharmonic in some neighbourhood of $\bar{E}_i$. It follows that if we choose $A$ to be a smooth function defined in a neighbourhood of $E$, such that $A \geqslant c_i$ on $\bar{E}_i \setminus E_{i-1}$, $i = 1, 2, \ldots$ (here $E_0 = \emptyset$), then the function $\sigma_1 = e^{A\varrho}\varrho$ is strictly plurisubharmonic in some neighbourhood $U_i$ of $E$ in $M$. Moreover, the set $U_2 = \bigcup_{j=1}^{\infty} \operatorname{int}(\sup \varphi_j)$ is a neighbourhood of $E$ such that $M \cap U_2 = \{z \in U_2 |\ \sigma_1(z) < 0\}$. Also, since $d\sigma_1 = e^{A\varrho} \sum_{j=1}^{\infty} \varphi_j d\sigma_{z_j} \neq 0$ on $E$, there exists a neighbourhood $U_3$ of $E$ with $U_3 \subset U_2$ such that $d\sigma_1(z) \neq 0$ for all $z \in U_3$. We then see that $U = U_1 \cap U_3$ and $\sigma = \sigma_1|_U$ satisfy the assertion.

The next lemma is a slightly modified version of the result from [21] on the smoothing of the intersection of two strictly pseudoconvex hypersurfaces.

LEMMA 2.2 ([21], p. 244). *Suppose that $U$ is an open subset of $M$ and let*

$\sigma_i$, $i = 1, 2$, *be smooth and strictly plurisubharmonic functions in* $U$ *such that, for every* $z \in U$, $d\sigma_i(z) \neq 0$, $i = 1, 2$, *the hypersurfaces* $N_i = \{z \in U|\ \sigma_i(z) = 0\}$, $i = 1, 2$, *intersect transversally and* $N_1 \cap N_2$ *is compact in* $U$. *Let* $A_i = \{z \in U|\ \sigma_i(z) < 0\}$. *Then there exists a neighbourhood* $W$ *of* $N_1 \cap N_2$ *such that* $\bar{W} \subset U$ *and an open subset* $A$ *of* $U$ *such that*

(2.1) $(A_1 \cap A_2) \setminus W \subset A \subset A_1 \cap A_2$, $(\partial A \cap U) \setminus (N_1 \cup N_2) \subset W$ *and* $\partial A$ *is smooth and strictly pseudoconvex at all points of* $\partial A \cap U$.

The proof of the above lemma is similar to that in [21].

We will also use the well-known fact that if $N$ is a Stein manifold, then there exists an increasing sequence of strictly pseudoconvex subdomains $\{N_i\}_{i=1}^{\infty}$ of $N$ such that

(2.2) $\bar{N}_i \subset N_{i+1}$, $i = 1, 2, \ldots$, $\bigcup_{i=1}^{\infty} N_i = N$, and for every $n, k \in \boldsymbol{N}$ with $n < k$, $(\bar{N}_n)^{\wedge}_{\mathcal{O}(N_k)} = \bar{N}_n$.

We need a lemma on the transversal intersection of a strictly pseudoconvex domain with a hypersurface; this lemma seems also to be well-known, although we cannot give any exact reference.

LEMMA 2.3. *Let* $D$ *and* $N$ *be a strictly pseudoconvex domain and a smooth hypersurface, respectively, in a closed complex submanifold* $M$ *of* $\boldsymbol{C}^n$. *Let* $U$ *be a neighbourhood of* $\partial D \cap N$. *Then there exists a strictly pseudoconvex domain* $C$ *in* $M$ *such that* $C \subset D \cup U$, $C \setminus U = D \setminus U$ *and* $\partial C$ *intersects* $N$ *transversally.*

Proof. Let $\sigma$ be a strictly plurisubharmonic defining function for $D$ in some neighbourhood $V$ of $\partial D$. We may assume that $U \subset V$. There exists $\varepsilon > 0$ such that, for every $t \in \boldsymbol{R}$ with $|t| \leqslant \varepsilon$, the domains $D_t = (D \setminus V) \cup \{z \in V|\ \sigma(z) < t\}$ are strictly pseudoconvex. Let $W$ be a neighbourhood of $\partial D \cap N$, relatively compact in $U$. Diminishing $\varepsilon$ if necessary, we may assume that $\partial D_t \cap N \subset W$ for $|t| < \varepsilon$. It follows from [6], Lemma 4.6, that for almost all $t \in [-\varepsilon, \varepsilon]$ (with respect to the Lebesgue measure) the hypersurfaces $\partial D_t$ intersect $N$ transversally. Choose a function $\varphi \in \mathscr{C}^{\infty}(M)$ such that $0 \leqslant \varphi \leqslant 1$, supp $\varphi \subset U$, and $\varphi = 1$ on $\bar{W}$. Set $\tau_t = (1-\varphi)\sigma + (\sigma - t)$, $|t| \leqslant \varepsilon$ (observe that $\sigma - t$ is a defining function for the domain $D_t$) and let $C_t = (D \setminus V) \cup \{z \in V|\ \tau_t(z) < 0\}$. Then $\partial C_t \cap N = \partial D_t \cap N$, and (in local coordinates) $d\tau_t = d\sigma - t\, d\varphi$ and $H(\tau_t) = H\sigma - t \cdot H\varphi$, i.e., $\tau_t$ is strictly plurisubharmonic and $d\tau_t \neq 0$ in a neighbourhood of $\partial C_t$ if $t$ is sufficiently close to 0. Therefore, any domain $C = C_t$ with $t$ sufficiently close to zero satisfies the assertion.

The next lemma is a kind of Andreotti–Grauert's "bumps lemma" [2]:

LEMMA 2.4. *Let* $M$, $D$ *and* $E$ *be as in the assumption of Theorem* 1.1. *Then there exists a domain of holomorphy* $C$ *in* $M$ *such that* $D \cup E \subset C$ *and* $\partial D \setminus E \subset \partial C$.

Proof. Choose an increasing sequence $\{K_i\}_{i=1}^{\infty}$ of compact subsets of $E$

such that $\bigcup_{i=1}^{\infty} K_i = E$ and a neighbourhood $U$ of $E$ in $M$ such that $U \cap \partial D = E$ and $\bar{U} \cap \partial D = \bar{E}$. We construct a sequence $\{C_i\}_{i=0}^{\infty}$ of domains of holomorphy in $M$ such that

(2.3) $D = C_0 \subset C_1 \subset C_2 \subset \ldots, \quad D \cup K_i \subset C_i, \quad \partial D \setminus E \subset \partial C_i, \quad \partial C_i \setminus \partial D \subset U,$ and $\partial C_i \setminus (\partial D \setminus E)$ is smooth and strictly pseudoconvex at every point.

Suppose that $C_0, \ldots, C_m$ are already constructed. Since $\partial C_m \setminus (\partial D \setminus E)$ is smooth and strictly pseudoconvex, there exist, by Lemma 2.1, a neighbourhood $V$ of $\partial C_m \setminus (\partial D \setminus E)$ in $M$ and a smooth and strictly plurisubharmonic function $\sigma$ in $V$ such that $d\sigma \neq 0$ in $V$ and $\{z \in V \mid \sigma(z) < 0\} = C_m \cap V$. Since $\partial C_m \setminus \partial D \subset U$, we may assume that $V \subset U$. Choose $\varphi \in \mathscr{C}^{\infty}(M)$ such that $\operatorname{supp} \varphi \subset V$ and $\varphi > 0$ in a neighbourhood of $K_{m+1} \cap C_m$. Then the function $\sigma = \varepsilon\varphi$, $\varepsilon > 0$, is smooth and strictly plurisubharmonic in $V$ and $d(\sigma - \varepsilon\varphi) \neq 0$ in $V$, provided that $\varepsilon$ is sufficiently small, and $K_{m+1} \subset C_m \cup \{z \in V \mid (\sigma - \varepsilon\varphi)(z) < 0\}$. Set $C_{m+1} = C_m \cup \{z \in V \mid (\sigma - \varepsilon\varphi)(z) < 0\}$. The sequence $\{C_i\}$ constructed in this way satisfies (2.3).

The set $C = \bigcup_{i=1}^{\infty} C_i$ is a domain of holomorphy, being a union of an increasing sequence of domains of holomorphy $C_i$ on a Stein manifold. It is easy to show that the conditions $D \cup E \subset C$ and $\partial D \setminus E \subset \partial C$ are also satisfied. (The last one can be verified by use of the properties $U \cap \partial D = E$ and $\bar{U} \cap \partial D = \bar{E}$.)

