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Nodes of eigenfunctions of a certain class of ordinary differential equations of the fourth order

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Abstract. In this paper we shall consider the problem of eigenvalues and eigenfunctions for a certain class of ordinary differential equations of the fourth order with boundary conditions of the first kind. We prove some theorems about multiplicity of zero-points of eigenfunctions and about multiplicity of eigenvalues. Finally we prove the main theorem: The n-th eigenfunction of this problem has in interval (a, b) exactly $n-1$ zero-points.

Introduction. Let $\mathfrak{M}$ be the set of functions of clas $C^4([a, b])$ satisfying the following boundary conditions:

$$u(a) = u'(a) = u(b) = u'(b) = 0. \tag{1}$$

Consider in $\mathfrak{M}$ a differential operator of the fourth order of the form

$$Lu = L_1 L_2 u, \tag{2}$$

where $L_k\varphi$ $(k = 1, 2)$ denotes the following differential operators of the Sturm–Liouville type, i.e.

$$L_k\varphi = -[p_k(x)\varphi']' + q_k(x)\varphi \quad (k = 1, 2). \tag{3}$$

Let us assume that $p_k(x) \in C^{2k-1}([a, b])$, $q_k(x) \in C^{2k-2}([a, b])$, $p_k(x) > 0$, $q_k(x) \geqslant 0$ for $x \in [a, b]$.

We shall consider the eigenvalues and eigenfunctions for the differential equation

$$Lu - \mu\varrho(x)u = 0 \tag{4}$$

with boundary conditions (1), where Lu is defined by (2), μ is a real parameter, and $\varrho(x) > 0$ is a continuous function in $[a, b]$.

DEFINITION. We say that a real number λ is an *eigenvalue* of problem (4), (1) if there exists a function $u(x) \not\equiv 0$ on $\mathfrak{M}$ such that (4) holds for $\mu = \lambda$, $u(x)$ is called an *eigenfunction* corresponding to the eigenvalue λ.

The present paper deals with some properties of eigenvalues and eigenfunctions of problem (4), (1).

Our main theorem reads as follows:

Under some assumptions about the operator L in equation (4), the n-th eigenfunction of problem (4), (1) has in interval (a, b) exactly $n-1$ zero-points (see Theorem 6).

1. The auxiliary lemmas and theorems. Let $f(x)$ be a continuous function in $[a, b]$. Put

$$A = \{x : x \in (a, b);\ f(x) = 0\} \quad \text{(cf. [5])}.$$

A is the sum of separate intervals closed in (a, b). Let M denote the set of all subintervals in A. $M = M_1 + M_2$, where M_1 contains subintervals reduced to a single point or such, which are not closed in $[a, b]$, $M_2 = M - M_1$. Let us put

$$Z(f) = \begin{cases} l+2k & \text{if } M_1 \text{ and } M_2 \text{ are finite sets,} \\ +\infty & \text{if } M_1 \text{ or } M_2 \text{ is an infinite set,} \end{cases}$$

where l and k represent the powers of M_1 and M_2 respectively. If $f(x)$ has a finite number of isolated zero-points in interval (a, b), then $Z(f)$ denotes this number.

LEMMA 1. *If $f(x) \in \mathfrak{M}$ has only a finite number of isolated zero-points in (a, b) equal to s (with multiplicity at most 4), then the number of zero-points of function $h(x) = Lf(x)$ is equal at least to $(s+r)$, where r denotes the number of zero-points of the function $f(x)$ in (a, b) with multiplicity greater than one.*

Proof. Let $a = x_0 < x_1 < \ldots < x_s < x_{s+1} = b$ denote zero-points of $f(x)$ in $[a, b]$. Put $g(x) = L_2 f(x)$. Since the multiplicity of $x_0 = a$ and $x_{s+1} = b$ is at least 2 then, by Lemma 2.3 and Theorem 3.1 of [5], we have $Z(g) \geqslant s+r+2$. Let $t_1, t_2, \ldots, t_{s+r+2}$ be zero-points of $g(x)$ in (a, b), where $a < t_1 < t_2 < \ldots < t_{s+r+2} < b$. By Lemma 2.4 of [5] applied to $g(x)$ in the interval $[t_1, t_{s+r+2}]$, we have that $h(x) = L_1 g(x)$ has at least $s+r$ zero-points in (t_1, t_{s+r+2}). On the other hand, we have $h(x) = L_1 g(x) = L_1 L_2 f(x) = Lf(x)$ and $(t_1, t_{s+r+2}) \subset (a, b)$, thus $h(x) = Lf(x)$ has at least $s+r$ zero-points in (a, b), which completes the proof.

