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Generic properties of infinite system of integral equations in Banach spaces

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Abstract. The present paper investigates generic properties of existence, uniqueness and convergence of successive approximations for infinite systems of integral equations in Banach spaces. A property is said to be generic on a metric space if the subset on which it fails of being true is of Baire first category. The study of generic properties of differential equations was started by Orlicz [13], where it was shown that the set of bounded continuous functions f: $[0, a] \times R^n \to R^n$ for which the equation $x' = f(t, x)$ does not have uniqueness of all solutions is a set of first category. An analogous result for hyperbolic equations was proved by Alexiewicz and Orlicz in [2]. More recently Lasota and Yorke [11], Vidossich [15] and De Blasi and Myjak [6] studied properties concerning existence, uniqueness and convergence of successive approximations for differential equations in Banach spaces. Other generic problems in connection with non-linear equations have been investigated in [3]-[5] and [7]-[9].

The paper is divided into three sections. Notation, definitions and basic lemmas are contained in Section 1. In Section 2 we prove that the set of all functions f for which approximate solutions of (2) do not converge is of first category in some metric function space M, introduced in Section 1. Section 3 contains a similar result for successive approximations.

1. Preliminaries. Assume that N is the set of positive integers, $J = [0, a]$ is a compact interval in R, and E_i is a Banach space with the norm $\|\cdot\|_i$ $(i = 1, 2, \ldots)$. We introduce the following denotations:

$E = E_1 \times E_2 \times \ldots$ – the Fréchet space of all infinite sequences $x = (x_i)$, $x_i \in E_i$ for $i = 1, 2, \ldots$, with the quasinorm

$$|x| = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x_i\|_i}{1+\|x_i\|_i};$$

$C_i = C(J, E_i)$ – the Banach space of all continuous functions u: $J \to E_i$ with the norm $\|u\|_{ic} = \sup\{\|u(t)\|_i: t \in J\}$;

$C = C(J, E)$ – the Fréchet space of all continuous functions u: $J \to E$ with the quasinorm $|u|_c = \sup\{|u(t)|: t \in J\}$;

$T = \{(t, s): 0 \leqslant s \leqslant t \leqslant a\}$; $D = T \times E$;

M – the set of all functions $f = (f_1, f_2, \ldots)$: $D \to E$ such that for any $i \in N$:

1° f_i is a continuous mapping of D into E_i;

2° there exists a constant m_i such that $\|f_i(t,s,x)\|_i \leq m_i$ for each $(t,s,x)\in D$;

3° there exists a real-valued function $(\tau,t,s)\to r_i(\tau,t,s)$ $(0\leq s\leq t \leq \tau\leq a)$ such that:

(i) for any fixed t,τ the function $s\to r_i(\tau,t,s)$ is L-integrable on $[0,t]$;

(ii) $\sup\{\|f_i(\tau,s,x)-f_i(t,s,x)\|_i\colon x\in E\}\leq r_i(\tau,t,s)$;

(iii) $\lim\limits_{\tau-t\to 0+}\int_0^t r_i(\tau,t,s)\,ds = 0$ for fixed t or τ.

L — the set of all $f\in M$ such that

4° there exists $m>0$ such that $\|f_i(t,s,x)\|_i\leq m$ for each $(t,s,x)\in D$, $i\in N$, and

5° for any $z\in E$ there exist a neighbourhood V of z and a constant $k>0$ such that $\|f_i(t,s,x)-f_i(t,s,y)\|_i\leq k|x-y|$ for all $(t,s)\in T$, $x,y\in V$ and $i\in N$.

For any $f,g\in M$ put

$$d(f,g) = \sup\{|f(t,s,x)-g(t,s,x)|\colon (t,s,x)\in D\}.$$

Then $\langle M,d\rangle$ is a complete metric space. Moreover, for any sequence (f^n) in M

$$(*)\qquad \lim_{n\to\infty} d(f^n,f) = 0 \Leftrightarrow \lim_{n\to\infty} f_i^n(t,s,x) = f_i(t,s,x)$$

uniformly on D for each $i\in N$.

LEMMA 1. *L is dense in M.*

Proof. Assume that $f\in M$ and $\varepsilon>0$. Choose $p\in N$ such that $1/2^p\leq\varepsilon/2$. For any $x\in E$ let

$$N(x,\varepsilon) = \{y\in E\colon \sup_{(t,s)\in T}\|f_i(t,s,y)-f_i(t,s,x)\|_i<\varepsilon/4 \text{ for } i=1,\ldots,p\}.$$

