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Partitions of spectral sets

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Let H be a complex Hilbert space with the inner product (f, g) $(f, g \in H)$ and the norm $\|f\| = \sqrt{(f,f)}$. $L(H)$ stands for the algebra of all linear bounded operators in H. For $T \in L(H)$, T^* is the adjoint of T and $T|S$ is the restriction of T to the invariant subspace S. $\|T\|$ is the norm of $T \in L(H)$. I stands for the identity operator.

Suppose X is a compact Hausdorff space. Let A be a function algebra on X, i.e. a uniformly closed subalgebra of $C(X)$, containing constants and separating points of X.

The algebra homomorphism $T: A \to L(H)$ is called a *representation of* A. The representation T is called a *K-representation* if

$$\|T(u)\| \leqslant K\|u\| \quad (\|u\| = \sup_X |u|) \quad \text{for } u \in A. \tag{1}$$

If $K = 1$, then T is a contractive representation. Without any loss of generality we assume that

$$T(1) = I. \tag{2}$$

Given the K-representation $T: A \to L(H)$ there are regular Borel measures $p(f, g)$ $(f, g \in H)$ on X such, that

$$\|p(f, g)\| \leqslant K\|f\|\|g\| \quad (f, g \in H), \tag{3}$$

$$\big(T(u)f, g\big) = \int_X u\,dp(f, g); \quad u \in A;\, f, g \in H. \tag{4}$$

Suppose closed $\sigma \subset X$. We say that the K-representation $T: A \to L(H)$ is *σ-supported* if it has a system of elementary measures vanishing outside σ. Write A_σ for the uniform closure (in $C(\sigma)$) of $A|\sigma$. If T is a σ-supported K-representation of A, then there is a unique representation $T_\sigma: A_\sigma \to L(H)$ such that $T(u) = T_\sigma(u|\sigma)$ for $u \in A$. T_σ has σ-elementary measures.

Let $P \in L(H)$ be a projection ($P = P^2$!) which commutes with all $T(u)$, where $T: A \to L(H)$ is a K-representation of A. Define $H' = PH$,

$H'' = (I-P)H$. Then $T'(u) = T(u)|H'$, $T''(u) = T(u)|H''$ define continuous representations of A; $H = H'+H''$ and $T = T'+T''$, both sums direct.

Assume $\sigma \subset X$ is a peak set for A. Then $A_\sigma = A|\sigma$ by a result of [7]. If $p(f, g)$ are X-elementary measures of the K-representation T: $A \to L(H)$, then by [12] there is a unique projection P_σ, such that

$$P_\sigma T(u) = T(u) P_\sigma \quad \text{for } u \in A, \tag{5}$$

$$(P_\sigma T(u) f, g) = \int_X u\chi_\sigma dp(f, g), \qquad u \in A; f, g \in H. \tag{6}$$

χ_σ is the characteristic function of σ. It follows that $T_\sigma(u) \overset{\text{df}}{=} T(\tilde{u})|P_\sigma H$ $(u = \tilde{u}|\sigma, \tilde{u} \in A)$ is a K-representation of A_σ in $P_\sigma H$. We call it the σ-part of T.

Suppose we are given a compact set $X \subset C$. Let $R(X)$ be the closure in $C(X)$ of restrictions to X of the algebra of rational functions having poles off X. X with the natural topology of the complex plane may be identified with the spectrum of $R(X)$ and ∂X, the topological boundary of X, with the Shilov boundary of $R(X)$.

Let $u_1(z) \overset{\text{df}}{=} z$ and let T be a K-representation of $R(X)$. It follows that the spectrum $\text{sp}(V)$ of $V \overset{\text{df}}{=} T(u_1)$ is in X. Conversely, if for some $V \in L(H)$, $\text{sp}(V) \subset X$, and

$$\|u(V)\| \leqslant K \sup_X |u| \tag{7}$$

for rational u with poles off X, then the representation $u \to u(V)$ extends uniquelly to a K-representation of $R(X)$. We then say that V generates this representation and that X is a K-spectral set of V. This definition extends the one of von Neumann [15] and already has appeared in literature — see [2] for example. If $K = 1$ we simply say that X is a spectral set of V — this is the von Neumann definition.