LEMMA 2. *If $f(x)$ satisfies the assumptions of Lemma 1 and the multiplicity of a or b is 3 or 4, then $Z(Lf) \geqslant s+r+1$ (s, r as in Lemma 1).*

Proof. Let $a = x_0 < x_1 < \ldots < x_s < x_{s+1} = b$ be zero-points of $f(x)$ in $[a, b]$. Write $g(x) = L_2 f(x)$. Suppose that the multiplicity of a is 3 or 4. Of course a is also a zero-point of $g(x)$. By Lemma 1 $g(x)$ has at least $s+r+2$ zero-points in (a, b), and $g(a) = 0$. Applying Lemma 2.4 of [5] to $g(x)$ in the interval $[a, t_{s+r+2}]$, we have that $h(x) = Lf(x) = L_1 g(x)$ has at least $s+r+1$ zero-point in (a, t_{s+r+2}). By reasoning as in Lemma 1, we have $Z(h) \geqslant s+r+1$. The proof is analogous for b of multiplicity of 3 or 4.

LEMMA 3. *If* $f(x) \in \mathfrak{M}$, *then* $Z(Lf \geqslant Z(f)$.

The proof of this lemma is quite similar to that of an analogous lemma in [5], and is omitted.

2. Multiplicity of zero-points of eigenfunctions and multiplicity of eigenvalues of problem (4), (1). Let $u(x)$ be an eigenfunction of problem (4), (1). By the well-known theorem on the uniqueness of solution of Cauchy's problem for equation (4), we have that $u(x)$ has no zero-point of multiplicity greater than 3 in $[a, b]$, and that the number of zero-points of $u(x)$ in $[a, b]$ is finite.

We shall prove the following:

THEOREM 1. *If $u(x)$ is an eigenfunction of problem* (4), (1) *corresponding to an eigenvalue $\lambda \neq 0$, then all zero-points of $u(x)$ in (a, b) are single.*

Proof. Suppose $x_0 \in (a, b)$ is a zero-point of $u(x)$ with multiplicity greater than one. By Lemma 1 we get $Z(Lu) \geqslant Z(u)+1$. On the other hand, according to (4), we have $Lu(x) = \lambda\varrho(x)u(x)$. Since $\varrho(x) > 0$ in $[a, b]$ and $\lambda \neq 0$, $Z(Lu) = Z(u)$ — a contradiction.

THEOREM 2. *Under the assumptions of Theorem* 1, *the points a and b are the zero-points of $u(x)$ with multiplicity equal to* 2.

The proof of the above theorem follows from Lemma 2, the same as Theorem 1 follows from Lemma 1.

THEOREM 3. *Any two eigenfunctions of problem* (4), (1) *corresponding to the same eigenvalue $\lambda \neq 0$, are linearly dependent.*

Proof. Let $u(x)$ and $v(x)$ be two eigenfunctions of problem (4), (1) corresponding to an eigenvalue $\lambda \neq 0$. Suppose $u(x)$ and $v(x)$ are linearly independent in $[a, b]$. Put $w(x) = u(x) - \alpha u(x)$, where $\alpha = u''(a)/v''(a)$. Suppose $w(x) \not\equiv 0$ in $[a, b]$, and $w(a) = w'(a) = w''(a) = 0$. Hence $w(x) \in \mathfrak{M}$, and satisfies (4) for $\mu = \lambda$. This means that $w(x)$ is an eigenfunction of problem (4), (1) corresponding to a eigenvalue $\lambda \neq 0$, which has at the point a a zero-point with multiplicity at least equal 3. We obtain a contradiction with Theorem 2.