LEMMA 2.5. *Let $M$, $D$ and $E$ be as in the assumption of Theorem 1.1. Given a compact subset $K$ of $D \cup E$, there exists a strictly pseudoconvex subdomain $N$ of $M$ such that $K \subset \bar{N} \subset D \cup E$, $K \cap \partial N \subset E$, $\partial N \cap \partial D \subset E$, and $\partial N \cap \partial D$ is a neighbourhood of $K \cap \partial D$ in $\partial D$.*

Proof. By Lemma 2.4, there exists a domain of holomorphy $C$ in $M$ such that $D \cup E \subset C$ and $\partial D \setminus E \subset \partial C$. By (2.2), there exists a sequence $\{C_i\}_{i=1}^{\infty}$ of strictly pseudoconvex subdomains of $C$ such that $\bigcup_{i=1}^{\infty} C_i = C$. Choose a neighbourhood $U$ of $E$ in $M$ and a strictly plurisubharmonic function $\sigma$, defining for $D \cap U$ in $U$, according to Lemma 2.1. Choose also a connected and compact subset $L$ of $D \cup E$, which is a neighbourhood of $K$ in $D \cup E$, a positive integer $i$ so that $L \subset C_i$, and a neighbourhood $V$ of $\partial C_i \cap E$ such that $\bar{V} \cap L = \emptyset$ and $\bar{V} \subset U$. By Lemma 2.3, there exists a strictly pseudoconvex domain $C'$ in $M$ such that $C' \subset C_i \cup V$, $C' \setminus V = C_1 \setminus V$, and $\partial C'$ and $E$ intersect transversally. Choose a defining strictly plurisubharmonic function $\tau$ for $C'$ in some neighbourhood $W$ of $\partial C'$. By Lemma 2.2, there exist a neighbourhood $U_1$ of $\partial C' \cap E$ with $\bar{U}_1 \subset V \cap W$ and an open subset $A$ of $V \cap W$ such that $(C' \cap D \cap V \cap W) \setminus U_1 \subset A \subset C' \cap D \cap V \cap W$, $(\partial A \cap V \cap W) \setminus (\partial C' \cup \partial D) \subset U_i$ and $\partial A$ is smooth and strictly pseudoconvex at all points of $\partial A \cap V \cap W$. Let $N$ be the connected component of

$[(C' \cap D) \setminus U_1] \cup A$ containing $L$; then $N$ has the properties listed in the assertion of the lemma.

COROLLARY 2.6. *There exists a sequence of strictly pseudoconvex domains* $\{D_i\}_{i=1}^{\infty}$ *in* $M$ *such that* $D_1 \subset D_2 \subset \ldots$, $\bigcup_{i=1}^{\infty} D_i = D$, $\bigcup_{i=1}^{\infty} \bar{D}_i = D \cup E$, *and for every* $i \in N$, $\partial D_i \cap \partial D \subset E$, $\partial D_i \cap \partial D_{i+1} \subset E$ *and* $\partial D_{i+1} \cap \partial D$ *is a neighbourhood of* $\partial D_i \cap \partial D$ *in* $\partial D$.

Proof. Choose a sequence $\{K_i\}_{i=1}^{\infty}$ of compact subsets of $D \cup E$ such that $K_1 \subset K_2 \subset \ldots$, and $\bigcup_{i=1}^{\infty} K_i = D \cup E$. Let $D_i$ be constructed as in Lemma 2.5 with respect to $K_i$. Then proceed inductively, by constructing $D_{i+1}$ as in Lemma 2.5, with respect to $K_{i+1} \cup \bar{D}_i$.

We need also a technical lemma on the approximation by holomorphic functions.

LEMMA 2.7. *Suppose that* $G$ *is a domain of holomorphy in* $M$ *and* $F$ *is an open subset of* $\partial G$ *such that* $\partial G$ *is smooth and strictly pseudoconvex at every point of* $F$. *Let* $C$, $C_1$ *and* $C_2$ *be domains of holomorphy in* $M$ *such that* $G \cup F \subset C$, $\partial G \setminus F \subset \partial C$, *and* $\bar{C}_1 \subset C_2 \subset \bar{C}_2 \subset C$. *Suppose also that* $W$ *is a neighbourhood of* $\bar{C}_2$ *such that* $(\bar{C}_1)\hat{}_{\mathcal{O}(W)} = \bar{C}_1$. *Then there exists a neighbourhood* $U$ *of* $\bar{G} \cap \bar{C}_2$ *such that every function holomorphic in a neighbourhood of* $\bar{G} \cap \bar{C}_1$ *can be approximated uniformly on* $\bar{G} \cap \bar{C}_1$ *by functions holomorphic in* $U$.

Proof. By Lemma 2.5 there exists a strictly pseudoconvex domain $N$ in $M$ such that $\bar{G} \cap \bar{C}_2 \subset \bar{N} \subset G \cup F$, $\partial(\bar{G} \cap \bar{C}_2) \setminus \partial G \subset N$, and $\partial N \cap \partial G \subset F$. Let $\sigma$ be a strictly plurisubharmonic defining function for $N$ in some neighbourhood $V$ of $\partial N$. We may assume that $V \subset C$. Then for some $\varepsilon > 0$ and for every $t$ with $0 \leqslant t < \varepsilon$, the sets $N_t = N \cup \{z \in V \mid \sigma(z) < t\}$ are strictly pseudoconvex and $\bar{N}_t \subset C$. It is well known that $(\bar{N})\hat{}_{\mathcal{O}(N_t)} = \bar{N}$ for $t > 0$. Choose any $t$ with $0 < t \leqslant \varepsilon$ and set $U = W \cap N_t$. Since by assumption $(\bar{C}_1)\hat{}_{\mathcal{O}(W)} = \bar{C}_1$, we have $(\bar{G} \cap \bar{C}_1)\hat{}_{\mathcal{O}(U)} = \bar{G} \cap \bar{C}_1$. The assertion now follows from [12], Corollary 5.2.9.

3. Proof of Theorem 1.1. As in [21], we consider separately the cases $q \geqslant 2$ and $q = 1$. Let us first assume that $q \geqslant 2$. Choose a sequence $\{D_i\}_{i=1}^{\infty}$ of strictly pseudoconvex domains in $M$ with the properties listed in Corollary 2.6. Set $N_i = D_{2i-1}$, $P_i = D_{2i}$, $i = 1, 2, \ldots$ We shall construct a sequence $\{u_i\}_{i=1}^{\infty}$ of differential forms such that

$$(3.1)\qquad u_i \in \mathscr{C}^{k+1/2}_{(0,q-1)}(\bar{P}_i) \cap \mathscr{C}^{\infty}_{(0,q-1)}(P_i), \quad \bar{\partial} u_i = f \text{ in } \bar{P}_i, \text{ and } u_{i+1}|_{\bar{N}_i} = u_i|_{\bar{N}_i}.$$

Suppose that $u_1, \ldots, u_m$ are constructed. By [17], Theorem 2, there exists $v \in \mathscr{C}^{k+1/2}_{(0,q-1)}(\bar{P}_{m+1}) \cap \mathscr{C}^{\infty}_{(0,q-1)}(P_{m+1})$ such that $\bar{\partial} v = f$ in $\overline{P_{m+1}}$. Then $\bar{\partial}(u_m - v)$

$= 0$ in $\bar{P}_m$. Therefore, again by [17], Theorem 2, there exists $w \in \mathscr{C}^{k+1/2}_{(0,q-2)}(\bar{P}_m) \cap \mathscr{C}^{\infty}_{(0,q-2)}(P_m)$ such that $\bar{\partial}w = u_m - v$. Choose a function $\chi \in \mathscr{C}^{\infty}(M)$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on $\bar{N}_m$, and $\chi = 0$ on $\bar{D} \setminus P_m$. Then the form $\chi w$, extended by zero to all of $\bar{D}$, is in $\mathscr{C}^{k+1/2}_{(0,q-2)}(\bar{D}) \cap \mathscr{C}^{\infty}_{(0,q-2)}(D)$, and $\bar{\partial}(\chi w) = (\bar{\partial}\chi) \wedge w + \chi(u_m - v) \in \mathscr{C}^{k+1/2}_{(0,q-1)}(\bar{D}) \cap \mathscr{C}^{\infty}_{(0,q-1)}(D)$. Set $u_{m+1} = v + \bar{\partial}(\chi w)$ on $\bar{P}_{m+1}$. Then $u_{m+1}$ satisfies (3.1).

The desired solution $u$ is defined by setting $u = u_m$ on $\bar{N}_m$.

The case $q = 1$ is more complicated. Fix $\varepsilon$ with $0 < \varepsilon < 1/2$. By Lemma 3.4 there exists a domain of holomorphy $C$ in $M$ such that $D \cup E \subset C$ and $\partial D \setminus E \subset \partial G$. Choose an increasing sequence $\{K_i\}_{i=1}^{\infty}$ of connected open subsets of $D \cup E$ such that $\bigcup_{i=1}^{\infty} K_i = D \cup E$. Let $\{C_j\}_{j=1}^{\infty}$ be a sequence of strictly pseudoconvex and bounded subdomains of $C$ satisfying (2.2). Set $P_i = C_{3i-2}$, $R_i = C_{3i-1}$ and $S_i = C_{3i}$, $i = 1, 2, \ldots$ We may assume, choosing a subsequence of $\{C_j\}$ if necessary, that for every $i$, $K_i \subset S_i$.