If $R(X)$ is viewed as a function algebra on X, then for every its peak set $\sigma \subset X$ we have $R(X)|\sigma = R(\sigma)$ — see [1]. We infer from the preceding story that the following theorem holds true:

THEOREM 1. *Let the compact $X \subset C$ be a K-spectral set of $V \in L(H)$. If $\sigma \subset X$ is a peak set of $R(X)$ and T_σ is the σ-part of the representation generated by V, then $V_\sigma = T_\sigma(u_1) = V|P_\sigma H$ has σ as a K-spectral set.*

COROLLARY 1. *Suppose σ is a peak set of $R(X)$ and X is K-spectral for $V \in L(H)$. Let P_σ be the corresponding projection and define an operator valued measure F on the ring generated by σ and $X-\sigma$ as follows:*

$$F(\emptyset) = 0, \quad F(X) = I, \quad F(\sigma) = P_\sigma, \quad F(X-\sigma) = I-P_\sigma,$$

F is a spectral measure, i.e. $F(\sigma_1)F(\sigma_2) = F(\sigma_1 \cap \sigma_2)$. It follows from a theorem of Sz.-Nagy-Dixmier [3], that there is an $S \in L(H)$, with inverse $S^{-1} \in L(H)$ such that

$$E(\beta) = SF(\beta)S^{-1}$$

is an orthogonal measure. We infer now from Theorem 1 that V is similar to a sum $V_1 \oplus V_2$, where V_1 has σ as a K_1-spectral set with $K_1 = \|K\|\|S\|\|S^{-1}\|$ and V_2 generates an $X-\sigma$ supported continuous representation of $R(X)$. If σ is an interpolation peak set, i.e. if $R(X)|\sigma = C(\sigma)$, then again by Sz. Nagy-Dixmier theorem V_1 is similar to a normal operator. This is a slight extension of a result of Douglas ([2], Th. 4).

If $K = 1$, then simply $V = V_1 \oplus V_2$.

COROLLARY 2. *If σ is a maximal set of antisymmetry for $R(X)$, then σ is a peak set for $R(X)$ by the results of Glicksberg* [7]. *It follows that if X is K-spectral for $V \in L(H)$, then σ is K-spectral of $V_\sigma = V|P_\sigma H$.*

Assume the compact $X \subset C$ is K-spectral for $V \in L(H)$ and let $X = \bigcup_{k|0}^{s} X_k$ $(s \leqslant \infty)$ be the Bishop decomposition of X relative to $R(X)$. $X_1, X_2, \ldots$ are peak sets with positive planar Lebesgue measures and X_0 is a union of peak point maximal sets of antisymmetry. For details we refer to [7] and [1]. Using the theorems of [12] we get by Theorem 1 that there is a sequence of projections $P_0, P_1, P_2, \ldots$ commuting with V such that $V_k = V|P_k H$ $(k = 1, 2, \ldots)$ has X_k as a K-spectral set. Moreover, $V_0 = V|P_0 H$ has X as a K-spectral set with X_0-supported elementary measures and $I = \sum_{k|0}^{s} P_k$ weakly. If $K = 1$, then $I = \bigoplus_{k|0}^{s} P_k$ and $V = \bigoplus_{k|0}^{s} V_k$.

In what follows we need the following proposition which belongs to I. Glicksberg [1]

PROPOSITION 1. *If $p \perp R(X)$, then p vanishes identically on X_0.*

Proof. Since there are no completely singular orthogonal measures for $R(X)$ ([18], Th. 3.2), the Corollary 1.3 of [8] yields that $p = \sum_n p_n$, where the measures $p_n \ll M_{z_n}$ with z_n running over non-trivial Gleason parts for $R(X)$, hence carried by X_k with $k > 0$. So p is carried by $\bigcup_{k|1}^{s} X_k$ which proves the assertion.

It follows now from Proposition 1 that for $f, g \in P_0 H$ there is a unique elementary measure $p(f, g)$ (carried by X_0) of the K-representation generated by V_0. Consequently $V_0 = \int z dF(z)$, F being a spectral measure.

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[1] I take here the opportunity to express my thanks to Professor Glicksberg for making me known this proposition and its proof — W. M.