Since T is compact and f_i is continuous,

$$\lim_{y\to x}\sup_{(t,s)\in T}\|f_i(t,s,y)-f_i(t,s,x)\|_i = 0 \quad \text{for any } i\in N,$$

and hence $N(x,\varepsilon)$ is open in E. As the space E is paracompact, there is a locally finite refinement $\{Q_\alpha\}_{\alpha\in A}$ of $\{N(x,\varepsilon)\colon x\in E\}$, where each Q_α is non-empty and open. For any $x\in E$ and $\alpha\in A$ put

$$\mu_\alpha(x) = \begin{cases} 0 & \text{if } x\notin Q_\alpha,\\ \operatorname{dist}(x,\partial Q_\alpha) & \text{if } x\in Q_\alpha,\end{cases}$$

and

$$p_\alpha(x) = \mu_\alpha(x)/\sum_{\beta\in A}\mu_\beta(x).$$

It can easily be verified (cf. [1], p. 238, [11]) that μ_α is Lipschitzian with constant 1 and p_α is locally Lipschitzian. For any $\alpha\in A$ choose $x_\alpha\in Q_\alpha$. Let

$$g_i(t,s,x) = \begin{cases} \sum_{\alpha\in A} p_\alpha(x) f_i(t,s,x_\alpha) & \text{for } (t,s,x)\in D,\ i = 1,\dots,p,\\ 0 & \text{for } (t,s,x)\in D,\ i \geqslant p+1,\end{cases}$$

and $g = (g_1, g_2, \dots)$.

Fix $x\in E$, and choose $\alpha_0\in A$ such that $x\in Q_{\alpha_0}$. Since $\{Q_\alpha\}$ is locally finite, there exists a ball $B(x,\eta)\subset Q_{\alpha_0}$ such that $B(x,\eta)\cap Q_\alpha \neq \emptyset$ only for α belonging to a finite subset $\{\alpha_1,\dots,\alpha_n\}$ of A. As the functions p_{α_j} are locally Lipschitzian, there exist a ball $V = B(x,\delta)\subset B(x,\eta)$ and a constant $k>0$ such that

$$|p_{\alpha_j}(y)-p_{\alpha_j}(z)| \leqslant k|y-z| \quad \text{for } y,z\in V,\ j = 0,1,\dots,n.$$

Hence

$$\|g_i(t,s,y)-g_i(t,s,z)\|_i = \Big\|\sum_{j=0}^{n}\big(p_{\alpha_j}(y)-p_{\alpha_j}(z)\big)f_i(t,s,x_{\alpha_j})\Big\|_i$$

$$\leqslant \sum_{j=0}^{n}|p_{\alpha_j}(y)-p_{\alpha_j}(z)|\cdot\|f_i(t,s,x_{\alpha_j})\|_i \leqslant \sum_{j=0}^{n} m_i k|y-z| \leqslant m(n+1)k|y-z|$$

for $y,z\in V$, $(t,s)\in T$ and $i = 1,\dots,p$, where $m = \max(m_1,\dots,m_p)$ and $m_i = \sup\{\|f_i(t,s,x)\|_i\colon (t,s,x)\in D\}$. Moreover,

$$\|g_i(\tau,s,x)-g_i(t,s,x)\|_i = \Big\|\sum_{\alpha\in A}\big(f_i(\tau,s,x_\alpha)-f_i(t,s,x_\alpha)\big)p_\alpha(x)\Big\|_i$$

$$\leqslant \sum_{\alpha\in A}\|f_i(\tau,s,x_\alpha)-f_i(t,s,x_\alpha)\|_i p_\alpha(x) \leqslant \sum_{\alpha\in A} r_i(\tau,t,s)p_\alpha(x) = r_i(\tau,t,s),$$

and

$$\|g_i(t,s,x)\|_i = \Big\|\sum_{\alpha\in A}p_\alpha(x)f_i(t,s,x_\alpha)\Big\|_i \leqslant \sum_{\alpha\in A}p_\alpha(x)\|f_i(t,s,x_\alpha)\|_i \leqslant m$$

for each $0\leqslant s\leqslant t\leqslant \tau\leqslant a$, $x\in E$ and $i = 1,\dots,p$.

Obviously, for any i, $1\leqslant i\leqslant p$, the function g_i is continuous at $(t,s,x)\in D$ as a linear combination of a finite family of continuous functions. As $g_i = 0$ for $i\geqslant p+1$, from the above argument we conclude that $g\in L$.

Furthermore, for any $\alpha\in A$ there exists a neighbourhood $N(y,\varepsilon)$ containing Q_α, and therefore

$$\|f_i(t,s,u)-f_i(t,s,x_\alpha)\|_i \leqslant \|f_i(t,s,u)-f_i(t,s,y)\|_i + $$
$$+\|f_i(t,s,y)-f_i(t,s,x_\alpha)\|_i \leqslant \varepsilon/2$$

for each $u \in Q_\alpha$, $(t,s) \in T$ and $i = 1, \ldots, p$. Hence

$$\|g_i(t,s,x)-f_i(t,s,x)\|_i = \Big\|\sum_{\alpha\in A} p_\alpha(x)\big(f_i(t,s,x_\alpha)-f_i(t,s,x)\big)\Big\|_i$$

$$\leqslant \sum_{\alpha\in A} p_\alpha(x)\,\|f_i(t,s,x_\alpha)-f_i(t,s,x)\|_i \leqslant \sum_{\alpha\in A} p_\alpha(x)\varepsilon/2 = \varepsilon/2$$

