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# On the Chern character of glued bundles

by JAN A. REMPAŁA (Warszawa)

**Abstract.** We give an expression for the differential form representing the Chern character of a bundle obtained by glueing together two vector bundles (one of which is trivial) in terms of the glueing isomorphism.

Let $M$ be a $C^\infty$-smooth paracómpact manifold and let $M_1$, $M_2$ be its open subsets with

$$M_1 \cup M_2 = M, \quad M_1 \cap M_2 = M_{12}.$$

Let $P_1$ and $P_2$ be smooth complex vector bundles over $M_1$ and $M_2$ respectively (smoothly), isomorphic over $M_{12}$. It is well known that there exists a vector bundle $P$ over $M$ with $P|\ M_i \cong P_i$, $i = 1, 2$, and any two such bundles are isomorphic.

Following (Fedosov(1)) we shall assume that $P_i$ is defined by a smooth family of projectors $P_i(x)$: $C^N \to C^N$, $x \in M_i$. Then any mutually inverse isomorphisms $a$: $P_1|\ M_{12} \to P_2|\ M_{12}$ and $b$: $P_2|\ M_{12} \to P_1|\ M_{12}$ can be realized by smooth families of linear maps $a(x), b(x)$: $C^N \to C^n$, $x \in M_{12}$, satisfying the relations

$$a(x)P_1(x) = P_2(x)a(x) = a(x),$$
$$b(x)P_2(x) = P_1(x)b(x) = b(x),$$
$$b(x)a(x) = P_1(x),$$
$$a(x)b(x) = P_2(x).$$

Let $f_1, f_2$ be smooth real-valued functions subject to the conditions

$$\operatorname{supp} f_i \subset M_1, \quad f_1^2 + f_2^2 = 1.$$

Then the bundle $P$ may be defined by the projectors

$$P(x)\colon C^N \times C^N \to C^N \times C^N, \quad x \in M,$$
$$P(x) = \begin{bmatrix} f_1^2(x)P_1(x) & f_1(x)f_2(x)b(x) \\ f_1(x)f_2(x)a(x) & f_2^2(x)P_2(x) \end{bmatrix}.$$

(1) B. V. Fedosov, *On the index of an elliptic system on a manifold* (in Russian), Funktsional. Anal. Prilozhen. 4 (4) (1970), 57–67.

We wish to express the differential form $\operatorname{ch} P$ representing the Chern character of $P$ by forms derived directly from $P_1$, $P_2$, $a$, $b$ (see footnote([1])). This is interesting in connection with the Atiyah–Singer index formula (Fedosov, footnote([1])).

According to

$$\operatorname{ch} P = \dim P + \sum_{k=1} \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \operatorname{Tr} w_p^k,$$

where $w_p = PdPdP$ (for simplicity we omit the sign of exterior product) and $\operatorname{Tr} w_p^k$ denotes the trace of the $k$-th exterior power of $w_p$. Thus all we need is to calculate $\operatorname{Tr} w_p^k$.

We do this under the assumption (not very restrictive, in fact) that $P_2$ is a trivial bundle, i.e., that $x \mapsto P_2(x)$ is a constant function.

Setting

$$Q_1 = \begin{bmatrix} d(f_1^2) P_1 & d(f_1 f_2) b \\ d(f_1 f_2) a & d(f_2^2) P_2 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} f_1^2 dP_1 & f_1 f_2 db \\ f_1 f_2 da & 0 \end{bmatrix},$$

after straightforward calculations we obtain the following relations:

$$dP = Q_1 + Q_2,$$

$$PQ_1 + Q_1 P = Q_1,$$

$$Q_2^{2k+1} P = Q_2^{2k+1} - PQ_2^{2k+1}, \quad k = 0, 1, \ldots,$$

$$Q_2^{2k} P = PQ_2^{2k}, \quad k = 1, 2, \ldots,$$

$$R_1 Q_1 R_2 Q_1 R_3 = 0 \quad \text{for all matrix-valued forms } R_1, R_2, R_3,$$

$$PQ_1 - Q_1 P = (f_2 df_1 - f_1 df_2) \begin{bmatrix} 0 & -b \\ a & 0 \end{bmatrix}.$$

Using the above relations we get by induction

$$\operatorname{Tr} w_p^k = \operatorname{Tr} Q_2^{2k} P + k \operatorname{Tr}(PQ_1 - Q_1 P) Q_2^{2k-1}, \quad k = 1, 2, \ldots \tag{1}$$

Now, let $A_l$, $B_l$, $C_l$, $D_l$ be matrix-valued forms such that

$$Q_2^l = \begin{bmatrix} A_l & C_l \\ B_l & D_l \end{bmatrix}, \quad l = 0, 1, \ldots \tag{2}$$