Applying the Sz.-Nagy-Dixmier theorem we infer that the following holds true:

PROPOSITION 2. *The part V_0 is similar to a normal operator. If $K = 1$, then V_0 is normal.*

Using the above proposition and the results of [12] we get easily the following

THEOREM 2. *Suppose $X \subset C$ is a spectral set of $V \in L(H)$. Let $X = \bigcup_{k|0}^{s} X_k$ be the Bishop decomposition of X relative to $R(X)$. Then $V = \bigoplus_{k|0}^{s} V_k$, where V_k has X_k as a spectral set for $k > 0$ and V_0 is normal with spectral measure carried by ∂X.*

COROLLARY 3. *If V is ∂X-pure* (²), *then $V_0 = 0$ in the above theorem.*

We will now consider the partitions of spectral sets induced by Gleason parts of $R(X)$. In what follows the algebra $R(X)$ is viewed as a function algebra on ∂X, i.e. consisting of the restrictions of functions of $R(X)$ to ∂X. A result of Wilken [18], Th. 3.3, says that representing measures for points in a part G for $R(X)$ have closed supports in $\bar{G}$ and represent this points for $R(\bar{G})$ as well. Since $R(X)$ is viewed as an algebra on ∂X, the representing measures for points in G are carried by $\bar{G} \cap \partial X \subset \partial \bar{G}$.

Suppose X is a K-spectral set of $V \in L(H)$. Then by Theorem 2.1 of [13] $H = H' + H''$ and $V = V_G + V_S$ (both sums direct), where $V_G \in L(H')$, $V_S \in L(H'')$ have X as a K-spectral set; V_G generates a G-continuous representation of $R(X)$, and V_S a G-singular one (³). The representation generated by V_G extends uniquelly to a $\partial\bar{G}$-supported K-representation of $\overline{(R(X)|\bar{G})}$. Since in general $\overline{(R(X)|\bar{G})} \neq R(\bar{G})$, we cannot just conclude at this point that $\bar{G}$ is K-spectral for V_G. If G has a connected complement, then $R(\bar{G}) = \overline{(R(X)|\bar{G})}$ by Mergelyan theorem, which proves that in this special case $\bar{G}$ is K-spectral for V_G (⁴). However, as proved by Sarason in [17], if $K = 1$ and X has connected complement, then $\bar{G}$ is spectral for V_G provided X is spectral for V. The key property (besides the essential dilatability of the representation generated by V) applied by Sarason was Proposition 13 of [17]. The proposition below is a generalization of this proposition (⁵).

PROPOSITION 3. *Suppose G is a Gleason part for $R(X)$ and let the measure $p \perp R(X)$. Then the G-continuous part p_G of p is orthogonal to $R(\bar{G})$.*

(²) See [5], [11] and [17] for related matter.

(³) See [13] for definitions.

(⁴) The proof of this property, erronously suggested as applicable in a general case may be found in [14].

(⁵) I would like to thank here Professor I. Glicksberg for pointing out to me the enclosed proof of Proposition 3. It is much simpler than that of mine — W. M.

Proof. The general M. and F. Riesz theorem of [8] yields that $p_G \perp R(X)$. Suppose $z_0 \in X - \bar{G}$. It follows that p_G is singular with respect to M_{z_0} and if $m \in M_{z_0}$, then

$$q = \frac{1}{z - z_0} p_G - m \int \frac{dp_G}{z - z_0} \perp \big(C + (z - z_0) R(X)\big).$$

For any $z_0 \in X$, $C + (z - z_0) R(X)$ $(z \in X)$ is dense in $R(X)$, so $q \perp R(X)$ and its M_{z_0}-singular part $\dfrac{1}{z - z_0} p_G \perp R(X)$. Hence

$$\hat{p}_G(z_0) \overset{\mathrm{df}}{=} \int \frac{dp_G}{z - z_0} = 0 \qquad \text{for } z_0 \in X - \bar{G}.$$

Since $p_G \perp R(X)$, $\hat{p}_G(z_0) = 0$ for $z_0 \notin X$ also, which completes the proof.

As already noticed, the elementary measures of the representation of $R(X)$ generated by V_G being G-continuous are supported by $\partial \bar{G}$. If $p'(f, g)$, $p''(f, g)$ $(f, g \in H')$ are two such measures, then $R(X) \perp p' - p'' = (p' - p'')_G$ and by Proposition 3, $p' - p'' \perp R(\bar{G})$. It follows that

$$\xi(u; f, g) = \int_{\bar{G}} u\, dp'(f, g) = \int_{\bar{G}} u\, dp''(f, g)$$

is a well defined functional for $u \in R(\bar{G})$, linear in f and antilinear in g and such that $|\xi(u; f, g)| \leqslant K \sup_{\bar{G}} |u|\, \|f\|\, \|g\|$. We conclude that there is linear mapping $T\colon R(\bar{G}) \to L(H')$ such that $\|T(u)\| \leqslant K \|u\|_G$ $(\|u\|_G = \sup_{\bar{G}} |u|)$ and