Given $i \in N$, let $V$ be a neighbourhood of $\partial S_i \cap E$ in $M$ such that $V$ is a relatively compact subset of $P_{i+1}$ and $\bar{V}_i \cap \bar{R}_i = \emptyset$. By Lemma 2.3, there exists a strictly pseudoconvex domain $T_i$ such that $T_i \subset S_i \cup V$, $T_i \setminus V = S_i \setminus V$, and $\partial T_i$ intersects $E$ transversally. It is easy to see that then $D \cap T_i$ has only a finite number of connected components, say $T_1^{(i)}, \ldots, T_{n_i}^{(i)}$, and that $\overline{T_m^{(i)}} \cap \overline{T_n^{(i)}} = \emptyset$ for $m \neq n$. Since $K_i$ is connected, there exists an integer $l_i$, $1 \leq l_i \leq n_i$, such that $K_i \subset \overline{T_{l_i}^{(i)}}$. Set $L_m^{(i)} = \partial T_i \cap E \cap \overline{T_m^{(i)}}$, $m = 1, \ldots, n_i$. Let $X$ and $Y$ be neighbourhoods of $E$ and $\partial T_i$ in $M$ such that there exist strictly plurisubharmonic functions $\sigma \in \mathscr{C}^{\infty}(X)$ and $\tau \in \mathscr{C}^{\infty}(Y)$ defining for $D \cap X$ in $X$ in the sense of Lemma 2.1 and defining for $T_i$ in $Y$ in the sense of (1.2), respectively. Choose a neighbourhood $U$ of $L_{l_i}^{(i)}$ such that $U \subset V \cap X \cap Y$, $\bar{U} \cap K_i = \emptyset$, and $\bar{U} \cap \overline{T_m^{(i)}} = \emptyset$, $m = 1, \ldots, n_i$, $m \neq l_i$. Apply Lemma 2.2 to $U$, $A_1 =: D \cap U$ and $A_2 =: T_i \cap U$, and let $W$ and $A$ be as in (2.1). Then the set $B = (T_{l_i}^{(i)} \setminus W) \cup A$ is a domain of holomorphy in $M$ such that $K_i \subset B$ and every connected component of $B$ is strictly pseudoconvex. Let $N_i$ be the component of $B$ which contains $K_i$. Then the sets $N_i$ just constructed are strictly pseudoconvex, and since $K_i \subset K_{i+1}$ for $i = 1, 2, \ldots$, it follows that $N_1 \subset N_2 \subset \ldots$

Choose a sequence $\{\eta_i\}_{i=1}^{\infty}$ of positive real numbers such that $\varepsilon + \eta_i < 1/2$, $\eta_1 > \eta_2 > \ldots$, and $\lim_{n \to \infty} \eta_n = 0$. We shall construct a sequence of functions $\{u_i\}_{i=2}^{\infty}$ such that

$$
\text{(3.2)} \qquad \begin{aligned} &u_i \in \mathscr{C}^{k+\varepsilon+\eta_i}(\bar{N}_i) \cap \mathscr{C}^{\infty}(N_i), \quad \bar{\partial}u_i = f \text{ in } \bar{N}_i, \quad \text{and} \\ &\|u_i - u_{i-1}\|_{\bar{N}_{i-2}, k+\varepsilon+\eta_i} < 2^{-i}, \quad i = 3, 4, \ldots \end{aligned}
$$

[Here, given $t \in N$, $t > 0$, the norms $\| \ \|_{\bar{N}_i, t}$ are understood in the following sense: Let $\{U_j\}_{j=1}^{\infty}$ and $\{W_j\}_{j=1}^{\infty}$ be two coverings of $D \cup E$ by open sets such that every $U_j$ is a coordinate patch, $W_j$ is relatively compact in $U_j$, and such that for every $i \in N$, only finitely many sets $U_j$'s intersect $\bar{N}_i$. For $i = 1, 2, \ldots$, let $I_i = \{j \in N | \ W_j \cap \bar{N}_i \neq \emptyset\}$. Then, given $g \in \mathscr{C}^t(\bar{N}_i)$, set

$$(3.3)\qquad \|g\|_{\bar{N}_i,t} = \sum_{j \in I_i} \Bigg( \sum_{|\alpha|+|\beta| \leqslant [t]} \sup_{z \in \bar{W}_j \cap \bar{N}_i} |D^\alpha \bar{D}^\beta g(z)| + \\ + \sum_{|\alpha|+|\beta|=[t]} \sup_{z,z' \in \bar{W}_j \cap \bar{N}_i} \frac{|D^\alpha \bar{D}^\beta g(z) - D^\alpha \bar{D}^\beta g(z')|}{|z-z'|^{t-[t]}} \Bigg),$$

where, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}, \qquad \bar{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial \bar{z}_1^{\alpha_1} \ldots \partial \bar{z}_n^{\alpha_n}}, \qquad n = \dim_C M,$$

and all the derivatives are computed with respect to the local coordinates in $U_j$; in the case of $t \in N$, the second sum in brackets on the right-hand side of (3.3) is to be left out. Any other choice of the families $\{U_j\}$ and $\{W_j\}$ with the same properties yields equivalent norms in $\mathscr{C}^t(\bar{N}_i)$ for every $i$.]

Take for $u_2$ any solution of the equation $\bar{\partial} u = f$ in $\bar{N}_2$ such that $u_2 \in \mathscr{C}^{k+\varepsilon+\eta_2}(\bar{N}_2) \cap \mathscr{C}^\infty(N_2)$ (it exists by [17], Theorem 2). Suppose that $u_2, \ldots, u_m$ are constructed. By [17], there exists $v \in \mathscr{C}^{k+\varepsilon+\eta_m}(\bar{N}_{m+1}) \cap \mathscr{C}^\infty(N_{m+1})$ such that $\bar{\partial} v = f$. Then $u_m - v \in \mathcal{O}(N_m) \cap \mathscr{C}^{k+\varepsilon+\eta_m}(\bar{N}_m)$. There exists a neighbourhood $Z$ of $\bar{N}_m$ and a function $w \in \mathcal{O}(Z)$ such that

$$(3.4)\qquad \|w - (u_m - v)\|_{\bar{N}_m, k+\varepsilon+\eta_{m+1}} < 2^{-(m+1)}.$$

This can be shown, e.g. as follows: By Fornaess' embedding theorem ([5], Theorem 9), there exist a neighbourhood $U$ of $\bar{N}_m$ in $M$, an integer $p \geqslant n = \dim_C M$, a holomorphic mapping $\psi$: $U \to \psi(U) \subset C^p$, which maps $U$ biholomorphically onto a closed complex submanifold $\psi(U)$ of $C^p$, and a strictly convex domain $G \subset C^p$, such that $\psi(\bar{N}_m) \subset G$, $\psi(U \setminus \bar{N}_m) \subset C^p \setminus \bar{G}$ and $\psi(U)$ intersects $\partial G$ transversally. The function $h = (u_m - v) \circ \psi^{-1}$ is holomorphic on $\psi(N_m)$ and is in $\mathscr{C}^{k+\varepsilon+\eta_m}(\psi(\bar{N}_m))$. By [13], Theorem 1, there exists a function $H \in \mathcal{O}(G) \cap \mathscr{C}^{k+\varepsilon+\eta_m}(\bar{G})$ such that $H|_{\psi(N_m)} = h$. We may assume that $0 \in G$. Since $\eta_{m+1} < \eta_m$, it is easy to see that the functions $H_r(z) =: H(rz)$, $0 < r < 1$, tend to $H$ in the $\| \ \|_{\bar{G}, k+\varepsilon+\eta_{m+1}}$-norm, as $r \to 1$. Hence also $H_r \circ \psi \to u_m - v$ in $\| \ \|_{\bar{N}_m, k+\varepsilon+\eta_{m+1}}$-norm, and one can choose $w$ to be a convenient function of the form $H_r \circ \psi$.

Choose a neighbourhood $Q$ of $\bar{R}_m \cap E \cap \overline{T_{l_m}^{(m)}}$ in $M$ such that $Q \subset C \cap X \cap Z$ (recall that $X$ is a neighbourhood of $E$ in $M$ with a defining strictly plurisubharmonic function $\sigma$ for $X \cap D$), $Q \cap ((\bar{R}_m \cap \bar{D}) \setminus \overline{T_{l_m}^{(m)}}) = \emptyset$, and $Q \cap \partial D \subset \partial D \cap N_m$, and a function $\varphi \in \mathscr{C}^\infty(M)$ such that $\operatorname{supp} \varphi \subset Q$ and $\varphi > 0$ in a neighbourhood of $\bar{R}_m \cap E \cap \overline{T_{l_m}^{(m)}}$. Consider the functions $\sigma - \eta\varphi$, $\eta > 0$. We see that if $\eta$ is sufficiently close to 0, then the set $G = D \cup \{z \in X | \ (\sigma - \eta\varphi)(z) < 0\}$ is a domain of holomorphy in $M$ such that $D \subset G$, $D \setminus Q = G \setminus Q$, $\overline{G \setminus D} \subset Q$, and $\partial G \setminus (\partial D \setminus E)$ is smooth and strictly pseudoconvex at every point. Also, $\bar{G} \cap \bar{R}_m = K \cup L$, where $K = \bar{G} \cap \bar{R}_m \cap (N_m \cup Q)$ and $L = \bar{G} \cap \bar{R}_m \cap (\bar{T}_m \setminus \overline{T_{l_m}^{(m)}})$ are both compact in $M$ and disjoint. Let $U_1$ and $U_2$ be neighbourhoods of $K$ and $L$, respectively, such that $U_1 \subset Z$ and $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Define $\tilde{w} \in \mathcal{O}(U_1 \cup U_2)$ by $\tilde{w}|_{U_1} = w|_{U_1}$ and $\tilde{w}|_{U_2} = 0$. Now apply Lemma 2.7 to $G$, $F =: \partial G \setminus (\partial D \setminus E)$, $C$, $C_1 =: \bar{R}_m$, $C_2 =: \bar{P}_{m+2}$, and $U =: U_1 \cup U_2$. It follows that the function $\tilde{w}$ can be approximated uniformly on $\bar{G} \cap \bar{R}_m$ by functions, which are holomorphic in a neighbourhood of $\bar{G} \cap \bar{P}_{m+2}$. But, by construction, $\bar{N}_{m-1} \subset G \cap R_m$. It is well known that then the function $\tilde{w}$ can be approximated on $\bar{N}_{m-1}$ by functions holomorphic in a neighbourhood of $\bar{N}_{m-1}$ in any one of the norms $\| \ \|_{\bar{N}_{m-1}, s}$, $s > 0$; in particular, also in $\| \ \|_{\bar{N}_{m-1}, k+\varepsilon+\eta_{m+1}}$-norm. Moreover, $\bar{N}_{m+1} \subset \bar{P}_{m+2}$. It follows that there exists a function $t$ holomorphic in a neighbourhood of $\bar{N}_{m+1}$ such that

$$\|t - w\|_{\bar{N}_{m-1}, k+\varepsilon+\eta_{m+1}} < 2^{-(m+1)}. \tag{3.5}$$

Hence, by (3.4) and (3.5),

$$\|t - (u_m - v)\|_{\bar{N}_{m-1}, k+\varepsilon+\eta_{m+1}} \leqslant 2^{-m}.$$

Set $u_{m+1} = t + v$ on $\bar{N}_{m+1}$. Then $u_{m+1} \in \mathscr{C}^{k+\varepsilon+\eta_{m+1}}(\bar{N}_{m+1})$, $\bar{\partial} u_{m+1} = f$ on $\bar{N}_{m+1}$ and $\|u_{m+1} - u_m\|_{\bar{N}_{m-1}, k+\varepsilon+\eta_{m+1}} \leqslant 2^{-m}$, and so (3.2) is satisfied.

