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On a conformal-mapping property

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Abstract. This paper states a conformal-mapping property and gives a proof that this property is equivalent to Ivory's theorem from the standpoint of conformal mapping. Furthermore, this paper determines, by using the above equivalence, all the entire functions satisfying the above property.

1. Introduction. Let $w = f(z)$ be a non-constant entire function of a complex variable z and let D be a non-empty simply connected domain where $w = f(z)$ is univalent. We denote the set of all domains satisfying the above conditions by $\mathscr{S}$.

Let $A_1 A_2 A_3 A_4$ be an arbitrary rectangle contained entirely in D whose sides are parallel to the real and imaginary axes on the z-plane. Here the four vertices A_1, A_2, A_3, A_4 are listed consecutively. We put $A'_k = f(A_k)$ $(k = 1, 2, 3, 4)$ on the w-plane. We consider the following two conditions:

(C.1) $\overline{A'_1 A'_3} = \overline{A'_2 A'_4}$.

(C.2) Let EF be an arbitrary line segment contained entirely in D with midpoint M and let $E' = f(E)$, $F' = f(F)$, $M' = f(M)$ on the w-plane.

(i) If EF is parallel to the real axis on the z-plane, then the tangent line to the arc $f(EF)$ at M' is parallel to the chord $E'F'$ joining its extremities on the w-plane.

(ii) If EF is parallel to the imaginary axis on the z-plane, then the tangent line to the arc $f(EF)$ at M' is parallel to the chord $E'F'$ joining its extremities on the w-plane. (See [10], p. 235.)

Ivory's theorem (see [1], [2], [4]–[10], [12] and [3], p. 32) reads:

For a family of confocal conics, let P_1, P_2, P_3, P_4 be the four vertices of a curvilinear rectangle formed by any four members of this family arbitrarily chosen. Then $\overline{P_1 P_3} = \overline{P_2 P_4}$ holds, in other words, the lengths of the two diagonals $P_1 P_3$, $P_2 P_4$ of the curvilinear rectangle $P_1 P_2 P_3 P_4$ are equal. Here the vertices P_1, P_2, P_3, P_4 are listed consecutively.

In many of the above references this was proved by the use of the mapping functions $\cos z$ and z^2 from the standpoint of conformal mapping and, moreover, it was proved that this property characterizes the (confocal) conic sections. In other words, the following theorem was proved:

THEOREM A. *If $w = f(z)$ is a non-constant entire function of z, then condition* (C.1) *holds in each D belonging to $\mathscr{S}$ if and only if*

$$f(z) = a \sin kz + b \cos kz + c,$$

or

$$f(z) = az^2 + bz + c,$$

where a, b, c are arbitrary complex constants and k is an arbitrary real or purely imaginary constant with $|a| + |b| > 0$ and $k \neq 0$.

The purpose of the present note is to prove the following theorem:

THEOREM. *Let $w = f(z)$ be a non-constant entire function of z.*

(a) *Condition* (C.1) *and condition* (C.2) (i) *are equivalent.*

(b) *Condition* (C.1) *and condition* (C.2) (ii) *are equivalent.*

In Section 3 we shall state a proof of the above theorem. Furthermore, in Section 4, by this theorem and by Theorem A we shall determine all the entire functions satisfying condition (C.2) (i) or condition (C.2) (ii).

2. Lemma. To prove the theorem in Section 1 we shall apply the following lemma (see [11], p. 15):

LEMMA. *Suppose that $w = f(z)$ is defined in a closed disk K with centre at $z = z_0$ and is differentiable at $z = z_0$. Suppose further that $f'(z_0) \neq 0$. If the point z moves along the ray R: $\arg(z - z_0) = \varphi$ (= const) emanating from the point $z = z_0$ on the z-plane, then the arc $f(R \cap K)$ possesses a directed tangent line at the point $w = f(z_0)$ which makes an angle $\varphi + \arg(f'(z_0))$ with the real axis on the w-plane.*

3. Proof of the theorem. We shall give a proof of the theorem in Section 1. First, we shall prove that (C.1) and (C.2) (i) are equivalent.

Let the four vertices of the rectangle $A_1A_2A_3A_4$ (see Section 1) represent the complex numbers $x + y$, $x - \bar{y}$, $x - y$, $x + \bar{y}$, respectively. Then we see that condition (C.1) is equivalent to the following functional equation:

$$|f(x+y) - f(x-y)| = |f(x+\bar{y}) - f(x-\bar{y})|, \tag{1}$$

where x, y are complex variables and $x + y$, $x - \bar{y}$, $x - y$, $x + \bar{y} \in D$.