Since $Q_2^{l+1} = Q_2 Q_2^l$ it is easy to get

$$A_{l+2} = f_1^2 dP_1 A_{l+1} + f_1^2 f_2^2 db\, da\, A_l, \tag{3}$$
$$A_0 = E \quad \text{(the unit matrix)},$$
$$A_1 = f_1^2 dP_1,$$

$$B_{l+1} = f_1 f_2 da\, A_l, \tag{4}$$
$$C_{l+2} = f_1^2 dP_1 C_{l+1} + f_1^2 f_2^2 db\, da\, C_l,$$

$$C_0 = 0, \quad C_1 = f_1 f_2 db, \tag{5'}$$

$$D_{l+1} = f_1 f_2 da\, C_l. \tag{6'}$$

By induction, from (3) and (5′) it easily follows that

$$C_{l+1} = f_1 f_2 A_l db, \quad l = 0, 1, \ldots, \tag{5}$$

and thus

$$D_{l+1} = f_1^2 f_2^2 da\, A_{l-1} db, \quad l = 1, 2, \ldots \tag{6}$$

Define matrix-valued forms on $M_{12}$ by

$$v_0 = bda, \quad v_1 = dba.$$

Then we have $dP_1| M_{12} = v_0 + v_1$, $dbda = v_1 v_0$. By (2) and (4)–(6)

$$\operatorname{Tr} Q_2^{2k} P = f_1^2 \operatorname{Tr} P_1 A_{2k} - f_2^2 \operatorname{Tr} A_{2k} + 2f_1^2 f_2^2 \operatorname{Tr} v_0 A_{2k-1},$$
$$\operatorname{Tr}(PQ_1 - Q_1 P) Q_2^{2k-1} = f_1 df_1 \operatorname{Tr}(v_1 - v_0) A_{2k-2}$$

and so (1) may be rewritten as

$$\operatorname{Tr} w_p^k = f_1^2 \operatorname{Tr} P_1 A_{2k} - f_2^2 \operatorname{Tr} A_{2k} + 2f_1^2 f_2^2 \operatorname{Tr} v_0 A_{2k-1} + \\ + kf_1 df_1 \operatorname{Tr} v_1 A_{2k-2} - kf_1 df_1 \operatorname{Tr} v_0 A_{2k-2}. \tag{7}$$

Now we pass to the calculation of $A_l$.

Let $F_l$ be the set of all $l$-tuples consisting of 0 or 1. For $(i_l, i_{l-1}, \ldots, i_1) = I \in F_l$ denote by $n(I)$ the number of all indices $p$, $2 \leqslant p \leqslant k$, such that $i_p = 1$ and $i_{p-1} = 0$.

The function $n(\cdot)$ has the following properties:

$$n(\emptyset) = 0, \quad n(0) = n(1) = 1,$$
$$n(0, I) = n(I), \tag{8}$$
$$n(1, 1, I) = n(1, I),$$
$$n(1, 0, I) = n(0, I) + 1 = n(I) + 1.$$

For $I = (i_l, i_{l-1}, \ldots, i_1) \in F_l$ we set

$$v^I = v_{i_l} v_{i_{l-1}} \ldots v_{i_1}, \qquad V_l(m) = \sum_{\{I \in F_l: n(I) = m\}} v^I.$$

Obviously,

$$(dP_1|\ M_{12})^l = (v_0 + v_1)^l = \sum_{m \geq 0} V_l(m). \tag{9}$$

LEMMA 1.

$$A_l = f_1^{2l}(dP_1)^l + \sum_{m \geq 1} (f_1^{2l-2m} - f_1^{2l}) V_l(m). \tag{10}$$

Proof. Let $A_l'$ denote the right-hand side of the above formula. We shall show that $A_l'$ satisfies (3).

Obviously, this is so for $l = 0$ and $l = 1$. Put

$$F_l^m = \{I \in F_l:\ n(I) = m\}, \qquad F_l^* = \{I \in F_l:\ n(I) > 0\}$$

and observe that in view of (8)

$$F_{l+2}^* = \{(0, I):\ I \in F_{l+1}^*\} \cup \{(1, 0, J):\ J \in F_l\} \cup \{(1, 1, J):\ (1, J) \in F_{l+1}^*\}.$$

We may write

$$A_l' = f_1^{2l}(dP_1)^l + \sum_{I \in F_l^*} (f_1^{2l-2n(I)} - f_1^{2l}) v^I,$$

and so, by (9),

$$f_2^2 A_l' = (1 - f_1^2) \sum_{I \in F_l} f_1^{2l-2n(I)} v^I.$$

Hence we have

$$\begin{aligned}
& f_1^2 dP_1 A_{l+1}' + f_1^2 f_2^2 v_0 A_l' \\
&= f_l^{2l+4}(dP_1)^{l+2} + \sum_{I \in F_{l+1}^*} (f_1^{2l+4-2n(I)} - f_1^{2l+4})(v_0 + v_1) v^I + \\
&\qquad + \sum_{J \in F_l} f_1^{2l+2-2n(J)} v_1 v_0 v^J - \sum_{J \in F_l} f_1^{2l+4-2n(J)} v_1 v_0 v^J \\
&= f_1^{2l+4}(dP_1)^{l+2} + \sum_{I \in F_{l+1}^*} (f_1^{2l+4-2n(0,I)} - f_1^{2l+4}) v^{(0,I)} + \\
&\qquad + \sum_{(1,J) \in F_{l+1}^*} (f_1^{2l+4-2n(1,J)} - f_1^{2l+4}) v^{(1,1,J)} + \\
&\qquad + \sum_{J \in F_l} f_1^{2l+4-2n(0,J)} v^{(1,0,J)} - \sum_{J \in F_l} f_1^{2l+4} v^{(1,0,J)} +
\end{aligned}$$