It follows that the sequence $\{u_i\}_{i=2}^\infty$ converges in the space $\mathscr{C}^{k+\varepsilon}(D \cup E)$ to a function $u \in \mathscr{C}^{k+\varepsilon}(D \cup E) \cap \mathscr{C}^\infty(D)$ such that $\bar{\partial} u = f$.

Remark. In the case of $k = \infty$, one should replace condition (3.2) by the requirement that $u_i \in \mathscr{C}^i(\bar{N}_i)$ and $\|u_i - u_{i-1}\|_{\bar{N}_{i-2}, i} \leqslant 2^{-i}$.

Note. Using the linearity and continuity of the solution operator for the $\bar{\partial}$-equation in [18], we conclude from the above proof that in the case $q \geqslant 2$, $\varepsilon$ can be replaced by 1/2, and the operator $f \mapsto u$, solving the $\bar{\partial}$-equation, can be chosen to be linear and continuous (in the natural topology of $\mathscr{C}^k_{(0,q)}(D \cup E)$, given by the norms $\| \ \|_{\bar{N}_i, k}$). On the other hand, one can prove, by use of (the proof of) Lemma 5.4 in [18], that if $D$ is a domain of holomorphy in a closed complex submanifold $M$ of $C^n$, $E$ is an open subset of $\partial D$ and if there exists a point $z_0 \in \partial D \setminus E$ such that $\partial D$ is smooth and strictly pseudoconvex in a neighbourhood of $z_0$, then there does not exist any linear and continuous operator $L$: $\mathscr{C}^k_{(0,1)}(D \cup E) \cap \mathscr{C}^\infty_{(0,1)}(D) \cap \ker \bar{\partial} \ni f \to Lf \in \mathscr{C}^\infty(D)$ satisfying $\bar{\partial} Lf = f$. (Hence, a fortiori, in such a case there exists no linear and continuous solution operator for the $\bar{\partial}$-equation from $\mathscr{C}^k_{(0,1)}(D \cup E) \cap \mathscr{C}^\infty_{(0,1)}(D) \cap \ker \bar{\partial}$ to $\mathscr{C}^{k+\varepsilon}(D \cup E) \cap \mathscr{C}^\infty(D)$.)

**4. Applications.** In this section we give some applications of Theorem 1.1. We first consider the problem of extension of holomorphic functions from complex submanifolds to domains of holomorphy, in such a way that the boundary regularity of functions after extension is preserved.

Given $k \in N \cup \{\infty\}$, a domain $D$ with $\mathscr{C}^1$ boundary in a complex submanifold $M$ of $C^n$ and an open subset $U$ of $\partial D$, set $A^k(D) = \mathcal{O}(D) \cap \mathscr{C}^k(\bar{D})$ and $A^k_U(D) = \mathcal{O}(D) \cap \mathscr{C}^k(D \cup U)$. We prove the following theorem, which is an extension of [13], Theorem 1 and [14], Theorem 4.2 and 5.2.

THEOREM 4.1. *Suppose that $D$ is a domain of holomorphy in $C^n$ and let $U$ be an open subset of $\partial D$ such that $\partial D$ is smooth and strictly pseudoconvex at all points of $U$. Let $D'$ be a domain in $C^n$ such that $D \cup U \subset D'$ and suppose that $M'$ is a complex closed submanifold of $D'$. Set $M = M' \cap D$, $E = \partial M \cap U$. Suppose that $U$ and $M'$ intersect transversally. Then for every function $f \in A^k_E(M)$ there exists $F \in A^k_U(D)$ such that $F|_M = f$.*

In order to prove this theorem we need the following result; its proof is based on the method used in [3], Theorem 4.

PROPOSITION 4.2. *Let $D$, $U$, $D'$, $M'$ and $M$ be as in the assumption of Theorem 4.1. Suppose that $f$ is a $\bar{\partial}$-closed differential form of type $(0, 1)$ with coefficients of class $\mathscr{C}^k(D \cup U)$ and smooth in $D$, such that $f|_M = 0$. Then, for every $\varepsilon$ with $0 \leqslant \varepsilon < 1/2$, there exists $u \in \mathscr{C}^{k+\varepsilon}(D \cup U)$ smooth in $D$ such that $\bar{\partial} u = f$ and $u|_M = 0$.*

Proof. Choose sequences $\{D_i\}_{i=1}^\infty$ and $\{U_i\}_{i=1}^\infty$ of domains in $C^n$ with the following properties: Every $U_i$ is strictly pseudoconvex, the family $\{U_i\}$ is locally finite, $\bigcup_{i=1}^\infty U_i \subset D'$, $\bigcup_{i=1}^\infty U_i \cap \partial D = U$, every $D_i$ is a domain of holomorphy in $C^n$, $D = D_0 \subset D_1 \subset D_2 \subset \ldots$, $\bar{D}_i \subset \bar{D} \cup \bigcup_{j=1}^i U_j$, $\partial D_i \setminus (\partial D \setminus U)$ is smooth and strictly pseudoconvex at every point, $\overline{D_i \setminus D_{i-1}} \subset U_i$, $D \cup U \subset \bigcup_{i=1}^\infty D_i$, $U_i \cap D_{i-1}$ is strictly pseudoconvex and $\partial(U_i \cap D_{i-1})$ intersects $M'$ transversally. (The existence of $\{D_i\}$ and $\{U_i\}$ can be proved by means of the "bumps lemma" [1] and the method used in the proof of Lemmas 2.3 and 2.4.)

Let $\varepsilon > 0$ be given. We will construct inductively sequences $\{f_i\}_{i=1}^{\infty}$ and $\{u_i\}_{i=1}^{\infty}$ such that:

(4.1) $f_i$ is a differential form of type $(0, 1)$ with coefficients in $\mathscr{C}^{k+\varepsilon}(D_i \cup (\partial D_i \setminus (\partial D \setminus U)))$, smooth in $D_i$, $f_i = f_{i-1}$ outside $U_i$ (where $f_0 = f$), $f_i|_{D_i \cap M'} = 0$, $\bar\partial f_i = 0$, $u_i \in \mathscr{C}^{k+\varepsilon}(D \cup U) \cap \mathscr{C}^{\infty}(D)$, $\operatorname{supp} u_i \subset U_i \cap \bar{D}$, $u_i|_M = 0$, and $f_{i-1} = f_i + \bar\partial u_i$ on $\overline{D_{i-1} \cap U_i}$.

By Theorem 1.1, there exists $v_1 \in \mathscr{C}^{k+\varepsilon}(D \cup U) \cap \mathscr{C}^{\infty}(D)$ such that $\bar\partial v_1 = f$; moreover, $\bar\partial v_1|_M = 0$. Thus $v_1$ is holomorphic on $U_1 \cap D \cap M'$ and of class $\mathscr{C}^{k+\varepsilon}(\overline{U_1 \cap D \cap M'})$. Since $U_1 \cap D$ intersects $M'$ transversally, it follows from [13], Theorem 1, that there exists $V_1 \in \mathcal{O}(D \cap U_1) \cap \mathscr{C}^{k+\varepsilon}(\overline{D \cap U_1})$ such that $V_1|_{M' \cap D \cap U_1} = v_1|_{M' \cap D \cap U_1}$. Let $w_1 = v_1|_{D \cap U_1} - V_1$. Then $w_1 \in \mathscr{C}^{k+\varepsilon}(\overline{D \cap U_1})$, $\bar\partial w_1 = f$ on $\overline{D \cap U_1}$, and $w_1 = 0$ on $D \cap U_1 \cap M'$. Choose $\eta_1 \in \mathscr{C}_0^{\infty}(U_1)$ such that $\eta_1 = 1$ in a neighbourhood of $\overline{D_1 \setminus D}$; further, define a function $f_1$ on $C_1 = D_1 \cup (\partial D_1 \setminus (\partial D \setminus U))$ by $f_1 = \bar\partial[(1-\eta_1) w_1]$ in $\overline{D \cap U_1}$, $f_1 = 0$ in $\overline{D_1 \setminus D}$, and $f_1 = f$ on $C_1 \setminus U_1$, and a function $u_1$ to be $\eta_1 w_1$ on $\overline{D \cap U_1}$ and zero outside this set. It is easily seen, by the assumptions on $\eta_1$ and the fact that $\bar\partial w_1 = f$ on $\overline{D \cap U_1}$, that these definitions are compatible and that $f_1 = \bar\partial w_1 - \bar\partial(\eta_1 w_1) = f - \bar\partial u_1$ on $\overline{D \cap U_1}$.

Having constructed $f_1, \ldots, f_m$ and $u_1, \ldots, u_m$, we construct $f_{m+1}$ and $u_{m+1}$ just as above, but now with respect to $D_m$, $\partial D_m \setminus (\partial D \setminus U)$, $U_{m+1}$, and $D_{m+1}$ in place of $D$, $U$, $U_1$ and $D_1$, respectively.