for $(t,s,x) \in D$, $i = 1, \ldots, p$, and consequently,

$$|g(t,s,x)-f(t,s,x)| = \sum_{i=1}^{\infty} \frac{1}{2^i}\cdot\frac{\|g_i(t,s,x)-f_i(t,s,x)\|_i}{1+\|g_i(t,s,x)-f_i(t,s,x)\|_i}$$

$$\leqslant \sum_{i=1}^{p} \frac{1}{2^i}\frac{\varepsilon/2}{1+\varepsilon/2}+\sum_{i=p+1}^{\infty}\frac{1}{2^i}$$

$$\leqslant \frac{\varepsilon}{2}+\frac{1}{2^p} \leqslant \varepsilon \quad \text{for } (t,s,x)\in D,$$

so that $d(f,g) \leqslant \varepsilon$. This completes the proof of Lemma 1.

Let $p = (p_1, p_2, \ldots)$ be any fixed continuous function from J into E. For any $f \in M$ and $i \in N$ put

$$F_i(x)(t) = p_i(t)+\int_0^t f_i\big(t,s,x(s)\big)ds \quad (x\in C,\ t\in J).$$

In the usual way (cf. [14]) we may prove that F_i is a continuous mapping of C into C_i, and the set $F_i(C)$ is equiuniformly continuous. Let

$$\omega_i(d) = \sup\{\|F_i(x)(t)-F_i(x)(s)\|_i\colon x\in C,\, t,s\in J,\, |t-s| \leqslant d\}.$$

It is clear that $\lim_{d\to 0}\omega_i(d) = 0$. From this it follows that the mapping $F\colon C\to C$, defined by

$$F(x) = \big(F_1(x), F_2(x), \ldots\big) \quad \text{for } x\in C,$$

is continuous, and

$$(1) \qquad \omega(d) = \sup\{|u(t)-u(s)|\colon u\in F(C),\, t,s\in J,\, |t-s| \leqslant d\}\to 0$$

when $d\to 0$, because

$$\omega(d) \leqslant \sum_{i=1}^{\infty}\frac{1}{2^i}\cdot\frac{\omega_i(d)}{1+\omega_i(d)}.$$

We consider the infinite system of Volterra integral equations

$$(2) \qquad x_i(t) = p_i(t)+\int_0^t f_i\big(t,s,x_1(s),x_2(s),\ldots\big)ds \quad (t\in J,\ i = 1,2,\ldots),$$

where $\int$ denotes the Bochner integral and $f\in M$.

Putting

$$\int_\alpha^\beta f(t,s,x(s))\,ds = \Big(\int_\alpha^\beta f_1(t,s,x(s))\,ds, \int_\alpha^\beta f_2(t,s,x(s))\,ds, \ldots\Big)$$

$(0 \leqslant \alpha \leqslant \beta \leqslant a,\ x \in C)$, we see that (2) is equivalent to the equation

$$x(t) = p(t) + \int_0^t f(t,s,x(s))\,ds \qquad (t \in J). \tag{3}$$

LEMMA 2. *If $f \in L$, then there exists a unique solution of* (2) *defined on J.*

Proof. Assume that $f \in L$ and m is a constant satisfying 4°. First we remark that

$$|\!|\!|\int_\alpha^\beta f(t,s,x(s))\,ds|\!|\!| \leqslant m(\beta-\alpha) \qquad \text{for } 0 \leqslant \alpha \leqslant \beta \leqslant a \text{ and } x \in C. \tag{4}$$

Indeed, as $\|f_i(t,s,x)\|_i \leqslant m$ for $(t,s,x) \in D$ and $i \in N$, we have

$$|\!|\!|\int_\alpha^\beta f(t,s,x(s))\,ds|\!|\!| = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{\|\int_\alpha^\beta f_i(t,s,x(s))\,ds\|_i}{1+\|\int_\alpha^\beta f_i(t,s,x(s))\,ds\|_i}$$

$$\leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{m(\beta-\alpha)}{1+m(\beta-\alpha)} \leqslant m(\beta-\alpha).$$

Denote by Q the set of all t, $0 < t \leqslant a$, for which there exists a unique solution of (2) defined on $[0,t]$. Let $\tau = \sup Q$. Then there exists a unique solution u of (2) defined on $[0,\tau)$. Since

$$|\!|\!|u(t_1)-u(t_2)|\!|\!| \leqslant \omega(|t_1-t_2|) \qquad \text{for each } t_1, t_2 \in [0,\tau),$$

there exists the limit

$$u(\tau^-) = \lim_{t\to\tau^-} u(t) = p(\tau) + \int_0^\tau f(\tau,s,u(s))\,ds,$$

and hence $\tau \in Q$. Suppose that $\tau < a$. Then there exist a closed ball $B = B(u(\tau),\varepsilon)$ and a constant $k > 0$ such that

$$\|f_i(t,s,x)-f_i(t,s,y)\|_i \leqslant k|\!|\!|x-y|\!|\!| \qquad \text{for } (t,s) \in T \text{ and } x, y \in B.$$