$$+\sum_{J\in F_l} f_1^{2l+2-2(n(1,0,J)-1)} v^{(1,0,J)} - \sum_{J\in F_l} f_1^{2l+4-2n(J)} v^{(1,0,J)}$$

$$= f_1^{2l+4}(dP_1)^{l+2} + \sum_{(0,I)\in F^*_{l+2}} (f_1^{2l+4-2n(0,I)} - f_1^{2l+4}) v^{(0,I)} +$$

$$+ \sum_{(1,1,J)\in F^*_{l+2}} (f_1^{2l+4-2n(1,1,J)} - f_1^{2l+4}) v^{(1,1,J)} +$$

$$+ \sum_{(1,0,J)\in F^*_{l+2}} (f_1^{2l+4-2n(1,0,J)} - f^{2l+4}) v^{(1,0,J)}$$

$$= f_1^{2l+4}(dP_1)^{l+2} + \sum_{I\in F^*_{l+2}} (f_1^{2l+4-n(I)} - f_1^{2l+4}) v^I = A'_{l+2}.$$

This implies the assertion of the lemma.

COROLLARY.

$$A_l|\ M_{12} = \sum_{m\geq 0} f_1^{2l-2m} V_l(m). \tag{10'}$$

Now consider formula (7) for $\mathrm{Tr}\, w_p^k$.

Since the supports of the forms $df_1$, $f_1^{2k-2m} - f_1^{2k}$ $(k \geq m \geq 1)$, $f_1 f_2$ are contained in $M_{12}$, we may use (10′). We obtain

$$\mathrm{Tr}\, w_p^k = f_1^{4k+2}\, \mathrm{Tr}\, P_1 (dP_1)^{2k} + \tag{11}$$

$$+ \sum_{m\geq 1} (f_1^{4k+2-2m} - f_1^{4k+2})\, \mathrm{Tr}\, ba\, V_{2k}(m) -$$

$$- \sum_{m\geq 0} f_2^2 f_1^{4k-2m}\, \mathrm{Tr}\, V_{2k}(m) + 2 \sum_{m\geq 0} f_2^2 f_1^{4k-2m}\, \mathrm{Tr}\, v_0 V_{2k-1}(m) +$$

$$+ k f_1\, df_1 \sum_{m\geq 0} f_1^{4k-4-2m} \big(\mathrm{Tr}\, v_1 V_{2k-2}(m) - \mathrm{Tr}\, v_0 V_{2k-2}(m)\big).$$

Let us define some new forms.

For positive integers $m, l, r_1, \ldots, r_m$ such that $r_1 + \ldots + r_m = [l/2]$ we put

$$S_l(r_1, \ldots, r_m) = \mathrm{Tr}\, v_1 v_0^{2r_1-1} v_1 v_0^{2r_2-1} \ldots v_1 v_0^{2r_{m-1}-1} v_1 v_0^{2r_m - e(l)},$$

where

$$e(l) = \begin{cases} 1 & \text{for } l \text{ even}, \\ 0 & \text{for } l \text{ odd}, \end{cases}$$

$$T_l(m) = \sum_{r_1+\ldots+r_m=[l/2]} r_m S(r_1, \ldots, r_m), \quad \tilde{T}_l(m) = \sum_{r_1+\ldots+r_m=[l/2]} S(r_1, \ldots, r_m).$$

Also, it is convenient to put

$$T_l(0) = \operatorname{Tr} v_0^l,$$
$$\tilde{T}_l(0) = 0,$$
$$T_l(m) = \tilde{T}_l(m) = 0 \quad \text{for } m > [l/2] \text{ or } m < 0,$$
$$Z_k = \sum_{1 \leqslant m \leqslant k} \frac{k}{2k-m} T_{2k-1}(m-1).$$

The summands in formula (11) may be expressed by $T_{2k}$. In the sequel we shall prove the following result.

LEMMA 2.

$$\operatorname{Tr} V_{2k}(m) = dT_{2k-1}(m-1) - dT_{2k-1}(m),$$
$$\operatorname{Tr} ba\, V_{2k}(m) = -\frac{k-m+1}{2k-m+1} dT_{2k-1}(m-2) + $$
$$+ 2\frac{k-m}{2k-m} dT_{2k-1}(m-1) - dT_{2k-1}(m),$$
$$\operatorname{Tr} v_0 V_{2k-1}(m) = \frac{k-m}{2k-m} dT_{2k-1}(m-1) - dT_{2k-1}(m),$$
$$\operatorname{Tr} v_0 V_{2k-2}(m) = T_{2k-1}(m) = -\operatorname{Tr} v_1 V_{2k-2}(m).$$