Then, by (4.1), for every $m \in N$ we have $f = \bar\partial(u_1 + u_2 + \ldots + u_m) + f_m$. Since the family $\{U_i\}$ is locally finite, it follows from (4.1) that the sequences $\{u_1 + \ldots + u_m\}_{m=1}^{\infty}$ and $\{f_m\}_{m=1}^{\infty}$ converge respectively to $t \in \mathscr{C}^{k+\varepsilon}(D \cup U) \cap \mathscr{C}^{\infty}(D)$ such that $t|_M = 0$ and to $g \in \mathscr{C}^{\infty}_{(0,1)}(C)$ (where $C = \bigcup_{i=1}^{\infty} D_i$) such that $g|_{C \cap M'} = 0$ and $\bar\partial g = 0$, and moreover, $f = \bar\partial t + g$ on $D$. Since $C$ is a domain of holomorphy, there exists, by [12], Corollary 4.2.6, a function $s \in \mathscr{C}^{\infty}(C)$ such that $\bar\partial s = g$. But $(\bar\partial s)|_{M' \cap C} = 0$; therefore $s|_{M' \cap C}$ is holomorphic. By the Oka–Cartan theorem, there exists $S \in \mathcal{O}(C)$ such that $S|_{M' \cap C} = s|_{M' \cap C}$. Then $\bar\partial(s - S) = g$, $s - S \in \mathscr{C}^{\infty}(C)$ and $s - S|_{M' \cap C} = 0$. Hence $f = \bar\partial u$, where $u = t + (s - S)|_{D \cup U} \in \mathscr{C}^{k+\varepsilon}(D \cup U) \cap \mathscr{C}^{\infty}(D)$ and $u|_M = 0$.

Having proved Proposition 4.2, we return to the proof of Theorem 4.1. Choose the family $\{U_i\}_{i=1}^{\infty}$ of domains in $C^n$ such that:

(4.2) For every $j$, $U_j \subset D$ is a strictly pseudoconvex domain such that either $\bar{U}_j \cap \bar{M} = \emptyset$ or $\partial U_j$ intersects $M'$ transversally, $\partial U_j \cap \partial D \subset U$ and $\bigcup_{j=1}^{\infty} \operatorname{int}(\partial U_j \cap U) = U$ (where the interior is taken with respect to $U$).

Let $i \in N$ and suppose that $\bar{U}_i \cap \bar{M} \neq \emptyset$. Since $\partial U_i$ intersects $M'$ transversally, there exists, by [13], Theorem 1, a function $g_i \in A^k(U_i)$ such that $g_i|_{U_i \cap M} = f|_{U_i \cap M}$. If $\bar{U}_i \cap \bar{M} = \emptyset$, we set $g_i = 0$. Restricting the domains $U_i$ if necessary we obtain a family $\{V_i\}_{i=1}^{\infty}$ of strictly pseudoconvex domains in $C^n$ such that $V_i \subset U_i$, $\partial V_i \cap \partial U_i \subset U$, each $\partial U_i \cap U$ is a neighbourhood in $U$ of $\partial V_i \cap U$, and $\bigcup_{i=1}^{\infty} \operatorname{int}(\partial V_i \cap U) = U$, where the interior is taken with respect to $U$, as before. Then the function $h_i = g_i|_{\bar{V}_i}$ is in $A^k(V_i)$, and moreover, $h_i \in \mathscr{C}^{\infty}(V_i \cap (\partial V_i \setminus U))$.

Since $\bigcup_{i=1}^{\infty} \bar{V}_i$ is a neighbourhood of $U$ in $D \cup U$, one can construct (by a similar method to that applied in the proof of Lemma 2.4) a domain of holomorphy $V_0$ such that $V_0 \subset D$; $D \setminus \bigcup_{i=1}^{\infty} V_i \subset V_0$, $\partial V_0 \cap U = \emptyset$, and $\partial V_0 \cap \bigcup_{i=1}^{\infty} V_i$ is smooth. By the Oka–Cartan theorem, there exists $g \in \mathcal{O}(D)$ such that $g|_M = f$. Set $h_0 = g|_{V_0 \cap (\partial V_0 \setminus \partial D)}$. Then $h_0$ is holomorphic in $V_0$ and smooth in $V_0 \cup (\partial V_0 \setminus \partial D)$.

Given $i, j = 0, 1, \ldots$, set $h_{ij} = h_j - h_i$ on $V_i \cap V_j$. Then, for every $i, j, l \in N$, $h_{ij}$ belongs to $A^k(V_i \cap V_j)$ and is smooth on the set $(V_i \cap V_j) \cup (\partial(V_i \cap V_j) \cap D)$, $h_{ij} = -h_{ji}$, and $h_{ij} + h_{jl} + h_{li} = 0$ on $V_i \cap V_j \cap V_l$. Choose a partition of unity $\{\varphi_s\}_{s=1}^{\infty}$ on $D \cup U$ subordinate to the covering $\{V_i \cap (\partial V_i \cap U)\}_{i=0}^{\infty}$ of $D \cup U$. We claim that there exist functions $f_i \in A^k(V_i) \cap \mathscr{C}^{\infty}(V_i \cup (\partial V_i \cap D))$, $i = 1, 2, \ldots$, and $f_0 \in A^{\infty}_{\partial V_0 \cap \partial D}(V_0)$, such that

$$f_i|_{V_i \cap M'} = 0 \quad \text{and} \quad f_i - f_j = h_{ij}. \tag{4.3}$$

Given $s \in N$, let $i_s \in N$ be such that $\operatorname{supp} \varphi_s \subset V_{i_s} \cup (\partial V_{i_s} \cap U)$. Let $u_l = -\sum_{s=1}^{\infty} \varphi_s h_{i_s l}$ (where $\varphi_s h_{i_s l}$ is defined to be zero outside $\operatorname{supp} \varphi_s$), $l = 0, 1, \ldots$ Then $u_l \in \mathscr{C}^k(\bar{V}_l) \cap \mathscr{C}^{\infty}(V_l \cup (\partial V_l \cap D))$, $l = 1, 2, \ldots$, $u_0 \in \mathscr{C}^{\infty}(V_0 \cup (\partial V_0 \setminus \partial D))$, $u_l|_{V_l \cap M'} = 0$, $l = 0, 1, \ldots$, and $u_l - u_j = h_{lj}$ on $V_l \cap V_j$. We are going to improve the functions $u_l$ to be holomorphic. Since $\bar{\partial}(u_l - u_j) = \bar{\partial} h_{lj} = 0$ on $V_l \cap V_j$, the (0, 1)-form $G$ defined as $\bar{\partial} u_l$ on $\bar{V}_l$ is well-defined and has coefficients of class $\mathscr{C}^k(D \cup U)$ and smooth in $D$; further, $G$ is $\bar{\partial}$-closed, and we have $G|_M = 0$. By Proposition 4.2, there exists $u \in \mathscr{C}^k(D \cup U) \cap \mathscr{C}^{\infty}(D)$ such that $\bar{\partial} u = G$ and $u|_M = 0$. Set $f_i = u_i - u|_{\bar{V}_i \setminus (\partial D \setminus U)}$. Then the functions $f_i$ so constructed satisfy (4.3). Now define $F$ by $F = h_i + f_i$ on $\bar{V}_i \setminus (\partial D \setminus U)$. Since $h_i - h_j = h_{ij} = f_j - f_i$ on $V_i \cap V_j$, it follows that $F$ is well-defined and $F \in A^k_U(D)$ because $h_i$ and $f_i$ are in $A^k(V_i) \cap \mathscr{C}^{\infty}(V_i \cup (\partial V_i \cap D))$, $i = 1, 2, \ldots$, and $h_0$ and

$f_0$ are in $A^{\infty}_{\partial V_0 \setminus \partial D}(V_0)$. Moreover, $F|_{M' \cap V_i} = h_i|_{M' \cap V_i} = f|_{M' \cap V_i}$, since $f_i|_{M' \cap V_i} = 0$. Hence $F|_{M'} = f$. This ends the proof.

Note. In the case of the spaces $A^{\infty}_E(M)$ and $A^{\infty}_U(D)$, Theorem 4.1 is valid under a weaker assumption on the intersection of $U$ and $M'$.

THEOREM 4.3. *Let $D$, $U$ and $D'$ be as in the assumption of Theorem* 4.1. *Let $M'$ be a complex closed submanifold of $D'$ such that $U$ and $M'$ are regularly separated (i.e., for every $z_0 \in M' \cap U$ there exist a neighbourhood $W$ of $z_0$ in $C^n$ and constants $k \in N$ and $c > 0$ such that* $\operatorname{dist}(z, W \cap U \cap M')^k \leqslant c \operatorname{dist}(z, W \cap M')$ *for every $z \in W \cap U$). Set $M = M' \cap D$, $E = M \cap U$. Then for every $f \in A^{\infty}_E(M)$ there exists $F \in A^{\infty}_U(D)$ such that $F|_M = f$.*

The proof follows the line of the proof of Theorem 4.1; therefore we only point out the necessary modifications. In the proof of Proposition 4.2, the condition "$\partial(U_i \cap D_{i-1})$ intersects $M'$ transversally" should be replaced by "$\partial(U_i \cap D_{i-1})$ and $M'$ are regularly separated", and to obtain the extension $V_i \in A^{\infty}(\overline{D_{i-1} \cap U_i})$ of $v_i|_{\overline{M' \cap D_{i-1} \cap U_i}} \in A^{\infty}(M' \cap D_{i-1} \cap U_i)$ in the inductive procedure from Proposition 4.2, we use [1], Theorem 1 instead of [13], Theorem 1. Similarly, in (4.2), one should replace the condition "$\partial U_i$ intersects $M'$ transversally", and the reference [13], Theorem 1 (used in finding $g_i \in A^{\infty}(U_i)$ such that $g_i|_{U_i \cap M} = f$) by the regular separability of $\partial U_i$ and $M'$, and [1], Theorem 1, respectively.