$$\big(T(u) f, g\big) = \int_{\bar{G}} u\, dp(f, g) \qquad (u \in R(\bar{G}), f, g \in H')$$

p being elementary measure. In fact $\int_{\bar{G}} u\, dp(f, g) = \int_{\partial \bar{G}} u\, dp(f, g)$. Suppose now that $z_0 \in X$ but $z_0 \notin \bar{G}$ and consider the measure $p\big(f, (V_G - z_0 I)^* f\big) \ll M_{\tilde{z}}$ $(\tilde{z} \in G)$ for $f \in H'$. Since $X - \bar{G}$ is a F_σ set, the Forelli lemma ([6], Ch. II. 7.3, p. 43) proves that there is a norm bounded sequence of rational functions $u_\nu \in R(X)$ such that

$$\begin{aligned} &u_\nu(z) \to 0 \quad &&\text{for } z \in X - \bar{G}, \\ &u_\nu(z) \to 1 \quad &&\text{a.e. } M_{\tilde{z}} \ (\tilde{z} \in G). \end{aligned} \tag{8}$$

We define

$$r_\nu(z) = \frac{u_\nu(z) - u_\nu(z_0)}{z - z_0}.$$

Then $r_\nu \in R(X)$. Since X is K-spectral for V_G with $\partial\bar{G}$-supported elementary measures, $T(r) = r(V_G)$ for every polynomial r, which for $f \in H'$ gives us

$$\big(r(V_G)\big(u_\nu(V_G)-u_\nu(z_0)I\big)f, f\big) = \big((V_G-z_0I)r(V_G)r_\nu(V_G)f, f\big)$$
$$= \int_{\bar{G}} r(z)r_\nu(z)\,dp_z\big(f, (V_G-z_0I)^*f\big) \overset{\text{df}}{=} \eta_\nu.$$

Since $z_0 \notin \bar{G}$ and u_ν are norm bounded

(9) $$|r_\nu(z)| \leqslant \varrho, \quad \nu = 1, 2, \ldots, \; z \in \bar{G}$$

for some finite constant ϱ. By dominated convergence and by (8) and (9) we have

$$\eta_\nu \to \int_{\bar{G}} r(z)\frac{1}{z-z_0}\,dp_z\big(f, (V_G-z_0I)^*f\big) \overset{\text{df}}{=} \eta$$

because $p\big(f, (V_G-z_0I)^*f\big)$ is G-continuous. The definition of $T(u)$ yields that

$$\eta = \left(T\left(\frac{r}{z-z_0}\right)f, (V_G-z_0I)^*f\right).$$

But $\eta_\nu \to \big(r(V_G)f, f\big)$ by (8). Since f is an arbitrary vector of H', we get that

(10) $$r(V_G) = (V_G-z_0I)T\left(\frac{r}{z-z_0}\right).$$

On the other hand, when considering for $f \in H'$ the G-continuous elementary measure $p\big((V_G-z_0I)f, f\big)$ we infer that

$$\big(r(V_G)\big(u_\nu(V_G)-u_\nu(z_0)I\big)f, f\big) = \int_{\bar{G}} r(z)r_\nu(z)\,dp_z\big((V_G-z_0I)f, f\big)$$
$$\to \left(T\left(\frac{r}{z-z_0}\right)(V_G-z_0I)f, f\right)$$

by (8) and (9). It follows that

(11) $$r(V_G) = (V_G-z_0I)T\left(\frac{r}{z-z_0}\right).$$

Taking $r(z) \equiv 1$ we infer by (10) and (11) that $z_0 \notin \operatorname{sp}(V_G)$ and

$$T\left(\frac{1}{z-z_0}\right) = (V_G-z_0I)^{-1}.$$

This together with (10) and (11) shows that for $z_0 \in X-\bar{G}$

(12) $$T\left(\frac{r}{z-z_0}\right) = r(V_G)(V_G-z_0I)^{-1}.$$

Since $T(u) = u(V_G)$ for rational u having poles off X, (12) holds for every $z_0 \notin \bar{G}$.