For the proof we introduce the function $g = g(z)$ defined as

$$g(z) = \overline{f(\bar{z})}. \tag{2}$$

We see that $g = g(z)$ is an entire function since $f = f(z)$ is an entire function.

Proof that (C.1) implies (C.2) (i). By squaring both sides of (1) and by the formula $|\gamma|^2 = \gamma\bar{\gamma}$ (γ complex) we have

$$(3)\quad (f(x+y)-f(x-y))(\overline{f(x+y)}-\overline{f(x-y)}) = (f(x+\bar{y})-f(x-\bar{y}))(\overline{f(x+\bar{y})}-\overline{f(x-\bar{y})}),$$

where x, y are complex variables and $x+y,\ x-\bar{y},\ x-y,\ x+\bar{y} \in D$.

By (2), (3) we have

$$(4)\quad (f(x+y)-f(x-y))(g(\bar{x}+\bar{y})-g(\bar{x}-\bar{y})) = (f(x+\bar{y})-f(x-\bar{y}))(g(\bar{x}+y)-g(\bar{x}-y)).$$

If we put $y = y_1+iy_2$ (y_1, y_2 real) in (4), then we have

$$(5)\quad (f(x+y_1+iy_2)-f(x-y_1-iy_2))(g(\bar{x}+y_1-iy_2)-g(\bar{x}-y_1+iy_2)) = (f(x+y_1-iy_2)-f(x-y_1+iy_2))(g(\bar{x}+y_1+iy_2)-g(\bar{x}-y_1-iy_2)),$$

where x is a complex variable and y_1, y_2 are real variables satisfying $x+y_1+iy_2,\ x-y_1+iy_2,\ x-y_1-iy_2,\ x+y_1-iy_2 \in D$.

By (5) and by the Identity Theorem we have for all complex z_1, z_2

$$(6)\quad (f(x+z_1+z_2)-f(x-z_1-z_2))(g(\bar{x}+z_1-z_2)-g(\bar{x}-z_1+z_2)) = (f(x+z_1-z_2)-f(x-z_1+z_2))(g(\bar{x}+z_1+z_2)-g(\bar{x}-z_1-z_2)),$$

where x is a complex variable belonging to D.

Let E_1F_1, E_2F_2 be arbitrary line segments which are contained entirely in D with common midpoint M and are parallel to the real axis on the z-plane. Furthermore, let the sense from E_1 to F_1 and the sense from E_2 to F_2 coincide with the positive sense of the real axis on the z-plane and let M, E_1, F_1, E_2, F_2 represent the complex numbers $x, x-s, x+s, x-t, x+t$, respectively. Here $s > 0$ and $t > 0$.

If we put $z_1 = \frac{1}{2}(s+t)$, $z_2 = \frac{1}{2}(s-t)$ in (6), then we have

$$(7)\quad (f(x+s)-f(x-s))(g(\bar{x}+t)-g(\bar{x}-t)) = (f(x+t)-f(x-t))(g(\bar{x}+s)-g(\bar{x}-s)),$$

where x is a complex variable and s, t are positive real variables with x, $x+s,\ x-s,\ x+t,\ x-t \in D$.

By (2), (7) we have

$$(8)\quad (f(x+s)-f(x-s))(\overline{f(x+t)-f(x-t)}) = (f(x+t)-f(x-t))(\overline{f(x+s)-f(x-s)}).$$

By the univalence of $w = f(z)$ in D we have

$$(9)\quad f(x+t)-f(x-t) \neq 0, \qquad \overline{f(x+t)-f(x-t)} \neq 0.$$

By (8), (9) we have

$$\frac{f(x+s)-f(x-s)}{f(x+t)-f(x-t)}=\overline{\left(\frac{f(x+s)-f(x-s)}{f(x+t)-f(x-t)}\right)}.$$

Hence $F(x, s, t)=\dfrac{f(x+s)-f(x-s)}{f(x+t)-f(x-t)}$ is real-valued and is non-zero by the univalence of $w=f(z)$ in D. Furthermore, $F(x, s, t)$ is a continuous function of two real variables s (>0), t (>0) with $F(x, s, s)=1$. Hence, by the intermediate-value property of continuous functions $F(x, s, t)$ is a positive function. Hence the two vectors $\overrightarrow{E_1'F_1'}$ $(E_1'=f(E_1),\ F_1'=f(F_1))$, $\overrightarrow{E_2'F_2'}$ $(E_2'=f(E_2),\ F_2'=f(F_2))$ on the w-plane are parallel.