Substituting the formulas of Lemma 2 to (11), we find after simple reductions

$$\operatorname{Tr} w_p^k = f_1^{4k+2} \operatorname{Tr} w_{p_1}^k + \sum_{1 \leqslant m \leqslant k} (f_1^{4k+2} - f_1^{4k-2m}) \frac{k}{2k-m} dT_{2k-1}(m-1) -$$
$$- 2kf_1\, df_1 \sum_{1 \leqslant m \leqslant k} f_1^{4k-2m-2} T_{2k-1}(m-1)$$
$$= f_1^{4k+2} \operatorname{Tr} w_{p_1}^k + \sum_{1 \leqslant m \leqslant k} (f_1^{4k+2} - f_1^{4k-2m}) \frac{k}{2k-m} dT_{2k-1}(m-1) +$$
$$+ \sum_{1 \leqslant m \leqslant k} d(f_1^{4k+2} - f_1^{4k-2m}) \frac{k}{2k-m} T_{2k-1}(m-1) -$$
$$- df_1^{4k+2} \sum_{1 \leqslant m \leqslant k} \frac{k}{2k-m} T_{2k-1}(m-1)$$
$$= \operatorname{Tr} w_{p_1}^k f_1^{4k+2} + Z_k\, df_1^{4k+2} +$$
$$+ d\left( \sum_{1 \leqslant m \leqslant k} (f_1^{4k+2} - f_1^{4k-2m}) \frac{k}{2k-m} T_{2k-1}(m-1) \right).$$

Since $\operatorname{supp}(f_1^{4k+2}-f_1^{4k-2m})$ is contained in $M_{12}$, the form

$$Y_k = \sum_{1\leqslant m\leqslant k} (f_1^{4k+2}-f_1^{4k-2m})\frac{k}{2k-m}T_{2k-1}(m-1)$$

is well defined on all of $M$ and has support in $M_{12}$.

On the other hand, we have

$$\operatorname{Tr}(w_{p_1}^k|\ M_{12}) = \operatorname{Tr} ba(v_0+v_1)^{2k} = \sum_{0\leqslant m\leqslant k} \operatorname{Tr} ba\, V_{2k}(m)$$
$$= -\sum_{1\leqslant m\leqslant k}\frac{k}{2k-m}dT_{2k-1}(m-1) = -dZ_k.$$

Now, let $g$ be any smooth function such that $gf_1 = f_1$. Then $\operatorname{supp}(f_1^{4k+2}-g) = \operatorname{supp} g(f_1^{4k+2}-1)$ is contained in $M_{12}$ and thus

$$Z_k\, d(f_1^{4k+2}-g)+\operatorname{Tr} w_{p_1}^k(f_2^{4k+2}-g) = -d(f_1^{4k+2}-g)Z_k-(f_1^{4k+2}-g)dZ_k$$
$$= -dY_k',$$

where $Y_k' = (f_1^{4k+2}-g)Z_k$.

Finally,

$$\operatorname{Tr} w_p^k = Z_k\, dg+g\operatorname{Tr} w_{p_1}^k+d(Y_k-Y_k').$$

Consequently, $\operatorname{Tr} w_p^k$ is homologous to $Z_k dg+g\operatorname{Tr} w_{p_1}^k$, and the Chern character of $P$ can be represented by the form

$$\operatorname{ch} P = g\operatorname{ch} P_1+Zdg, \quad \text{where } Z = \sum_{k=1}^{\infty}\left(\frac{i}{2\pi}\right)^k\frac{1}{k!}Z_k.$$

This is just the formula we wish to get.

If the bundle $P_1$ is trivial (see footnote([1])), then $w_{p_1} = 0$ and thus $\operatorname{ch} P = Zdg$. Moreover, in this case $v_1 = -v_0$ and so we can obtain a more explicit formula for $Z_k$:

$$Z_k = \sum_{1\leqslant m\leqslant k}\frac{k}{2k-m}\sum_{r_1+\ldots+r_{m-1}=k-1}(-1)^{m-1}r_{m-1}\operatorname{Tr} v_0^{2k-1}$$
$$= \sum_{1\leqslant m\leqslant k}(-1)^{m-1}\binom{k-1}{m-1}\frac{k}{2k-m}\operatorname{Tr} v_0^{2k-1}$$
$$= (-1)^{k-1}\frac{k!\,(k-1)!}{(2k-1)!}\operatorname{Tr}(bda)^{2k-1}.$$

This formula has been obtained by Fedosov (see footnote([1]), p. 65).

To complete our calculation we have to prove Lemma 2. For this we need some auxiliary results.

First, we show that the forms $V_l(m)$ are closed.