Consider the equation

$$x(t) = h(t) + \int_\tau^t f(t,s,x(s))\,ds \qquad (\tau \leqslant t \leqslant a), \tag{5}$$

where $h(t) = p(t) + \int_0^\tau f(t,s,u(s))\,ds$. The function h is continuous on $[\tau,a]$

and $h(\tau) = u(\tau)$. Choose $\delta, 0 < \delta \leqslant a-\tau$, in this way that $k\delta < 1$ and $|h(t)-h(\tau)|+m\delta \leqslant \varepsilon$ for $t \in [\tau, \tau+\delta]$. Denote by $\tilde{B}$ the set of all continuous functions from $[\tau, \tau+\delta]$ into B. Obviously $\tilde{B}$ is a complete metric subspace of $C([\tau, \tau+\delta], E)$. Let

$$G(x)(t) = h(t)+\int_\tau^t f(t, s, x(s))\,ds \quad \text{for } x \in \tilde{B},\ \tau \leqslant t \leqslant \tau+\delta.$$

Then, by (4),

$$|G(x)(t)-u(\tau)| \leqslant |h(t)-h(\tau)|+\Big|\int_\tau^t f(t, s, x(s))\,ds\Big|$$
$$\leqslant |h(t)-h(\tau)|+m\delta \leqslant \varepsilon \quad \text{for } x \in \tilde{B} \text{ and } t \in [\tau, \tau+\delta].$$

Moreover,

$$\|G_i(x)(t)-G_i(y)(t)\|_i \leqslant \int_\tau^t \|f_i(t, s, x(s))-f_i(t, s, y(s))\|_i\,ds$$
$$\leqslant \int_\tau^t k|x(s)-y(s)|\,ds$$
$$\leqslant k\delta|x-y|_c \quad \text{for } i \in N,\ x, y \in \tilde{B} \text{ and } t \in [\tau, \tau+\delta],$$

and therefore

$$|G(x)(t)-G(y)(t)| = \sum_{i=1}^{\infty} \frac{1}{2^i}\cdot\frac{\|G_i(x)(t)-G_i(y)(t)\|_i}{1+\|G_i(x)(t)-G_i(y)(t)\|_i} \leqslant k\delta|x-y|_c$$
$$\text{for } x, y \in \tilde{B} \text{ and } t \in [\tau, \tau+\delta].$$

This proves that G is a mapping of $\tilde{B}$ into itself, and

$$|G(x)-G(y)|_c \leqslant k\delta|x-y|_c \quad \text{for each } x, y \in \tilde{B}.$$

Applying the Banach fixed point theorem we conclude that there exists exactly one function $w \in \tilde{B}$ which satisfies (5) on $[\tau, \tau+\delta]$. Hence the function v defined by

$$v(t) = \begin{cases} u(t) & \text{for } 0 \leqslant t \leqslant \tau, \\ w(t) & \text{for } \tau \leqslant t \leqslant \tau+\delta, \end{cases}$$

is a solution of (3) on $[0, \tau+\delta]$, because

$$v(t) = w(t) = h(t)+\int_\tau^t f(t, s, w(s))\,ds$$
$$= p(t)+\int_0^\tau f(t, s, u(s))\,ds+\int_\tau^t f(t, s, w(s))\,ds$$
$$= p(t)+\int_0^\tau f(t, s, v(s))\,ds+\int_\tau^t f(t, s, v(s))\,ds$$
$$= p(t)+\int_0^t f(t, s, v(s))\,ds \quad \text{for } t \geqslant \tau.$$

Conversely, if a function z is a solution of (3) on $[0, \tau+\delta]$, then z is a solution of (2) on $[0, \tau]$, and therefore $z(t) = u(t)$ for $0 \leqslant t \leqslant \tau$. Moreover, for $t \in [\tau, \tau+\delta]$ we have

$$\begin{aligned} z(t) &= p(t)+\int_0^t f(t, s, z(s))\, ds \\ &= p(t)+\int_0^\tau f(t, s, u(s))\, ds+\int_\tau^t f(t, s, z(s))\, ds \\ &= h(t)+\int_\tau^t f(t, s, z(s))\, ds, \end{aligned}$$

and, by (4),

$$|z(t)-u(\tau)| \leqslant |h(t)-h(\tau)|+\Big|\int_\tau^t f(t, s, z(s))\, ds\Big| \leqslant |h(t)-h(\tau)|+m\delta \leqslant \varepsilon,$$

so that $G(z|[\tau, \tau+\delta]) = z|[\tau, \tau+\delta]$ and $z|[\tau, \tau+\delta] \in \tilde{B}$. This implies $z(t) = w(t)$ for $\tau \leqslant t \leqslant \tau+\delta$, and consequently $z = \vartheta$. Therefore $\tau+\delta \in Q$, in contradiction with the definition of τ. Hence $\tau = a$.