Theorem 3 implies

COROLLARY 1. *Every non-zero eigenvalue of problem* (4), (1) *is a single eigenvalue.*

In the sequel we shall need the following lemmas.

LEMMA 4. *Suppose $\{f_n(x)\}$ is a sequence of functions such that*:

1° $f_n(x) \in \mathfrak{M}$;

2° $\{f_n(x)\}$, $\{f_n'(x)\}$ *and* $\{f_n''(x)\}$ *tend uniformly to* $f(x), f'(x)$ *and* $f''(x)$ *respectively*;

3° $f(x) \in \mathfrak{M}$ *and $f(x)$ has p zero-points in (a, b), $p < \infty$, which are not zero-points of $f'(x)$ and $f''(a)f''(b) \neq 0$.*

Then the number of zero-point in (a, b) for every $f_n(x)$, for sufficiently large n, is equal to p.

The proof is similar to the analogous lemma in [1], and is omitted.

LEMMA 5 (cf. [5]). *Let $\{f_n(x)\}$ be a sequence of continuous functions in $[a, b]$ tending uniformly in $[a, b]$ to the function $f(x)$ of class $C^1([a, b])$, and let $f(x)$ have a finite number of single zero-points in (a, b). Then for sufficiently large n we have $Z(f_n) \geqslant Z(f)$.*

Let $u_m(x), u_{m+1}(x), \ldots, u_n(x)$ $(n \geqslant m)$ be eigenfunctions of problem (4), (1) corresponding to eigenvalues $\lambda_m, \lambda_{m+1}, \ldots, \lambda_n$, where $0 < |\lambda_m| < |\lambda_{m+1}| < \ldots < |\lambda_n|$. Put

$$f(x) = c_m u_m(x) + c_{m+1} u_{m+1}(x) + \ldots + c_n u_n(x), \tag{5}$$

$c_m, c_{m+1}, \ldots, c_n$ being real constants such that $c_m^2 + c_{m+1}^2 + \ldots + c_n^2 > 0$ and $c_m c_n \neq 0$.

Analogously as in [1] we shall prove the following:

THEOREM 4. *If $f(x)$ denotes the function defined by* (5), *then we have*

$$Z(u_m) \leqslant Z(f) \leqslant Z(u_n). \tag{6}$$

Proof. It follows from (5) that $f(x)$ satisfies the assumptions of Lemma 3. Let us put

$$f_1(x) = \frac{1}{\lambda_n \varrho(x)} Lf(x) = \frac{\lambda_m}{\lambda_n} c_m u_m(x) + \ldots + c_n u_n(x).$$

By Lemma 3, we get $Z(f_1) \geqslant Z(f) \cdot f_1(x)$ has the same form as $f(x)$ and satisfies the assumptions of Lemma 3. Applying again Lemma 3 to $f_1(x)$, we have for

$$f_2(x) = \left(\frac{\lambda_m}{\lambda_n}\right)^2 c_m u_m(x) + \ldots + c_n u_n(x),$$

the inequality $Z(f_2) \geqslant Z(f_1)$.

Proceeding thus further we get an infinite sequence of functions

$$f(x), f_1(x), f_2(x), \ldots, \tag{7}$$

for which the number of zero-points does not decrease with the increase of the index.

Put

$$f_\nu(x) = \left(\frac{\lambda_m}{\lambda_n}\right)^\nu c_m u_m(x) + \left(\frac{\lambda_{m+1}}{\lambda_n}\right)^\nu c_{m+1} u_{m+1}(x) + \ldots + c_n u_n(x). \tag{8}$$

Since $|\lambda_s/\lambda_n| < 1$ $(s = m, m+1, \ldots, n-1)$ and $u_s(x)$, $u_s'(x)$ and $u_s''(x)$ $(s = m, m+1, \ldots, n-1)$ are bounded in $[a, b]$, we find that (7) and its limit $c_n u_n(x)$ (for $\nu \to +\infty$) satisfy the assumptions of Lemma 4. Hence

$$Z(f) \leqslant Z(u_n). \tag{9}$$

The inequality

$$Z(u_m) \leqslant Z(f) \tag{10}$$

is proved in a similar way by putting

$$g_1(x) = c_m u_m(x) + \frac{\lambda_m}{\lambda_{m+1}} c_{m+1} u_{m+1}(x) + \ldots + \frac{\lambda_m}{\lambda_n} c_n u_n(x).$$