Let us introduce the following forms on $M_{12}$:

$$u_1 = d(ba), \quad u_2 = v_1 v_0,$$

$$\tilde{U}_l(p) = \begin{cases} \sum\limits_{a_p+\ldots+a_1=k} u_{a_p} u_{a_{p-1}} \ldots u_{a_1} & \text{for } l \geqslant 1,\ [(l+1)/2] \leqslant p \leqslant l, \\ 0 & \text{for } l < 0 \text{ or } p < [(l+1)/2] \text{ or } p > l, \\ E & \text{for } l = p = 0, \end{cases}$$

$$U_l(t) = \tilde{U}_l(l-t).$$

For $U_l(t)$ we have the inductive formula

$$\begin{gathered} U_l(t) = u_1 U_{l-1}(t) + u_2 U_{l-2}(t-1), \\ U_l(t) = 0 \quad \text{for } l < 0 \text{ and all } t, \\ U_0(0) = E, \quad U_0(t) = 0 \quad \text{for all } t \neq 0. \end{gathered} \tag{12}$$

Hence for $l \geqslant 1$

$$U_l(t) = d\big(U^*_{l-1}(t)\big),$$

where

$$U^*_{l-1}(t) = baU_{l-1}(t) + v_0 U_{l-2}(t-1). \tag{13}$$

From (12) and (3) it follows by induction that

$$A_l| \ M_{12} = \sum_{0 \leqslant t \leqslant [l/2]} f_1^{2l-2t} f_2^{2t} U_l(t).$$

Comparing the two formulas for $A_l$ we obtain a relation between $U_l$ and $V_l$.

LEMMA 3. *The following equalities are valid on* $M_{12}$:

$$U_l(t) = \sum_{t \leqslant m \leqslant [l/2]} \binom{m}{t} V_l(m), \tag{14}$$

$$V_l(m) = \sum_{m \leqslant t \leqslant [l/2]} (-1)^{m+t} \binom{t}{m} U_l(t). \tag{15}$$

Proof.

$$\sum_{m \geqslant 0} f_1^{2l-2m} V_l(m) = A_l| \ M_{12} = \sum_{0 \leqslant t \leqslant [l/2]} f_1^{2l-2t} (1-f_1^2)^t U_l(t)$$

$$= \sum_{0 \leqslant m \leqslant [l/2]} f_1^{2l-2m} \sum_{m \leqslant t \leqslant [l/2]} (-1)^{t+m} \binom{t}{m} U_l(t).$$

In a small neighbourhood of any given point $x \in M_{12}$ the function $f_1$ may be chosen arbitrarily. Thus in the above equality the coefficients at the same powers of $f_1$ must be equal. This proves the first assertion of lemma.

The second assertion follows directly from the first.

COROLLARY. $dV_l(m) = 0$.

Proof.

$$dV_l(m) = \sum_{m \leqslant t \leqslant [l/2]} (-1)^{m+t} \binom{t}{m} dU_l(t) = 0.$$

Remark. From (15) and (13) it follows easily that $V_l(m) = dV^*_{l-1}(m)$, where

$$V^*_{l-1}(m) = baV_{l-1}(m) - v_0 V_{l-2}(m) + v_0 V_{l-2}(m-1).$$

LEMMA 4.

$$\operatorname{Tr} V_l(m) = \bigl(2T_l(m) - \tilde{T}_l(m) - 2T_l(m+1) + \tilde{T}_l(m+1)\bigr)e(l),$$

$$\operatorname{Tr} baV_l(m) = \bigl(-T_l(m-1) + \tilde{T}_l(m-1) + 2T_l(m) - {}$$

$$- 2\tilde{T}_l(m) - 2T_l(m+1) + \tilde{T}_l(m+1)\bigr)e(1),$$

$$\operatorname{Tr} v_0 V_{l-1}(m) = T_l(m) - \bigl(\tilde{T}_l(m) + 2T_l(m+1) - \tilde{T}_l(m+1)\bigr)e(l)$$

$$= -\operatorname{Tr} v_1 V_{l-1}(m),$$

$$\operatorname{Tr} v_1 v_0 V_{l-2}(m) = \bigl(2T_{2k}(m+1) - \tilde{T}_{2k}(m+1)\bigr)e(l).$$

*All these formulas hold for* $m \geqslant 0$.

Proof. Write $V\binom{p}{q} = v_1^p v_0^q$ for $p, q \geqslant 0$. Then

$$V_l^{(1)}(m) = \sum V\binom{0}{q_0} V\binom{p_1}{q_1} \ldots V\binom{p_m}{q_m} V\binom{p_{m+1}}{0},$$

$$V_l^{(2)}(m) = \sum V\binom{0}{q_0} V\binom{p_1}{q_1} \ldots V\binom{p_m}{q_m},$$

$$V_l^{(3)}(m) = \sum V\binom{p_1}{q_1} \ldots V\binom{p_m}{q_m} V\binom{p_{m+1}}{0},$$

$$V_l^{(4)}(m) = \sum V\binom{p_1}{q_1} \ldots V\binom{p_m}{q_m}.$$

Here in each sum all $p_i$, $q_i$ are positive (integers) and their sum is equal to $l$.