LEMMA 3. *Let $f \in L$, and let v be the unique solution of* (2) *on J. Then there exist positive numbers η, k such that*

$$\|f_i(t, s, x)-f_i(t, s, y)\|_i \leqslant k|x-y| \tag{6}$$

for every $(t, s) \in T$, $i \in N$, $x, y \in B(v(\tau), \eta)$ and $\tau \in J$.

Proof. As $f \in L$, for any $\tau \in J$ there are positive numbers η_τ, k_τ such that

$$\|f_i(t, s, x)-f_i(t, s, y)\|_i \leqslant k_\tau |x-y| \quad \text{for every } (t, s) \in T,\ i \in N$$

and $x, y \in B(v(\tau), \eta_\tau)$. Since the set $v(J)$ is compact, there exists a finite subset $\{\tau_1, \ldots, \tau_n\}$ of J such that

$$v(J) \subset \bigcup_{j=1}^n B(v(\tau_j), \eta_{\tau_j}/2).$$

Putting $k = \max(k_{\tau_1}, \ldots, k_{\tau_n})$ and $\eta = \frac{1}{2}\min(\eta_{\tau_1}, \ldots, \eta_{\tau_n})$ we obtain (6).

Moreover, we shall use Lemma 2 from the first version of [11] (Technical Note BN-655, University of Maryland, 1970):

Let X be a complete metric space, and let $h\colon X \to [0, \infty)$ be a function which is continuous at each point of a set Y which is dense in X. If h vanishes on Y, then the set $\{x \in X\colon h(x) > 0\}$ is of first category in X (see also [12], Lemma 1.2).

2. The generic property of convergence of approximate solutions. For any $n \in N$ and $f \in M$ denote by $S_n(f)$ the set of all $u \in C$ such that

$$\Big|u(t)-p(t)-\int_0^t f(t, s, u(s))\, ds\Big| < 1/n \quad \text{for every } t \in J.$$

LEMMA 4. *If $f \in L$, then* $\lim_{n\to\infty} \delta(S_n(f)) = 0$ *(δ is the diameter).*

Proof. Assume that $f \in L$. Let v denote the unique solution of (2) on J, and let k, η be the constant from Lemma 3. Suppose that $\overline{\lim_{n\to\infty}}\, \delta(S_n(f)) > 0$. Thus there are $\varepsilon > 0$ and a sequence (u^n) such that

$$u^n \in S_n(f) \quad \text{and} \quad \|u^n - v\|_c \geqslant \varepsilon \quad \text{for } n = 1, 2, \ldots \tag{7}$$

Since the sequences $(u^n - F(u^n))$ and $(F(u^n))$ are equicontinuous, the sequence (u^n) is also equicontinuous, and therefore the numbers

$$\beta(d) = \sup\{\|u^n(t) - u^n(s)\|, \|v(t) - v(s)\|:\ n \in N, t, s \in J, |t-s| \leqslant d\} \to 0$$

as $d \to 0$. Denote by Q the set of all $t \in J$ such that $\lim_{n\to\infty} u^n(s) = v(s)$ uniformly on $[0, t]$. Obviously $0 \in Q$, because $\lim_{n\to\infty} \|u^n(0) - p(0)\| \leqslant \lim_{n\to\infty} 1/n = 0$ and $p(0) = v(0)$.

Choose $\delta > 0$ such that $\beta(\delta) \leqslant \eta/4$ and $k\delta < 1$. Assume that $\tau \in Q$. We shall show that $q = \min(a, \tau + \delta) \in Q$. As $\tau \in Q$, we can choose n_0 such that $\|u^n(s) - v(s)\| \leqslant \eta/2$ for $n \geqslant n_0$ and $0 \leqslant s \leqslant \tau$. Hence for $n \geqslant n_0$ and $s \in [\tau, q]$ we have $\|u^n(s) - v(s)\| \leqslant \|u^n(s) - u^n(\tau)\| + \|u^n(\tau) - v(\tau)\| + \|v(\tau) - v(s)\| \leqslant \eta/2 + 2\beta(\delta) \leqslant \eta$, so that

$$\|u^n(s) - v(s)\| \leqslant \eta \quad \text{for } n \geqslant n_0 \text{ and } s \in [0, q].$$

Applying (6) we obtain

$$\Big\| \int_0^t \big(f_i(t, s, u^n(s)) - f_i(t, s, v(s))\big)\, ds \Big\|_i \leqslant \int_0^t k \|u^n(s) - v(s)\|\, ds$$

for $n \geqslant n_0$, $i \in N$ and $t \in [0, q]$, and consequently

$$\Big\| \int_0^t \big(f(t, s, u^n(s)) - f(t, s, v(s))\big)\, ds \Big\| \leqslant \int_0^t k \|u^n(s) - v(s)\|\, ds$$

for $n \geqslant n_0$ and $t \in [0, q]$. Let $a_n = \sup_{0 \leqslant t \leqslant q} \|u^n(t) - v(t)\|$. By the inequality

$$\begin{aligned} \|u^n(t) - v(t)\| &\leqslant \Big\|u^n(t) - p(t) - \int_0^t f(t, s, u^n(s))\, ds\Big\| + \\ &\quad + \Big\| \int_0^t \big(f(t, s, u^n(s)) - f(t, s, v(s))\big)\, ds\Big\| \\ &\leqslant 1/n + \int_0^t k \|u^n(s) - v(s)\|\, ds \end{aligned}$$