Note. Given $k = 1, 2, \ldots$, or a positive real number $t$, $t \neq 1, 2, \ldots$, and a domain $D \subset C^n$ with $\mathscr{C}^1$ boundary, let $L^{\infty,k}(D) = \{f \in \mathscr{C}^{k-1}(\bar{D})|$ all derivatives of $f$ of order $k-1$ are bounded in $D\}$, $H^{\infty,k}(D) = L^{\infty,k}(D) \cap \mathscr{O}(D)$, and $\Lambda_t(D) = \mathscr{C}^t(\bar{D}) \cap \mathscr{O}(D)$. If $D$ is a domain in $C^n$ and $U$ is an open subset of $\partial D$ such that $\partial D$ is $\mathscr{C}^1$ at all points of $U$, we say that $f \in L^{\infty,k}(D \cup U)$ (resp. $f \in H^{\infty,k}_U(D)$ or $f \in (\Lambda_t)_U(D)$) iff, for every domain $G \subset D$ such that $\partial G$ is $\mathscr{C}^1$ and $\partial G \cap \partial D \subset U$, $f \in L^{\infty,k}(G)$ (resp. $f \in H^{\infty,k}(G)$ or $f \in \Lambda_t(G)$). The notions of the spaces $L^{\infty,k}(D \cup U)$, $H^{\infty,k}_U(D)$ and $(\Lambda_t)_U(D)$, similarly to those of $\mathscr{C}^t(D \cup U)$ and $A^k_U(D)$, can also be defined for domains $D$ with a $\mathscr{C}^1$-open part $U$ of the boundary on complex submanifolds of $C^n$.

Claim. Theorem 4.1 is valid also if the spaces $A^k_E(M)$ and $A^k_U(D)$ are replaced respectively by $H^{\infty,k}_E(M)$ and $H^{\infty,k}_U(D)$, or $(\Lambda_t)_E(M)$ and $(\Lambda_t)_U(D)$.

This claim can be proved similarly to Theorem 4.1. It follows from [16], corollary 4.7.1, that if $D$ is a strictly pseudoconvex domain in $C^n$ with a smooth boundary then, given a $\bar{\partial}$-closed (0, 1)-form $f$ with coefficients in $\mathscr{C}^t(\bar{D})$ and smooth in $D$, there exists $u \in \mathscr{C}^{t+\varepsilon}(\bar{D}) \cap \mathscr{C}^{\infty}(D)$, $\varepsilon < 1/2$ arbitrary, such that $\bar{\partial} u = f$. The method given in [15] allows to prove a similar assertion for strictly pseudoconvex smoothly bounded domains in a closed complex submanifold $M$ of $C^n$. (A similar result for $f \in L^{\infty,k}_{(0,1)}(\bar{D}) \cap \mathscr{C}^{\infty}_{(0,1)}(D)$ with $u \in \mathscr{C}^{k+1/2}(\bar{D}) \cap \mathscr{C}^{\infty}(D)$ satisfying $\bar{\partial} u = f$, $k = 1, 2, \ldots$, can be found already in [17].) Applying the proof of Theorem 1.1, we obtain (in the notation as in the assumption of Theorem 4.1) that

(4.4) for every $\varepsilon < 1/2$ and every $f \in L^{\infty,k}_{(0,1)}(D \cup E) \cap \mathscr{C}^{\infty}_{(0,1)}(D)$ (resp. $f \in \mathscr{C}^{l}_{(0,1)}(D \cup E) \cap \mathscr{C}^{\infty}_{(0,1)}(D)$) with $\bar{\partial} f = 0$ there exists $u \in \mathscr{C}^{k+\varepsilon}(D \cup E) \cap \mathscr{C}^{\infty}(D)$ (resp. $u \in \mathscr{C}^{l+\varepsilon}(D \cup E) \cap \mathscr{C}^{\infty}(D)$) such that $\bar{\partial} u = f$.

Then the proof of Proposition 4.2 and of Theorem 4.1 as well apply without a change.

Theorem 4.1 can be applied to the approximation of holomorphic functions in pseudoconvex domains. Suppose that $D$ is a domain in a complex closed submanifold $M$ of $C^n$ and $E$ is an open subset of $\partial D$ such that $\partial D$ is $\mathscr{C}^1$ at every point of $E$, and let $k \in N$ be fixed. Equipe $A^k_E(D)$ with the Fréchet space topology defined by the family of norms $\| \ \|_{\bar{D}_i,k}$ (see (3.3)), where $\{D_i\}_{i=1}^{\infty}$ is any sequence of subdomains of $D$ such that $\partial D_i$ are $\mathscr{C}^1$, $D_1 \subset D_2 \subset \ldots$, $\bigcup_{i=1}^{\infty} D_i = D$ and $\bigcup_{i=1}^{\infty} \bar{D}_i = D \cup E$. (Any other choice of the sequence $\{D_i\}$ and of the coverings $\{U_j\}$ and $\{V_j\}$ in (3.3) gives the same topology.)

THEOREM 4.4. *Let $D$ be a strictly pseudoconvex domain in a closed complex submanifold $M$ of $C^n$ and let $E$ be an open subset of $\partial D$. Fix $k \in N$. Then every $f \in A^k_E(D)$ can be approximated in the topology of the space $A^k_E(D)$ by holomorphic functions defined in a neighbourhood of $\bar{D}$ in $M$.*

Proof. By Fornaess' embedding theorem ([5], Theorem 9), there exist a Stein neighbourhood $N$ of $\bar{D}$ in $M$, a non-negative integer $m \geqslant \dim_C M$, a holomorphic mapping $h\colon N \to h(N) \subset C^n$, which maps $N$ biholomorphically onto a closed complex submanifold $h(N)$ of $C^n$, and a strictly convex domain $C \subset C^n$, such that $h(D) \subset C$, $h(N \setminus \bar{D}) \subset C^n \setminus \bar{C}$ and $h(N)$ intersects $\partial C$ transversally. Let $f \in A^k_E(D)$. Then $g = f \circ h^{-1} \in A^k_{h(E)}(h(D))$. By Theorem 4.1, there exists $G \in A^k_{\partial C \setminus K}(C)$ (where $K = h(\partial D \setminus E)$) such that $G|_{h(D)} = g$. We may assume that $0 \in G$. Then the functions $G_r(z) = G(rz)$, $r < 1$, tend to $G$ in the topology of the space $A^k_{\partial C \setminus K}(C)$, as $r \nearrow 1$. It follows that the functions $f_r = G_r \circ h$ are defined in a neighbourhood of $\bar{D}$ in $M$ and tend to $f$ in the topology of the space $A^k_E(D)$ as $r \nearrow 1$.

Now, we show an application of Theorem 1.1 to the proof of some results on the decomposition of certain ideals in algebras of holomorphic functions.

Bonneau Cumenge and Zériahi have proved the following division theorem.

THEOREM ([4], Theorem 1). *Let $D$ be a strictly pseudoconvex domain in $C^n$ with a boundary of class $\mathscr{C}^l$, $l \geqslant 2$, given by $D = (D \setminus V) \cup \{z \in V \mid \sigma(z) < 0\}$, where $\sigma$ is a strictly plurisubharmonic function of class $\mathscr{C}^l$ in some neighbour-*

*hood $V$ of $\partial D$, with a non-vanishing gradient. Let $M'$ be the submanifold of $D \cup V$ defined by $M' = \{z \in D \cup V \mid g_1(z) = \ldots = g_N(z) = 0\}$, where $g_1, \ldots, g_N$ are of class $\mathscr{C}^m(D \cup V)$, $m \geq 1$, and are holomorphic in $D$. Suppose that $\partial\sigma(z) \wedge \partial g_1(z) \wedge \ldots \wedge \partial g_N(z) \neq 0$ for $z \in M' \cap \partial D$. Set $M = M' \cap D$. Let $r = \min\{l-2, m-1\}$. Then, for every $t$ with $[t]+1 \leq r$ and $f \in \Lambda_t(D)$ such that $f|_M = 0$, there exist functions $f_1, \ldots, f_N \in \Lambda_{t-1/2}(D)$ such that $f = \sum_{i=1}^{N} g_i f_i$.*

We prove here the following "local" version of the above theorem, but with more restrictive conditions on $\partial D$ and $M'$, and with a weaker assertion.

THEOREM 4.5. *Let $D$ be a domain of holomorphy in $\boldsymbol{C}^n$ and let $U$ be an open subset of $\partial D$ such that $\partial D$ is smooth and strictly pseudoconvex at all points of $U$. Let $D'$ be a domain in $\boldsymbol{C}^n$ such that $D \cup U \subset D'$ and suppose that $M'$ is a complex submanifold of $D'$. Set $M = M' \cap D$. Suppose that $M'$ intersects $U$ transversally. Assume that $g_1, \ldots, g_N \in \mathcal{O}(D')$ are such that for every $z \in D'$ the germs of $g_1, \ldots, g_N$ at $z$ generate the ideal of germs at $z$ of holomorphic functions vanishing on $M'$. Then for every function $f \in (\Lambda_t)_U(D)$, $t \in \boldsymbol{R}$, $t > 1/2$, such that $f|_M = 0$ and for every $\eta$ with $1/2 < \eta < t$ there exist functions $f_1, \ldots, f_N \in (\Lambda_{t-\eta})_U(D)$ such that $f = \sum_{i=1}^{N} g_i f_i$.*

As in the proof of Theorem 4.1, we first prove an auxiliary result on the solution of the $\bar{\partial}$-equation.