Writing ($i = 1, \ldots, k$)

$$V_l^{(i0)}(m) = \operatorname{Tr} V_l^{(i)}(m), \qquad V_l^{(i1)}(m) = \operatorname{Tr} ba V_l^{(i)}(m), \qquad V_l^{(i2)}(m) = \operatorname{Tr} v_0 V_{l-1}^{(i)}(m),$$
$$V_l^{(i3)}(m) = \operatorname{Tr} v_1 V_{l-1}^{(i)}(m), \qquad V_l^{(i4)}(m) = \operatorname{Tr} v_1 v_0 V_{l-2}^{(i)}(m),$$

we see that $\sum_{l=1}^{4} V_l^{(ij)}(m)$ for $j = 0, 1, \ldots, 4$ is equal to $\operatorname{Tr} V_l(m)$, $\operatorname{Tr} ba V_l(m)$, $\operatorname{Tr} v_0 V_{l-1}(m)$, $\operatorname{Tr} v_1 V_{l-1}(m)$ and $\operatorname{Tr} v_1 v_0 V_{l-1}(m)$ respectively.

To compute $V_l^{(ij)}(m)$ we use the following easy facts:

(a) $bav_0 = v_0$, $v_1 ba = v_1$, $v_0 ba = -bav_1$.

(b) $V\binom{p}{q} = (-1)^{p-1} V\binom{1}{p+q-1}$ for $p, q \geqslant 1$.

(c) $\operatorname{Tr} AB = (-1)^{\deg A \deg B} \operatorname{Tr} BA$ for all matrix-valued forms $A$, $B$.

It is now easy to see that each $V_l^{(ij)}(m)$ can be written as a sum of the form

$$V_l^{(ij)}(m) = \sum_{p_1 + \ldots + p_{\bar{m}} + q_1 + \ldots + q_{\bar{m}} = l} c(p_{\bar{m}}, q_{\bar{m}}) \operatorname{Tr} V\binom{p_1}{q_1} \ldots V\binom{p_{\bar{m}}}{q_{\bar{m}}}$$

with $\bar{m}$ equal to one of the numbers: $m-1$, $m$, $m+1$.

On account of (b) and the equality $\sum_{1 \leqslant p \leqslant s-1} (-1)^{p-1} = e(s)$ we have

$$V_l^{(ij)}(m) = \sum_{p_1 + \ldots + p_{\bar{m}} + q_1 + \ldots + q_{\bar{m}} = l} (-1)^{p_1 + \ldots + p_{\bar{m}} - \bar{m}} c(p_{\bar{m}}, q_{\bar{m}}) \times$$
$$\times \operatorname{Tr} V\binom{1}{p_1 + q_1 - 1} \ldots V\binom{1}{p_{\bar{m}} + q_{\bar{m}} - 1}$$
$$= \sum_{s_1 + \ldots + s_{\bar{m}} = l} e(s_1) \ldots e(s_{\bar{m}-1}) c(s_{\bar{m}}) \operatorname{Tr} V\binom{1}{s_1 - 1} \ldots V\binom{1}{s_{\bar{m}} - 1}$$
$$= \sum_{r_1 + \ldots + r_m = [l/2]} c_{ij}(r_{\bar{m}}) S_l(r_1, \ldots, r_{\bar{m}}).$$

Here $c(p_{\bar{m}}, q_{\bar{m}})$, $c(s_{\bar{m}})$, $c_{ij}(r_{\bar{m}})$ are some numbers.

For instance, for $i = 1$, $j = 0$, $m \geqslant 1$ we have

$$V_l^{(10)}(m) = \sum_{\sum p_i + \sum q_i = l} \operatorname{Tr} V\binom{0}{q_0} V\binom{p_1}{q_1} \ldots V\binom{p_m}{q_m} V\binom{p_{m+1}}{0}$$
$$= \sum_{p + \sum p_i + q + \sum q_i = l} (-1)^{q(l+1)} \operatorname{Tr} V\binom{p_1}{q_1} \ldots V\binom{p_m}{q_m} V\binom{p}{q}$$

$$= \sum_{p+\sum p_i+q+\sum q_i=l} (-1)^{ql+q+p-1}(-1)^{p_1+\dots+p_m-1}\times$$

$$\times \operatorname{Tr} V\begin{pmatrix}1\\ p_1+q_1-1\end{pmatrix}\dots V\begin{pmatrix}1\\ p+q-1\end{pmatrix}$$

$$= \sum_{\substack{s_1+\dots+s_{\bar{m}+1}=l\\ p+q=s_{m+1}}} (-1)^{ql+s_{m+1}-1}(-1)^{p_1-1}\dots(-1)^{p_m-1}\times$$

$$\times \operatorname{Tr} V\begin{pmatrix}1\\ s_1-1\end{pmatrix}\dots V\begin{pmatrix}1\\ s_{m+1}-1\end{pmatrix}$$

$$= \sum_{\substack{s_1+\dots+s_{m+1}=l\\ p+q=s_{m+1}}} (-1)^{ql+s_{m+1}-1} e(s_1)\dots e(s_m) \operatorname{Tr} V\begin{pmatrix}1\\ s_1-1\end{pmatrix}\dots V\begin{pmatrix}1\\ s_{m+1}-1\end{pmatrix}$$

