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A new characterization of the sphere in R^3

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Abstract. Let M be a closed connected surface in R^3 with positive Gaussian curvature K and let K_{II} be the curvature of its second fundamental form. It is shown that M is a sphere if $K_{II} = c\sqrt{H}K^r$, for some constants c and r, where H is the mean curvature of M.

1. Introduction. A closed connected surface in R^3 with Gaussian curvature $K > 0$ throughout, an ovaloid in short, possesses a positive-definite second fundamental form II, if appropriately oriented. We denote by K_{II} the Gaussian curvature of the second fundamental form II and by H the mean curvature of M. Many authors have concerned themselves with the problem of characterization of the sphere by the curvature of the second fundamental form. In [2] Schneider has shown that the constancy of the curvature K_{II} implies that M is a sphere. Koutroufiotis [1] has shown that M is a sphere if $K_{II} = cK$ or $K_{II} = \sqrt{K}$ for some constant c. Stamou [3] has shown that M is a sphere if $K_{II} = cH\sqrt{K}$ or $K_{II} = cH/\sqrt{K}$ for some constant c. Also Stamou shows in [3] that M is a sphere if $K_{II} = cK^r$, for some constants c and r. This last result gives the results of [1] and [2] for appropriate constants c and r.

The purpose of this note is to prove the following

MAIN THEOREM. *Let M be an ovaloid in R^3. If $K_{II} = c\sqrt{H}K^r$, where c and r are constants, then M is a sphere.*

Remark. Obviously the constant c must be positive, as follows from the Gauss–Bonnet theorem.

2. Preliminaries. Let (u^i) be local coordinates on M and let Γ_{ij}^k, ∇_I [resp. ${}_{II}\Gamma_{ij}^k$, ∇_{II}] denote the Christoffel symbols and the first Beltrami operator relative to the first fundamental form I [resp. to the second fundamental form II]. If

$$T_{ij}^k = \Gamma_{ij}^k - {}_{II}\Gamma_{ij}^k \quad (i, j, k = 1, 2),$$

then by making use of the second fundamental tensor b_{ij} for "raising and lowering indices" we conclude ([2]) that $T_{ijk} = T_{ij}^h b_{hk}$ is totally symmetric and

$$K_{II} = H + \tfrac{1}{2} T_{ijk} T^{ijk} - \left(\frac{1}{8K^2}\right) \nabla_{II} K. \tag{1}$$

3. Main results. If we denote by dA and dA_{II} the area elements of M with respect to the first and the second fundamental form, it is obvious that $dA_{II} = \sqrt{K}\,dA$. Then by the Gauss–Bonnet theorem we have

$$4\pi = \int_M K\,dA = \int_M K_{II}\,dA_{II} = \int_M K_{II}\sqrt{K}\,dA$$

or

$$\int_M \sqrt{K}(\sqrt{K} - K_{II})\,dA = 0. \tag{2}$$

Proof of the Main Theorem. From equation (2) for $K_{II} = c\sqrt{H}K^r$ we get

$$\int_M \sqrt{K}(K - c\sqrt{H}K^r)\,dA = 0. \tag{3}$$

Now, let P_1 be a critical point of K; then by (1) we have

$$K_{II}(P_1) \geqslant H(P_1)$$

or

$$c\sqrt{H(P_1)}K^r(p_1) \geqslant H(P_1) \geqslant \sqrt{K(P_1)}, \qquad \text{since } H \geqslant \sqrt{K}.$$

Hence

$$\sqrt{H(P_1)}\left(cK^r(P_1) - \sqrt{H(P_1)}\right) \geqslant 0,$$

and so

$$cK^r(P_1) \geqslant \sqrt{H(P_1)}, \qquad \text{since } \sqrt{H(P_1)} > 0.$$

Moreover, since $\sqrt{H} \geqslant K^{1/4}$ everywhere on M, we obtain the following relation:

$$cK^r(P_1) \geqslant \sqrt{H(P_1)} \geqslant K^{1/4}(P_1),$$

or

$$K^r(P_1)\left(c - K^{1/4}(P_1)\right) \geqslant 0,$$

that is

$$K^{1/4-r}(P_1) \leqslant c, \tag{4}$$

because $K^r(P_1) > 0$.

We distinguish the following two cases.

Case 1. Let $r \leqslant \frac{1}{4}$ and so $\frac{1}{4}-r \geqslant 0$. In that case, we choose as P_1 a point such that $K(P_1) = \operatorname{Sup}_{P\in M} K(P)$ (at least one such point $P \in M$ necessarily exists, since M is closed and K is a continuous function). Then from (4) we conclude that

$$K^{1/4-r} \leqslant c \quad \text{everywhere on } M. \tag{5}$$

Case 2. Let $r > \frac{1}{4}$ and thus $\frac{1}{4}-r < 0$. In that case, we choose as P_1 a point such that $K(P_1) = \min_{P\in M} K(P)$. Then from (4) we again obtain

$$K^{1/4-r} \leqslant c \quad \text{everywhere on } M. \tag{6}$$

Generally, we conclude that for all values of r it holds

$$K^{1/4-r} \leqslant c \quad \text{everywhere on } M. \tag{7}$$

Moreover, since $\sqrt{H} \geqslant K^{1/4}$, we have

$$K^{1/4}\cdot K^{1/4-r}\cdot K^{r} \leqslant c\sqrt{H}K^{r},$$

that is

$$\sqrt{K} \leqslant c\sqrt{H}K^{r},$$

or, equivalently,

$$\sqrt{K} - c\sqrt{H}K^{r} \leqslant 0.$$

Since the function $\sqrt{K} - c\sqrt{H}K^{r}$ is non-positive, we get from (3)

$$\sqrt{K} = c\sqrt{H}K^{r}.$$

Thus

$$K_{II} = c\sqrt{H}K^{r} = \sqrt{K}.$$

But from a well-known result ([1], p. 177) we conclude that M is a sphere, because $K_{II} = \sqrt{K}$.

This completes the proof of the theorem.

References

[1] D. Koutroufiotis, *Two characteristic properties of the sphere*, Proc. Amer. Math. Soc. 44 (1974), p. 176–178.

[2] R. Schneider, *Closed convex hypersurfaces with second fundamental form of constant curvature*, ibidem 35 (1972), p. 230–233.

[3] G. Stamou, *Global characterizations of the sphere* (to appear).

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