Let u be a rational function having merely simple poles $z_1, \ldots, z_m$ all lying in $\bar{G}$. Then u may be written in the form

$$u(z) = \sum_{i|1}^{m} \frac{u_i(z)}{z-z_i}$$

with suitable polynomials u_i. The linearity of the mapping T implies that

$$T(u) = \sum_{i|1}^{m} T\left(\frac{u_i}{z-z_i}\right)$$

which by (12) gives us

$$T(u) = \sum_{i|1}^{m} u_i(V_G)(V_G - z_i I)^{-1} = \sum_{i|1}^{m} (V_G - z_i I)^{-1} u_i(V_G) = u(V_G).$$

Since $\|T(u)\| \leqslant K\|u\|_G$, $\bar{G}$ is a K-spectral set of V_G, because such u are dense in $R(X)$. Summing up we get the following

THEOREM 3. *Suppose that G is the Gleason part for $R(X)$. If X is a K-spectral set of $V \in L(H)$, then $\bar{G}$ is a K-spectral set of $V_G \in L(H')$.*

Combining Theorem 3 with Theorem 1, Theorem 3 of [14] we get the following

THEOREM 4. *Suppose the compact $X \subset C$ is a spectral set of $V \in L(H)$. Let G_i $(i = 1, 2, \ldots)$ be the sequence of all non-peak point parts for $R(X)$. Then*

$$V = \bigoplus_{i|1}^{s} V_i \oplus V_0 \quad (s \leqslant +\infty),$$

where:

(α) *V_i has $\bar{G}_i$ as a spectral set for $i > 0$.*

(β) *The representation of $R(\bar{G}_i)$ generated by V_i $(i > 0)$ has elementary measures absolutely continuous with respect to M_z for $z \in G_i$.*

(γ) *V_0 is a normal operator with spectrum carried by ∂X. Moreover, $V_0 = V_0' \oplus V_0''$, where V_0' is normal having completely singular spectral measure and $V_0'' = \bigoplus_{l} z_l I_l$ (I_l = identity in suitable H_l), where z_l ranges over the set of all peak points for $R(X)$.*

Both Theorem 3 and Theorem 4 have been proved in a completely dilation free way. Theorem 4 generalizes the Theorem 1 of Sarason paper [17], which concerned X having connected complement. Any V having such X as a spectral set generates a uniquelly ∂X-dilatable representation of $R(X)$ (see [4] and also [10]). This fact has been used essentially in [17].

COROLLARY 4. *If V has no non-zero normal part with spectrum on ∂X, then simply $V = \bigoplus_{i|1}^{s} V_i$.*

References

[1] A. Browder, *Introduction to function algebras*, New York 1969.
[2] R. G. Douglas, *On operators similar to normal operators*, Rev. Math. Pures et Appl. (Roumaine) 2, Tome XIV (1969), p. 193-197.
[3] J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leurs applications*, Acta Sci. Math. (Szeged) 12 (1950), p. 213-227.
[4] C. Foiaş, *Certain applications of spectral sets*, Studii si cercetari mat. 10 (1959), p. 365-401 (in Roumanian).
[5] — and I. Suciu, *Szegö measures*, Rev. Roumaine Math. Pure Appl. 11 (1966), p. 147-159.
[6] T. W. Gamelin, *Uniform algebras*, Englewood Cliffs, N. J. 1969.
[7] I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, T.A.M.S. 105 (1969), p. 415-435.
[8] — *The abstract F. and M. Riesz theorem*, J. Funct. An. 1 (1967), p. 109-122.
[9] — *Dominant representing measures and rational approximation*, T.A.M.S. 130 (1968), p. 425-462.
[10] A. Lebow, *On von Neumann's theory of spectral sets*, J. Math. Anal. Appl. 7, No. 1 (1963), p. 64-90.
[11] W. Mlak, *A note on Szegö type properties of semi-spectral measures*, Studia Math. 31 (1968), p. 241-251.
[12] — *Decomposition of operator-valued representations of function algebras*, ibidem 36 (1970), p. 111-123.
[13] — *Decompositions and extensions of operator-valued representations of function algebras*, Acta Sci. Math. (Szeged) 30 (1969), Fasc. 3-4, p. 181-193.
[14] — *Absolutely continuous operator-valued representations of function algebras*, Bull. Acad. Sci. Polon. 17 (9) (1969), p. 547-550.
[15] J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. 4 (1951), p. 258-281.
[16] R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand Math. St., Princeton, N. Y. 1966.
[17] D. Sarason, *On spectral sets having connected complement*, Acta Sci. Math. (Szeged) (1965) 26, 3-4, p. 289-299.
[18] D. R. Wilken, *The support of representing measures for $R(X)$*, Pacific J. Math. 26 (3), (1968), p. 621-626.

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