$$= \sum_{\substack{r_1+\dots+r_{m+1}=[l/2]\\ p+q=2r_{m+1}+1-e(l)}} (-1)^{ql+2r_{m+1}+e(l)} \operatorname{Tr} V\begin{pmatrix}1\\ 2r_1-1\end{pmatrix}\dots V\begin{pmatrix}1\\ 2r_{m+1}-e(l)\end{pmatrix}$$

$$= \sum_{\substack{r_1+\dots+r_{m+1}=[l/2]\\ 1\leqslant q\leqslant 2r_{m+1}-e(l)}} (-1)^{ql+l+1} S_l(r_1, \dots, r_{m+1})$$

$$= \sum_{r_1+\dots+r_{m+1}=[l/2]} (1-2r_{m+1})e(l) S_l(r_1, \dots, r_{m+1}).$$

Thus

$$c_{10}(r_{\bar{m}}) = c_{10}(r_{m+1}) = (1-2r_{m+1})e(l)$$

and

$$V_l^{(10)}(m) = e(l) \sum_{r_1+\dots+r_{m+1}=[l/2]} S_l(r_1, \dots, r_{m+1}) -$$

$$-2e(l) \sum_{r_1+\dots+r_{m+1}=[l/2]} r_{m+1} S_l(r_1 \dots r_{m+1})$$

$$= \big(\tilde{T}_l(m+1) - 2T_l(m+1)\big)e(l).$$

In a similar way we obtain (for $m \geqslant 1$)

$$c_{11} = c_{10} = (1-2r_{m+1})e(l),$$

$$V_l^{(1,1)}(m) = V_l^{(10)}(m) = \big(\tilde{T}_l(m+1) - 2T_l(m+1)\big)e(l),$$

$$c_{12} = \sum_{p+q=2r_{m+1}-e(l)} (-1)^{(q+1)(l+1)+p-1} = 1+(1-2r_{m+1})e(l),$$

$$V_l^{(12)}(m) = \tilde{T}_l(m+1) + \big(\tilde{T}_l(m+1) - 2T_l(m+1)\big)e(l),$$

$$c_{13} = \sum_{p+q=2r_{m+1}-e(l)} (-1)^{(l+1)(q+1)+p} = -1+(2r_{m+1}-1)e(l),$$

$$V_l^{(13)}(m) = -\tilde{T}_l(m+1)+\big(2T_l(m+1)-\tilde{T}_l(m+1)\big)e(l),$$

$$c_{20} = c_{21} = \sum_{p+q+t=2r_m+1-e(l)} (-1)^{q(l+1)+p-1} = r_m - e(l),$$

$$V_l^{(20)}(m) = T_l(m)-\tilde{T}_l(m)e(l) = V_l^{(21)}(m),$$

$$c_{22} = \sum_{p+q+s=2r_m-e(l)} (-1)^{(q+1)(l+1)+p-1} = r_m-1,$$

$$V_l^{(22)}(m) = T_l(m)-\tilde{T}_l(m),$$

$$c_{23} = \sum_{q=2r_{m+1}-e(l)} (-1)^{(q+1)(l+1)} = 1,$$

$$V_l^{(23)}(m) = \tilde{T}_l(m+1),$$

$$c_{30} = c_{31} = \sum_{p+q+s=2r_m+1-e(l)} (-1)^{(p+q)(l+1)+p+s-1}$$
$$= -r_m+(2r_m-1)e(l),$$

$$V_l^{(30)}(m) = V_l^{(31)}(m) = -T_l(m)+\big(2T_l(m)-\tilde{T}_l(m)\big)e(l),$$

$$c_{32} = \sum_{p=2r_{m+1}-e(l)} (-1)^{l+p} = -1,$$

$$V_l^{(32)}(m) = -\tilde{T}_l(m+1),$$

$$c_{33} = \sum_{p+q+s=2r_m-e(l)} (-1)^{(p+q+1)(l+1)+p+s} = 1-r_m,$$

$$V_l^{(33)}(m) = \tilde{T}_l(m)-T_l(m),$$

$$c_{40} = \sum_{p+q=2r_m+1-e(l)} (-1)^{p-1} = e(l),$$

$$V_l^{(40)}(m) = e(l)\,\tilde{T}_l(m),$$

$$c_{41} = \sum_{p+q+s+t=2r_{m-1}+1-e(l)} (-1)^{(p+q)(l+1)+p+s-1} = (1-r_{m-1})e(l),$$

$$V_l^{(41)}(m) = \big(\tilde{T}_l(m-1)-T_l(m-1)\big)e(l),$$

$$c_{42} = \sum_{p+q=2r_m-e(l)} (-1)^{l+p} = 1-e(l),$$

$$V_l^{(42)}(m) = \tilde{T}_l(m)-e(l)\,\tilde{T}_l(m),$$

$$c_{43} = \sum_{p+q=2r_m-e(l)} (-1)^{(p+q+1)(l+1)+p} = e(l)-1,$$

$$V_l^{(43)}(m) = e(l)\,\tilde{T}_l(m)-\tilde{T}_l(m).$$

Summing up the suitable terms we get the first, second and third formulas of Lemma 4 for $m \geqslant 1$. For $m = 0$ the calculation needs slight modifications but is very simple. Also, the calculation of $\operatorname{Tr} v_1 v_0 V_{l-2}$ can be done in a very similar way, but now new forms appear:

$$T_l'(m) = \sum_{r_1 + \ldots + r_{m-1} = [l/2]-1} S_l(r_1, \ldots, r_{m-1}, 1).$$