$(n \geqslant n_0,\ t \in [0, q])$ we have

$$a_n \leqslant 1/n + k\tau \sup_{0 \leqslant t \leqslant \tau} \|u^n(t) - v(t)\| + k\delta a_n \quad \text{for } n \geqslant n_0.$$

As $\tau \in Q$, this implies $\overline{\lim}\, a_n \leqslant k\delta\, \overline{\lim}\, a_n$, and consequently $\lim\limits_{n\to\infty} a_n = 0$, because $k\delta < 1$. Thus $q \in Q$. Since δ does not depend on τ and $0 \in Q$, this proves that $\lim\limits_{n\to\infty} u^n(t) = v(t)$ uniformly on J, in contradiction with (7). Hence $\lim\limits_{n\to\infty} \delta(S_n(f)) = 0$.

THEOREM 1. *The set* $\{f \in M\colon \lim\limits_{n\to\infty} \delta(S_n(f)) > 0\}$ *is of first category in* M.

Proof. Put $h(f) = \lim\limits_{n\to\infty} \delta(S_n(f))$ for any $f \in M$. Assume that $f \in L$ and $\varepsilon > 0$. By Lemma 4 we have $h(f) = 0$, and therefore there is n_0 such that $\delta(S_{n_0}(f)) \leqslant \varepsilon$. Choose $\gamma > 0$ and $p \in N$ such that $1/p+\gamma < 1/n_0$ and $1/2^p \leqslant \gamma/2$. Next choose $\eta > 0$ in this way that $2^p\eta < 1$ and $2^p\eta a/(1-2^p\eta) \leqslant \gamma/2$. Assume that $g \in M$ and $d(g,f) \leqslant \eta$. Then

$$\|g_i(t,s,x)-f_i(t,s,x)\|_i \leqslant 2^p\eta/(1-2^p\eta) \leqslant \gamma/2a \quad \text{for } i = 1,\dots,p \text{ and } (t,s,x) \in D.$$

If $u \in S_n(g)$, $n \geqslant p$, then

$$\begin{aligned}
&\Big|u(t)-p(t)-\int_0^t f(t,s,u(s))\,ds\Big| \\
&\leqslant \Big|u(t)-p(t)-\int_0^t g(t,s,u(s))\,ds\Big| + \Big|\int_0^t \big(g(t,s,u(s))-f(t,s,u(s))\big)\,ds\Big| \\
&\leqslant \frac{1}{n}+\sum_{i=1}^{p}\frac{1}{2^i}\cdot\frac{\int_0^t \|g_i(t,s,u(s))-f_i(t,s,u(s))\|_i\,ds}{1+\int_0^t \|g_i(t,s,u(s))-f_i(t,s,u(s))\|_i\,ds}+\sum_{i=p+1}^{\infty}\frac{1}{2^i} \\
&\leqslant \frac{1}{n}+\sum_{i=1}^{p}\frac{1}{2^i}\cdot\frac{\gamma/2}{1+\gamma/2}+\frac{1}{2^p} \leqslant \frac{1}{n}+\gamma < \frac{1}{n_0} \quad \text{for } t \in J,
\end{aligned}$$

and hence $u \in S_{n_0}(f)$. Therefore $S_n(g) \subset S_{n_0}(f)$ and $\delta(S_n(g)) \leqslant \delta(S_{n_0}(f)) \leqslant \varepsilon$ for $n \geqslant p$. Consequently $h(g) \leqslant \varepsilon$ for every $g \in M$ such that $d(g,f) \leqslant \eta$. This proves that the function h is continuous at each point $f \in L$. Moreover, $h(f) = 0$ for any $f \in L$ and, by Lemma 1, L is dense in M. Applying Lemma 2 of [11], we see that the set $\{f \in M\colon h(f) > 0\}$ is of first category in M.

Remark. It is clear that for any $f \in M$

$$\lim_{n\to\infty} \delta(S_n(f)) = 0 \Leftrightarrow \text{there is } v \in C \text{ such that for any sequence } (u^n),\, u^n \in S_n(f) \text{ for } n = 1,2,\dots,\ u^n \text{ converges to } v \text{ uniformly on } J.$$

COROLLARY 1. *The set of all* $f \in M$ *for which there is not precisely one solution of* (2) *is a set of first category in* M.

COROLLARY 2. *Let P denote the subset of M consisting of all $f \in M$ such that*

(i) *equation* (2) *has at least one solution x;*

(ii) *this solution is unique;*

(iii) *the solution x depends continuously upon the data, that is if the sequence $(f^n) \subset M$ converges to f and, for n large enough, equation* (2) *with f^n has a solution x^n, then x^n converges to x.*

Then $M \setminus P$ is of first category in M.