PROPOSITION 4.6. *Let $D$, $U$, $D'$, $M'$, $M$ and $g_1, \ldots, g_N$ be as in the assumption of Theorem 4.5. Let $t > 0$ be given and suppose that $f_1, \ldots, f_N$ are $\bar{\partial}$-closed differential forms of type $(0, 1)$ with coefficients of class $\mathscr{C}^t(D \cup U)$ and smooth in $D$, such that $\sum_{l=1}^{N} g_l f_l = 0$. Then for every $r$ with $0 < r < t$ there exist functions $u_1, \ldots, u_N \in \mathscr{C}^{t-r}(D \cup U)$ smooth in $D$, such that $\bar{\partial} u_l = f_l$, $l = 1, \ldots, N$, and $\sum_{l=1}^{N} g_l u_l = 0$.*

Proof. We construct sequences $\{D_i\}_{i=1}^{\infty}$ and $\{U_i\}_{i=1}^{\infty}$ of domains in $\boldsymbol{C}^n$ with properties listed at the begining of the proof of Proposition 4.2. Moreover, the domains $D_i$ and $U_i$ can be chosen so that, for every $i = 1, 2, \ldots$, either the (strictly pseudoconvex) set $\overline{U_i \cap D_{i-1}}$ does not intersect $M'$ (here $D_0 = D$) or

(4.5) there exists a neighbourhood $V_i$ of $\overline{U_i \cap D_{i-1}}$ in $\boldsymbol{C}^n$, which is a domain of holomorphy, and functions $h_{1,i}, \ldots, h_{k,i} \in \mathcal{O}(V_i)$ (where $k = \dim_{\boldsymbol{C}} M'$), such that $M' \cap V_i = \{z \in V_i \mid h_{1,i}(z) = \ldots = h_{k,i}(z) = 0\}$ and $\partial\sigma_i(z) \wedge \partial h_{1,i}(z) \wedge \ldots \wedge \partial h_{k,i}(z) \neq 0$ for $z \in M' \cap \partial(U_i \cap D_{i-1})$, where $\sigma_i$ is a defining strictly plurisubharmonic function for $\partial(U_i \cap D_{i-1})$.

Choose a strictly increasing sequence of positive numbers $\{r_i\}_{i=1}^{\infty}$ such that $r_i \nearrow r$. As in the proof of Proposition 4.2, we will construct inductively sequences $\{f_{l,i}\}_{i=1}^{\infty}$ and $\{u_{l,i}\}_{i=1}^{\infty}$, $l = 1, \ldots, N$, such that

(4.6) $f_{l,i}$ is a differential form of type (0, 1) with coefficients in $\mathscr{C}^{t-r_i}(D_i \cup (\partial D_i \setminus (\partial D \setminus U)))$ and smooth in $D_i$, $f_{l,i} = f_{l,i-1}$ outside $U_i$ (where $f_{l,0} = f_l$), $\sum_{l=1}^{N} g_l f_{l,i} = 0$, $\bar{\partial} f_{l,i} = 0$, $u_{l,i} \in \mathscr{C}^{t-r_i}(D \cup U) \cap \mathscr{C}^{\infty}(D)$, $\operatorname{supp} u_{l,i} \subset U_i \cap \bar{D}$, $\sum_{l=1}^{N} g_l u_{l,i} = 0$, and $f_{l,i-1} = f_{l,i} + \bar{\partial} u_{l,i}$ on $\overline{D_{i-1} \cap U_i}$, $l = 1, \ldots, N$, $i = 1, 2, \ldots$

By Theorem 1.1 and (4.4) there exist the functions $v_{1,1}, \ldots$ $\ldots, v_{N,1} \in \mathscr{C}^{t+1/2-r_1}(D \cup U) \cap \mathscr{C}^{\infty}(D)$ such that $\bar{\partial} v_{l,1} = f_l$, $l = 1, \ldots, N$. Since $\bar{\partial}(\sum_{l=1}^{N} g_l v_{l,1}) = \sum_{l=1}^{N} g_l \bar{\partial} v_{l,1} = \sum_{l=1}^{N} g_l f_l = 0$, the function $F_1 = \sum_{l=1}^{N} g_l v_{l,1}$ is in $A_{t+1/2-r_1}(D \cup U)$, and moreover, $F_1|_{M'} = 0$. By the theorem of Bonneau, Cumenge and Zériahi and (4.5), there exist the functions $F_{1,1}, \ldots, F_{k,1} \in A_{t-r_1}(U_1 \cap D)$ such that $F_1 = \sum_{s=1}^{k} h_{s,1} F_{s,1}$ on $\overline{U_1 \cap D}$. Since every $h_{s,1}$ vanishes on $V_1 \cap M'$ and $V_1$ is a domain of holomorphy, it follows from the assumptions imposed on the functions $g_l$, $l = 1, \ldots, N$, and from [12], Theorem 7.2.9, that there exist functions $G_{l,s,1} \in \mathscr{O}(V_1)$, $l = 1, \ldots, N$, such that $h_{s,1} = \sum_{l=1}^{N} g_l G_{l,s,1}$ in $V_1$, $s = 1, \ldots, k$. Then $F_1|_{\overline{U_1 \cap D}} = \sum_{l=1}^{N} g_l H_{l,1}$ with $H_{l,1} = \sum_{s=1}^{k} G_{l,s} F_{s,1} \in A_{t-r_1}(U_1 \cap D)$. Set $w_{l,1} = v_{l,1} - H_{l,1}$, $l = 1, \ldots, N$.

Then $w_{l,1} \in \mathscr{C}^{t-r_1}(D \cap U_1)$, $\bar{\partial} w_{l,1} = f_l$, and $\sum_{l=1}^{N} g_l w_{l,1} = 0$. Choose $\eta_1 \in \mathscr{C}_0^{\infty}(U_1)$ such that $\eta_1 = 1$ in a neighbourhood of $\overline{D_1 \setminus D}$ and define $f_{l,1}$ on $C_1 = D_1 \cup (\partial D_1 \setminus (\partial D \setminus U))$ as follows: $f_{l,1} = \bar{\partial}[(1-\eta_1) w_{l,1}]$ in $\overline{D \cap U_1}$, $f_{l,1} = 0$ in $\overline{D_1 \setminus D}$, and $f_{l,1} = f_l$ on $C_1 \setminus U_1$; further, let $u_{l,1}$ be defined as $\eta_1 w_{l,1}$ on $\overline{D \cap U_1}$, and zero outside this set. Then, as in the proof of Proposition 4.2, one can check that these definitions are correct and that (4.6) is satisfied.

Having constructed $f_{l,1}, \ldots, f_{l,m}$ and $u_{l,1}, \ldots, u_{l,m}$, $l = 1, \ldots, N$, we construct $f_{l,m+1}$ and $u_{l,m+1}$ in a similar way as above. Namely, by Theorem 1.1, (4.4) and by the inductive hypothesis, there exist the functions $v_{l,m+1} \in \mathscr{C}^{t+1/2-r_{m+1}}(D_m \cup (\partial D_m \setminus (\partial D \setminus U))) \cap \mathscr{C}^{\infty}(D_m)$ such that $\bar{\partial} v_{l,m+1} = f_{m,l}$ for

$l = 1, \ldots, N$. The function $F_{m+1} = \sum_{l=1}^{N} g_l v_{l,m+1}$ is in

$$A_{t+1/2-r_{m+1}}\big(D_m \cup (\partial D_m \setminus (\partial D \setminus U))\big), \quad \text{and} \quad F_{m+1}|_{M'} = 0.$$

By the theorem of Bonneau, Cumenge and Zériahi, (4.5), and [12], Theorem 7.2.9, we can find functions $H_{l,m+1} \in A_{t-r_{m+1}}(U_{m+1} \cap D_m)$ such that $F_{m+1}|_{\overline{U_{m+1} \cap D_m}} = \sum_{l=1}^{N} g_l H_{l,m+1}$. Then the functions $w_{l,m+1} = v_{l,m+1} - H_{l,m+1} \in \mathscr{C}^{t-r_{m+1}}(U_{m+1} \cap D_m)$, $\bar\partial w_{l,m+1} = f_{m,l}$ and $\sum_{l=1}^{N} g_l w_{l,m+1} = 0$. Choose $\eta_{m+1} \in \mathscr{C}_0^{\infty}(U_{m+1})$ so that $\eta_{m+1} = 1$ in a neighbourhood of $\overline{D_{m+1} \setminus D_m}$, and define $f_{l,m+1}$ on $C_{m+1} := D_{m+1} \cup \big(\partial D_{m+1} \setminus (\partial D \setminus U)\big)$ as follows: $f_{l,m+1} \doteq \bar\partial[(1-\eta_{m+1}) w_{l,m+1}]$ in $\overline{U_{m+1} \cap D_m}$, $f_{l,m+1} = 0$ in $\overline{D_{m+1} \setminus D_m}$, and $f_{l,m+1} = f_{l,m}$ on $C_{m+1} \setminus U_{m+1}$; set $u_{l,m+1} = \eta_{m+1} w_{l,m+1}$ on $D \cap U_{m+1}$ and zero outside this set. As before, the definitions of $f_{l,m+1}$ and $u_{l,m+1}$ are compatible and (4.6) is satisfied.