We get

$$V_l^{(14)}(m) = e(l)\big(2T_l(m+1) - 3T_l(m+1) + T_l'(m+1)\big),$$
$$V_l^{(24)}(m) = \tilde{T}_l(m+1) - T_l'(m+1),$$
$$V_l^{(34)}(m) = \big(\tilde{T}_l(m+1) - T_l'(m+1)\big)\big(2e(l) - 1\big),$$
$$V_l^{(44)}(m) = e(l)\, T_l'(m+1),$$

and thus

$$\operatorname{Tr} v_1 v_0 V_{l-2}(m) = V_l^{(14)}(m) + \ldots + V_l^{(44)}(m) = e(l)\big(2T_l(m+1) - \tilde{T}_l(m+1)\big)$$

as asserted.

LEMMA 5. *For all* $m \geqslant 0$, $k \geqslant 0$

$$\tilde{T}_{2k}(m) = \frac{m}{k} T_{2k}(m).$$

Proof. It is easy to see that for any fixed $j$, $1 \leqslant j \leqslant m$,

$$T_{2k}(m) = \sum_{r_1 + \ldots + r_m = k} r_m S_{2k}(r_1, \ldots, r_m) = \sum_{r_1 + \ldots + r_m = k} r_j S_{2k}(r_1, \ldots, r_m).$$

Thus

$$k\tilde{T}_{2k}(m) = \sum_{r_1 + \ldots + r_m = k} k S_{2k}(r_1, \ldots, r_m) = \sum_{r_1 + \ldots + r_m = k} (r_1 + \ldots + r_m) S_{2k}(r_1, \ldots, r_m) = m \sum_{r_1 + \ldots + r_m = k} r_m S_{2k}(r_1, \ldots, r_m) = mT_{2k}(m)$$

as asserted.

LEMMA 6.

$$dT_{2k}(m) = 0,$$

$$dT_{2k-1}(m) = 2T_{2k}(m+1) - \tilde{T}_{2k}(m+1) = \frac{2k-m-1}{k} T_{2k}(m+1).$$

Proof. Since $dv_1 = -v_1 v_0 = -dv_0$, a simple computation gives $dV\binom{1}{1} = 0$ and for $r > 1$

$$dV\binom{1}{2r-1} = \sum_{\substack{s+t=2r-1\\ s,t\geq 1}} (-1)^{s+1} V\binom{1}{s} V\binom{1}{t}$$
$$= \sum_{\substack{p+q=r\\ p,q\geq 1}} \left(V\binom{1}{2p-1} V\binom{1}{2q} - V\binom{1}{2p} V\binom{1}{2q-1}\right).$$

Using this, it is easy to get $d\tilde{T}_{2k}(m) = 0$ and thus by Lemma 5, $dT_{2k}(m) = 0$.

The second assertion follows from Lemmas 4 and 3. We have $T_{2k-1}(m) = \operatorname{Tr} v_0 V_{2k-2}(m)$ and hence

$$dT_{2k-1}(m) = \operatorname{Tr} dv_0 V_{2k-2}(m) - \operatorname{Tr} v_0 dV_{2k-2}(m) = \operatorname{Tr} v_1 v_0 V_{2k-2}(m)$$
$$= 2T_{2k}(m+1) - \tilde{T}_{2k}(m+1) = \frac{2k-m-1}{k} T_{2k}(m+1).$$

Now we can give the proof of Lemma 2. By Lemmas 4, 5 and 6 we have

$$\operatorname{Tr} V_{2k}(m) = 2T_{2k}(m) - \tilde{T}_{2k}(m) - \left(2T_{2k}(m+1) - \tilde{T}_{2k}(m+1)\right)$$
$$= dT_{2k-1}(m-1) - dT_{2k-1}(m),$$

$$\operatorname{Tr} baV_{2k}(m) = -\frac{k-m+1}{k} T_{2k}(m-1) + $$
$$+2\frac{k-m}{k} T_{2k}(m) - \frac{2k-m-1}{k} T_{2k}(m+1)$$
$$= -\frac{k-m+1}{2k-m+1} dT_{2k-1}(m-2) +$$
$$+2\frac{k-m}{2k-m} dT_{2k-1}(m-1) - dT_{2k-1}(m),$$

$$\operatorname{Tr} v_0 V_{2k-1}(m) = \frac{k-m}{k} T_{2k}(m) - \frac{2k-m-1}{k} T_{2k}(m+1)$$
$$= \frac{k-m}{2k-m} dT_{2k-1}(m-1) - dT_{2k-1}(m),$$

$$\operatorname{Tr} v_0 V_{2k-2}(m) = -\operatorname{Tr} v_1 V_{2k-2}(m) = T_{2k-1}(m).$$

The proof is complete.

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