We verify that $Lg_1(x) = \lambda_m \varrho(x) f(x)$. Because $g_1(x)$ satisfies the assumptions of Lemma 3, $Z(g_1) \leqslant Z(f)$. Similary we construct the functions

$$g_\nu(x) = c_m u_m(x) + \left(\frac{\lambda_m}{\lambda_{m+1}}\right)^\nu c_{m+1} u_{m+1}(x) + \ldots + \left(\frac{\lambda_m}{\lambda_n}\right)^\nu c_n u_n(x), \tag{11}$$

such that

$$Lg_\nu(x) = \lambda_m \varrho(x) g_{\nu-1}(x), \qquad \nu = 2, 3, \ldots$$

By reasoning similarly as in the proof of (9) we get inequality (10), considering that for the functions of sequence (11) the number of each of their zero-points does not increase with the increase of the index. But inequalities (9) and (10) are just the thesis of Theorem 4.

THEOREM 5. *Let $u(x)$ and $v(x)$ be eigenfunctions of problem* (4), (1) *corresponding to the eigenvalues λ and μ respectively, where $|\lambda| < |\mu|$. Then we have the inequality*

$$Z(u) < Z(v). \tag{12}$$

Proof. By Theorem 4 we have $Z(u) \leqslant Z(v)$. Suppose that $Z(u) = Z(v)$, and let $w(x) = u(x) - \gamma v(x)$, where $\gamma = u''(a)/v''(a)$. By definition of $w(x)$ we get $w(a) = w'(a) = w''(a) = 0$. By Theorem 4 we have that $Z(w) = Z(u) = Z(v)$. Since a is a zero-point of $w(x)$ of multiplicity of at least 3, by Lemma 2 we get $Z(Lw) \geqslant Z(w) + 1$.

Let us put

$$w_1(x) = \frac{1}{\mu \varrho(x)} Lw(x) = \frac{\lambda}{\mu} u(x) - \gamma v(x),$$

whence $Z(w_1) \geqslant Z(w) + 1 = Z(v) + 1$. Applying Lemma 3 to $w_1(x)$ we get the inequality $Z(w_2) \geqslant Z(w_1)$, where

$$w_2(x) = \left(\frac{\lambda}{\mu}\right)^2 u(x) - \gamma v(x).$$

Proceeding thus further we get an infinite sequence of functions

$$w(x),\ w_1(x),\ w_2(x),\ \ldots, \tag{13}$$

for which

$$(14) \qquad Z(w_\nu) \geqslant Z(w_{\nu-1}) \geqslant Z(w)+1 = Z(v)+1, \qquad \nu = 2, 3, \ldots,$$

where

$$w_\nu(x) = \left(\frac{\lambda}{\mu}\right)^\nu u(x) - \gamma v(x), \qquad \nu = 2, 3, \ldots$$

On the other hand, since the sequence (13) and its limit $\gamma v(x)$, for $\nu \to +\infty$, satisfy the assumptions of Lemma 4, we find that $Z(w_\nu) = Z(v)$ for sufficiently large ν, which contradicts (14).

4. Nodes of the n-th eigenfunction of problem (4), (1). Let us make some additional assumptions:

ASSUMPTION Z. *The operator L in equation* (4) *is symmetric and positive definite in $\mathfrak{M}$, i.e.*

$$(Lu, u) = (u, Lu) \geqslant \beta(u, u) \quad \textit{for } u \in \mathfrak{M}, \ \beta > 0.$$

Remark 1. Assumption Z is satisfied if for instance we have $p_1(x) \equiv p_2'(x) > 0$ and $q_1(x) \equiv q_2(x) \geqslant 0$ for $x \in [a, b]$.

It is known (cf. [4]) that if Assumption Z is satisfied, then there exists an infinite sequence of eigenvalues of problem (4), (1)

$$(15) \qquad 0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots, \qquad \lim \lambda_n = +\infty$$

and a corresponding sequence of eigenfunctions

$$(16) \qquad u_1(x), u_2(x), u_3(x), \ldots,$$

which form a complete system in $L_2([a, b])$.