It follows from (4.6) that for every $m \in N$ the equalities $f_l = \bar\partial(u_{l,1} + \ldots + u_{l,m}) + f_{l,m}$, $l = 1, \ldots, N$, hold. As in the proof of Proposition 4.2, the sequences $\{u_{l,1} + \ldots + u_{l,m}\}_{m=1}^{\infty}$ and $\{f_{l,m}\}_{m=1}^{\infty}$ converge to $v_l \in \mathscr{C}^{t-r}(D \cup U) \cap \mathscr{C}^{\infty}(D)$ so that $\sum_{l=1}^{N} g_l v_l = 0$, and to $h_l \in \mathscr{C}_{0,1}^{\infty}(C)$ (where $C = \bigcup_{i=1}^{\infty} D_i$) with $\sum_{l=1}^{N} g_l h_l = 0$ and $\bar\partial h_l = 0$, respectively, and moreover, $f_l = \bar\partial v_l + h_l$ on $D$, $l = 1, \ldots, N$. Since $C$ is a domain of holomorphy, there exist, by [12], Corollary 4.2.6, functions $s_1, \ldots, s_N \in \mathscr{C}^{\infty}(C)$ such that $\bar\partial s_l = h_l$, $l = 1, \ldots, N$. Since $\bar\partial(\sum_{l=1}^{N} g_l s_l) = \sum_{l=1}^{N} g_l h_l = 0$, i.e. the function $S = \sum_{l=1}^{N} g_l s_l$ is holomorphic in $C$, and since $S|_{M' \cap C} = 0$, it follows from the assumption on the functions $g_l$ and from [12], Theorem 7.2.9, that there exist functions $S_1, \ldots, S_N \in \mathscr{O}(C)$ such that $S = \sum_{l=1}^{N} g_l S_l$. Then the functions $s_l - S_l$ are smooth in $C$, $\bar\partial(s_l - S_l) = h_l$, and $\sum_{l=1}^{N} g_l(s_l - S_l) = 0$.

Set $u_l = v_l + (s_l - S_l)$. Then $u_l \in \mathscr{C}^{t-r}(D \cup U) \cap \mathscr{C}^{\infty}(U)$, $\bar\partial u_l = \bar\partial v_l + h_l = f_l$ and $\sum_{l=1}^{N} g_l u_l = \sum_{l=1}^{N} g_l v_l + \sum_{l=1}^{N} g_l(s_l - S_l) = 0$. This ends the proof.

Proof of Theorem 4.5. Choose a sequence $\{U_j\}_{j=1}^{\infty}$ of strictly pseudoconvex domains in $C^n$ such that $\bigcup_{j=1}^{\infty} \bar U_j = D \cup U$ and, for every $j$, either

$\bar{U}_j \cap M' = \emptyset$ or $\partial U_j$ intersects $M'$ transversally and there exist a neighbourhood $V_j$ of $\bar{U}_j$ in $C^n$, which is a domain of holomorphy, and functions $h_{1,j}, \ldots, h_{k,j} \in \mathcal{O}(V_j)$ ($k = \dim_C M'$) such that $M' \cap V_j = \{z \in V_j | \; h_{1,j}(z) = \ldots \ldots = h_{k,j}(z) = 0\}$ and $\partial\sigma_j(z) \wedge \partial h_{1,j}(z) \wedge \ldots \wedge \partial h_{k,j}(z) \neq 0$ for $z \in M' \cap \partial U_j$, where $\sigma_j$ is a defining strictly plurisubharmonic function for $\partial U_j$. As in the proof of Proposition 4.6, it follows from the theorem of Bonneau, Cumenge and Zériahi, and from [12], Theorem 7.2.9, that there exist functions $p_{i,j} \in A_{t-1/2}(U_j)$, $i = 1, \ldots, N$, such that $f|_{\bar{U}_j} = \sum_{i=1}^{N} g_i p_{i,j}$. Let $p_i^{(kj)} = p_{i,j} - p_{i,k}$ on $\bar{U}_j \cap \bar{U}_k$, $j, k \in N$. Suppose that for every $j \in N$ one can choose $\{h_{i,j}\}_{i=1}^{N}$ such that

$$(4.7) \qquad h_{i,j} \in A_{t-\eta}(U_j), \quad \sum_{i=1}^{N} g_i h_{i,j} = 0, \quad \text{and} \quad h_{i,k} - h_{i,j} = p_i^{(kj)} \quad \text{on } U_j \cap U_k.$$

Then the functions $f_i$ defined by $f_i = p_{i,j} + h_{i,j}$ on $\bar{U}_j$ are well-defined, in $D \cup U$, $f_i \in (A_{t-\eta})_U(D)$ and $\sum_{i=1}^{N} g_i f_i = f$. Therefore, to end the proof, it suffices to find functions $h_{i,j}$ with properties listed in (4.7).

Let $\{\varphi_s\}_{s=1}^{\infty}$ be a partition of unity on $D \cup U$, subordinate to the covering $\{U_j \cup (\partial U_j \cap U)\}_{j=1}^{\infty}$ of $D \cup U$. Let $u_{i,k} = -\sum_{s=1}^{\infty} \varphi_s p_i^{(j_s k)}$ (where $\operatorname{supp} \varphi_s \subset U_{j_s} \cup (\partial U_{j_s} \cap U)$ and $\varphi_s p_i^{(j_s k)}$ is defined as zero outside $\operatorname{supp} \varphi_s$), $i = 1, \ldots, N$, $k \in N$. Then $u_{i,k} \in \mathscr{C}^{t-1/2}(\bar{U}_k) \cap \mathscr{C}^{\infty}(U_k \cup (\partial U_k \setminus U))$, $\sum_{i=1}^{N} g_i u_{i,k} = 0$ in $U_k$ and $u_{i,k} - u_{i,j} = p_i^{(kj)}$, but $u_{i,k}$ are not holomorphic, in general. However, $\bar{\partial}(u_{i,k} - u_{i,j}) = \bar{\partial} p_i^{(kj)} = 0$ on $U_k \cap U_j$, so, given $i = 1, \ldots, N$, the (0, 1)-form $G_i$ defined by $G_i = \bar{\partial} u_{i,k}$ in $\bar{U}_k$ is well-defined, $\bar{\partial}$-closed, and has coefficients of class $\mathscr{C}^{t-1/2}(D \cup U)$ and smooth in $D$; moreover, $\sum_{i=1}^{N} g_i G_i = \sum_{i=1}^{N} g_i \bar{\partial} u_{i,k} = \bar{\partial}(\sum_{i=1}^{N} g_i u_{i,k}) = 0$ in $U_k$, and so in all of $D$. Hence, by Proposition 4.6, there exist $u_1, \ldots, u_N \in \mathscr{C}^{t-\eta}(D \cup U) \cap \mathscr{C}^{\infty}(D)$ such that $\bar{\partial} u_i = G_i$, $i = 1, \ldots, N$, and $\sum_{i=1}^{N} g_i u_i = 0$. Set $h_{i,k} = u_{i,k} - u_i|_{\bar{U}_k}$, $k \in N$. Then $h_{i,k} \in A_{t-\eta}(U_k)$, $\sum_{i=1}^{N} g_i h_{i,k} = \sum_{i=1}^{N} g_i u_{i,k} - \sum_{i=1}^{N} g_i u_i = 0$ and $h_{i,k} - h_{i,j} = u_{i,k} - u_{i,j} = p_i^{(kj)}$ on $U_j \cap U_k$. Hence (4.7) is satisfied. This ends the proof.

Note. If $t = \infty$, the assertion of Theorem 4.5 (with $f$ and $f_1, \ldots, f_N \in A_U^{\infty}(D)$) holds also under the assumption that $M'$ and $U$ are only regularly separated (not necessarily transversal). This follows from the fact that, in this case, the domains $U_i$ in the proof of Proposition 4.6 and of Theorem 4.5 can be chosen so that $\partial U_j$ and $M'$ are regularly separated, and one can use the theorem of de Bartolomeis and Tomassini ([3], Theorem 6) instead of the theorem of Bonneau, Cumenge and Zériahi. This gives a "local" version of the theorem of de Bartolomeis and Tomassini.

Note. If $D$ is strictly pseudoconvex and bounded, and $U = \partial D$, then the assertion of Theorem 4.5 holds also with $\eta = 1/2$. This can be proved by the method used in the proof of the Main Theorem in [11].

Using Theorem 1.1 and the construction described by Øvrelid in [19], we can also prove the following decomposition theorem, which generalizes [14], Theorem 4.1.

THEOREM 4.7. *Suppose that $D$ is a domain of holomorphy on a complex closed submanifold $M$ of $C^n$. Let $E$ be an open subset of $\partial D$, such that $\partial D$ is smooth and strictly pseudoconvex at every point of $E$, and let $\{s_i\}_{i=1}^{\infty}$ be a sequence of points in $D$ without a cluster point in $D \cup E$. Let $\tilde{A}_E(M)$ denote any one of the spaces $A_E^k(M)$, $H_E^{\infty,k}(M)$ or $(\Lambda_t)_E(M)$, and let $g_1, \ldots, g_N \in \tilde{A}_E(M)$ be such that:*

(i) $\{z \in D \cup E \mid g_1(z) = \ldots = g_N(z) = 0\} = \{s_i\}_{i=1}^{\infty}$;

(ii) *for every $i \in N$, the germs of the functions $g_i$ at the point $s_i$ generate the ideal of germs at $s_i$ of holomorphic functions on $M$, vanishing at $s_i$.*

*Then for every $f \in \tilde{A}_E(M)$ such that $f(s_i) = 0$, $i = 1, 2, \ldots$, there exist functions $f_1, \ldots, f_N \in \tilde{A}_E(M)$ such that $f = \sum_{i=1}^{N} g_i f_i$.*

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