In other words, we have

$$\text{(10)}\qquad \arg\big(f(x+s)-f(x-s)\big)=\arg\big(f(x+t)-f(x-t)\big).$$

Observing that

$$\begin{aligned}\lim_{t\to+0}\arg\big(f(x+t)-f(x-t)\big)&=\lim_{t\to+0}\arg\left(\frac{f(x+t)-f(x-t)}{2t}2t\right)\\&=\lim_{t\to+0}\left(\arg\left(\frac{f(x+t)-f(x-t)}{2t}\right)+\arg(2t)\right)\\&=\lim_{t\to+0}\arg\left(\frac{f(x+t)-f(x-t)}{2t}\right)=\arg\big(f'(x)\big)\end{aligned}$$

($f'(x)\neq 0$ in D by the univalence of f in D), by (10) we have

$$\text{(11)}\qquad \arg\big(f(x+s)-f(x-s)\big)=\arg\big(f'(x)\big).$$

Applying the lemma in Section 2 with $\varphi=0$, by (11) we see that the tangent line to the arc $f(E_1F_1)$ at $M'=f(M)$ is parallel to the chord $E_1'F_1'$. Thus condition (C.2) (i) holds.

Proof that (C.2) (i) implies (C.1).

Let E_1F_1, E_2F_2 be arbitrary line segments which are contained entirely in D with common midpoint M and are parallel to the real axis on the z-plane. Furthermore, let M, E_1, F_1, E_2, F_2 represent the complex numbers $x, x-s, x+s, x-t, x+t$, respectively. Here $s\neq 0$ and $t\neq 0$.

By hypothesis the tangent lines to the arcs $f(E_1F_1)$, $f(E_2F_2)$ at $M'=f(M)$ are parallel to the chords $E_1'F_1'$ $(E_1'=f(E_1),\ F_1'=f(F_1))$, $E_2'F_2'$ $(E_2'=f(E_2),\ F_2'=f(F_2))$, respectively. Therefore, the two chords $E_1'F_1'$, $E_2'F_2'$ are parallel. Hence, by taking the univalence of $w=f(z)$ in D into account, $\dfrac{f(x+s)-f(x-s)}{f(x+t)-f(x-t)}$ is a real number. Hence we have

$$\text{(12)}\qquad \frac{f(x+s)-f(x-s)}{f(x+t)-f(x-t)}=\overline{\left(\frac{f(x+s)-f(x-s)}{f(x+t)-f(x-t)}\right)}.$$

By (2), (12) and by the fact that s, t are real we have

$$(13) \quad (f(x+s)-f(x-s))(g(\bar{x}+t)-g(\bar{x}-t)) \\ = (f(x+t)-f(x-t))(g(\bar{x}+s)-g(\bar{x}-s)),$$

where x is a complex variable and s, t are real variables with $x, x+s, x-s, x+t, x-t \in D$.

By (13) and by the Identity Theorem we have for all complex z_1, z_2

$$(14) \quad (f(x+z_1)-f(x-z_1))(g(\bar{x}+z_2)-g(\bar{x}-z_2)) \\ = (f(x+z_2)-f(x-z_2))(g(\bar{x}+z_1)-g(\bar{x}-z_1)),$$

where x is a complex variable belonging to D.

Let the four vertices of the rectangle $A_1A_2A_3A_4$ (see Section 1) represent the complex numbers $x+y,\ x-\bar{y},\ x-y,\ x+\bar{y}$.

If we put $z_1 = y,\ z_2 = \bar{y}$ in (14), then we have

$$(15) \quad (f(x+y)-f(x-y))(g(\bar{x}+\bar{y})-g(\bar{x}-\bar{y})) \\ = (f(x+\bar{y})-f(x-\bar{y}))(g(\bar{x}+y)-g(\bar{x}-y)),$$

where x, y are complex variables and $x+y,\ x-\bar{y},\ x-y,\ x+\bar{y} \in D$.

By (2), (15) we have

$$|f(x+y)-f(x-y)|^2 = |f(x+\bar{y})-f(x-\bar{y})|^2,$$

or

$$|f(x+y)-f(x-y)| = |f(x+\bar{y})-f(x-\bar{y})|.$$

Thus condition (C.1) $\overline{A_1'A_3'} = \overline{A_2'A_4'}$ (see Section 1) holds.

The proof of (b) is similar to that given for (a). Thus the proof of the theorem is now completed.

4. Corollary to the theorem in Section 1. If $w = f(z)$ is a non-constant entire function of z, then either condition (C.1) (i) or (C.1) (ii) holds in each D belonging to $\mathscr{S}$ if and only if

$$f(z) = a\sin kz + b\cos kz + c,$$

or

$$f(z) = az^2 + bz + c,$$

where a, b, c are arbitrary complex constants and k is an arbitrary real or purely imaginary constant with $|a|+|b| > 0$ and $k \neq 0$.

Proof. The proof is clear from Theorem A and the theorem in Section 1.

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