By Theorem 3 we have that (15) is a strongly increasing sequence. According to Theorem 5 we get the inequality

$$(17) \qquad Z(u_n) \geqslant n-1.$$

Our main pourpose now is to prove that $Z(u_n) = n-1$.

We shall prove the following:

THEOREM 6. *Let Assumption Z hold. Then the n-th eigenfunction $u_n(x)$ of problem* (4), (1) *has exactly $n-1$ zero-points in (a, b).*

Proof. By (17) it is sufficient to prove that $Z(u_n) \leqslant n-1$ for each n. Suppose that there exists $n = m$ such that $Z(u_m) \geqslant m$. Let us denote zero-points of $u_m(x)$ in $[a, b]$ by $a = x_0 < x_1 < \ldots < x_{m+1} = b$. Put

$$(18) \qquad U_j(x) = \begin{cases} u_m(x) & \text{in } [x_{j-1}, x_j], \\ 0 & \text{in } [a, b]-[x_{j-1}, x_j], \end{cases} \qquad j = 1, \ldots, m+1.$$

It is evident that $U_1(x), \ldots, U_{m+1}(x)$ are linearly independent in $[a, b]$. Put

$$\varphi(x) = c_1 U_1(x) + \ldots + c_{m+1} U_{m+1}(x), \tag{19}$$

where $c_1, \ldots, c_{m+1}$ are real numbers such that $c_1^2 + \ldots + c_{m+1}^2 > 0$, and $\varphi(x)$ is orthogonal to $u_1(x), \ldots, u_m(x)$ in $L_2([a, b])$, i.e.

$$(\varphi, u_i) = 0, \quad i = 1, \ldots, m. \tag{20}$$

Let us note that $\varphi(x)$ defined by (19) is continuous in $[a, b]$, and

$$Z(\varphi) \leqslant Z(u_m). \tag{21}$$

Hence it can be expanded into Fourier's series of functions (15) convergent in the norm of $L_2([a, b])$.

By (20) we get

$$\varphi(x) = \sum_{k=m+1}^{\infty} a_k u_k(x), \quad a_k = (\varphi, u_k). \tag{22}$$

We can assume that $a_{m+1} \neq 0$.

By [5] we find that the function

$$\varphi_1(x) = \sum_{k=m+1}^{\infty} \left(\frac{\lambda_{m+1}}{\lambda_k}\right) a_k u_k(x) \tag{23}$$

is an element of $\mathfrak{M}$, since the series in (23) is uniformly and absolutely convergent in $[a, b]$, and that

$$L\varphi_1(x) = \lambda_{m+1}\varrho(x)\varphi(x).$$

Hence by Lemma 3 we get $Z(\varphi_1) \leqslant Z(\varphi)$. Let us put

$$\varphi_\nu(x) = \sum_{k=m+1}^{\infty} \left(\frac{\lambda_{m+1}}{\lambda_k}\right)^\nu a_k u_k(x), \quad \nu = 2, 3, \ldots \tag{24}$$

The following holds

$$L\varphi_\nu(x) = \lambda_{m+1}\varrho(x)\varphi_{\nu-1}(x), \quad \nu = 2, 3, \ldots$$

Hence

$$Z(\varphi) \geqslant Z(\varphi_1) \geqslant Z(\varphi_2) \geqslant \ldots \tag{25}$$

It follows from the definition of the sequence (24) and from previous remarks, that $\{\varphi_\nu(x)\}$ tends uniformly to $a_{m+1}u_{m+1}(x)$ in $[a, b]$. Since $a_{m+1} \neq 0$, by Lemma 5, from (21) and (25) we find that

$$Z(u_m) \geqslant Z(\varphi) \geqslant Z(u_{m+1}), \tag{26}$$

which contradicts Theorem 5.

COROLLARY 2. *When Assumption Z holds, the first eigenfunction of problem* (4), (1) *does not vanish at any point of the interval* (a, b).

Remark 2. Corollary 2 contradicts the theorem of paper [3] in the case of an ordinary differential equation of the fourth order.

References

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