3. The generic property of convergence of successive approximations. For $f \in M$ and $z \in C$ denote by $(x^n(\cdot, f, z))$ the sequence of the successive approximations defined by

$$x^{n+1}(t,f,z) = p(t) + \int_0^t f\big(t, s, x^n(s,f,z)\big)ds, \qquad x^0(t,f,z) = z(t) \ (t \in J,\ n \in N).$$

Under the stated hypothesis each $x^n(\cdot,f,z)$ is well defined and $x^n(\cdot,f,z) \in C$.

LEMMA 5. *If $f \in L$, then for any $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in N$ such that*

$$\|x^n(\cdot, g, z) - v\|_c \leqslant \varepsilon \quad \textit{for } n \geqslant n_0,\ g \in M,\ d(g,f) \leqslant \delta, \textit{ and } z \in C,$$

where v is the unique solution of (2) *on J.*

Proof. Let $f \in L$, let v denote the unique solution of (2) on J, and let k, η be the constant from Lemma 3. Suppose that Lemma 5 is not true. Then there exist $\varepsilon > 0$ and sequences $(z^l), (g^l), (n_j)$ such that $z^l \in C$, $g^l \in B(f, 1/l)$, $n_j \to \infty$, and

$$\|x^{n_j}(\cdot, g^l, z^l) - v\|_c > \varepsilon \quad \text{for } j, l \in N. \tag{8}$$

For simplicity put $u^{nl} = x^n(\cdot, g^l, z^l)$. Denote by Q the set of all $t \in J$ such that $\lim\limits_{l,n \to \infty} u^{nl}(s) = v(s)$ uniformly on $[0, t]$. Obviously $0 \in Q$, because $u^{nl}(0) = p(0) = v(0)$ for $l, n \geqslant 1$. Let

$$G^l(x)(t) = p(t) + \int_0^t g^l\big(t, s, x(s)\big)ds \qquad (x \in C,\ t \in J),$$

and let $r_l = \sup\{\|G^l(x) - F(x)\|_c \colon x \in C\}$. Since $d(g^l, f) \leqslant 1/l$, by $(*)$ we see that the numbers

$$r_{li} = \sup\{\|G_i^l(x) - F_i(x)\|_{ic} \colon x \in C\} \to 0 \quad \text{as } l \to \infty$$

for every $i \in N$. As

$$r_l \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{r_{li}}{1 + r_{li}},$$

this shows that $\lim\limits_{l \to \infty} r_l = 0$.

Choose $\delta > 0$ such that $\omega(\delta) \leqslant \eta/6$, where ω is defined by (1). Assume that $q \in Q$. Let $h = \min(a, q+\delta)$. Next choose n_0, l_0 in this way that $r_l \leqslant \eta/6$ for $l \geqslant l_0$, and

(9) $\quad |u^{nl}(t) - v(t)| \leqslant \eta/3 \quad$ for $n \geqslant n_0$, $l \geqslant l_0$ and $t \in [0, q]$.

Then for $n \geqslant n_0$, $l \geqslant l_0$ and $t \in [q, h]$ we have

$$\begin{aligned} |u^{nl}(t) - v(t)| &\leqslant |u^{nl}(t) - u^{nl}(g)| + |u^{nl}(q) - v(q)| + |v(q) - v(t)| \\ &\leqslant \eta/3 + \omega(\delta) + |G^l(u^{n-1,l})(t) - F(u^{n-1,l})(t)| + \\ &\quad + |F(u^{n-1,l})(t) - F(u^{n-1,l})(q)| + |F(u^{n-1,l})(q) - G^l(u^{n-1,l})(q)| \\ &\leqslant \eta/3 + \omega(\delta) + 2r_l + \omega(\delta) \leqslant \eta, \end{aligned}$$

and hence, by (9),

(10) $\quad u^{nl}(t) \in B(v(t), \eta) \quad$ for $n \geqslant n_0$, $l \geqslant l_0$ and $t \in [0, h]$.

Now we shall show that

$$(11) \qquad |u^{nl}(t) - v(t)| \leqslant r_l\left(1 + kt + \ldots + \frac{(kt)^{n-n_0-1}}{(n-n_0-1)!}\right) + \eta\frac{(kt)^{n-n_0}}{(n-n_0)!}$$

for $n \geqslant n_0 + 1$, $l \geqslant l_0$ and $t \in [0, h]$.

By (10) and (6) we have

$$\|F_i(u^{n_0 l})(t) - F_i(v)(t)\|_i \leqslant \int_0^t k|u^{n_0 l}(s) - v(s)|\,ds \leqslant k\eta t \quad \text{for } i \in N,$$

so that

$$|F(u^{n_0 l})(t) - F(v)(t)| \leqslant k\eta t \quad \text{for } l \geqslant l_0 \text{ and } t \in [0, h].$$

Hence

$$\begin{aligned} |u^{n_0+1,l}(t) - v(t)| &\leqslant |G^l(u^{n_0 l})(t) - F(u^{n_0 l})(t)| + |F(u^{n_0 l})(t) - F(v)(t)| \\ &\leqslant r_l + k\eta t \quad \text{for } l \geqslant l_0 \text{ and } t \in [0, h]. \end{aligned}$$

Suppose that (11) holds for any $n > n_0$. Then, using (10) and (6), we obtain

$$\begin{aligned} \|F_i(u^{n+1,l})(t) - F_i(v)(t)\|_i &\leqslant \int_0^t k|u^{nl}(s) - v(s)|\,ds \\ &\leqslant \int_0^t k\left(r_l\left(1 + ks + \ldots + \frac{(ks)^{n-n_0-1}}{(n-n_0-1)!}\right) + \eta\frac{(ks)^{n-n_0}}{(n-n_0)!}\right)ds \\ &= r_l\left(kt + \frac{(kt)^2}{2!} + \ldots + \frac{(kt)^{n-n_0}}{(n-n_0)!}\right) + \eta\frac{(kt)^{n-n_0+1}}{(n-n_0+1)!} \end{aligned}$$

for each $i \in N$, which implies

$$|F(u^{n+1,l})(t) - F(v)(t)| \leqslant r_l\left(kt + \ldots + \frac{(kt)^{n-n_0}}{(n-n_0)!}\right) + \eta\frac{(kt)^{n-n_0+1}}{(n-n_0+1)!}$$

for $l \geqslant l_0$ and $t \in [0, h]$. By the inequality

$$\|u^{n+1,l}(t)-v(t)\| \leqslant \|G^l(u^{nl})(t)-F(u^{nl})(t)\|+\|F(u^{nl})(t)-F(v)(t)\|$$
$$\leqslant r_l+\|F(u^{nl})(t)-F(v)(t)\|,$$

this implies (11) for $n+1$. From (11) it follows that

$$\lim_{l,n\to\infty} u^{nl}(t) = v(t) \quad \text{uniformly on } [0, \min(q+\delta, a)].$$

As δ does not depend on q and $0 \in Q$, this proves that

$$\lim_{l,n\to\infty} u^{nl}(t) = v(t) \quad \text{uniformly on } [0, a],$$

in contradiction with (8). Hence Lemma 5 is true.

THEOREM 2. *Let K be the subset of M of all those $f \in M$ such that the corresponding sequence $(x^n(t, f, z))$ of the successive approximations converges uniformly in $(t, z) \in J \times C$. Then $M \setminus K$ is of first category in M.*

Proof. Put $q(f) = \overline{\lim_{m,n\to\infty}} \sup_{z \in C} \|x^n(\cdot, f, z)-x^m(\cdot, f, z)\|_c$ for $f \in M$. Assume that $f \in L$. Let v be the unique solution of (2) on J. From Lemma 5 it follows that for any $\varepsilon > 0$ there exist $n_0 \in N$ and $\delta > 0$ such that

$$\|x^n(\cdot, g, z)-v\|_c \leqslant \varepsilon/2 \quad \text{for each } g \in B(f, \delta),\ n \geqslant n_0 \text{ and } z \in C.$$

Hence

$$\|x^n(\cdot, g, z)-x^m(\cdot, g, z)\|_c \leqslant \varepsilon \quad \text{for each } g \in B(f, \delta),\ m, n \geqslant n_0 \text{ and } z \in C,$$

which implies $q(g) \leqslant \varepsilon$ for $g \in B(f, \delta)$. As, by Lemma 5, $q(f) = 0$ for $f \in L$, this proves that the function q is continuous at each point $f \in L$. Moreover, by Lemma 1, the set L is dense in M. Applying Lemma 2 of [11] we see that the set $\{f \in M\colon q(f) > 0\}$ is of first category in M. Obviously, $M \setminus K = \{f \in M\colon q(f) > 0\}$, because $q(f) = 0$ is equivalent to the convergence of $(x^n(t, f, z))$ to a solution x_z of (2) uniformly in $(t, z) \in J \times C$.

THEOREM 3. *There exists a subset M_0 of M such that the set $M \setminus M_0$ is of first category in M and for every $f \in M_0$ the successive approximations $x^n(t, f, z)$ converge uniformly in $(t, z) \in J \times C$ to a unique solution $x(\cdot, f)$ of (2).*

Proof. Let $H = \{f \in M\colon \lim_{n\to\infty} \delta(S_n(f)) = 0\}$, and let K be the set from Theorem 2. Then $M_0 = H \cap K$.

COROLLARY 3. *For any $f \in M_0$ there exists a metric d_f in C such that d_f is equivalent to the metric generated by $\|\cdot\|_c$, and the integral mapping F, corresponding to f, is a contraction in the metric space $\langle C, d_f \rangle$.*

Proof. This follows from the Gerstein–Sadowski theorem ([10], Theorem 3.5, p. 58).

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