ANNALES
POLONICI MATHEMATICI
XLVI (1985)

On the Theorems of Hartogs and Bishop

by John Wermer (Providence)

Dedicated to the memory of Franciszek Leja in grateful recognition of his work in polynomial approximation

Abstract. A result of F. Hartogs [7] on singularity sets in C^2 and a result of E. Bishop on the existence of analytic structure in maximal ideal spaces of certain Banach algebras [5] are discussed. The validity of the maximum principle for certain associated algebras of functions turns out to connect these two results.

Let X be a relatively closed subset of a domain Ω in C^n. We call X a *singularity set in* Ω if $\exists$ function f analytic on $\Omega \setminus X$ such that f admits no analytic extension to any point of X. When $n = 1$, such singularity sets are completely arbitrary closed sets.

In one of the early papers on analytic functions of several complex variable, F. Hartogs considered singularity sets in the cylinder domain

$$\Omega = \{(z, w) |\ |z| < 1\} \quad \text{in } C^2$$

and showed that they were far from arbitrary. Simple examples are obtained by choosing a polynomial P in z, w and taking $X = \{(z, w) \in \Omega |\ P(z, w) = 0\}$. Then the function $f = 1/P$ is analytic on $\Omega \setminus X$ and singular on X so X is a singularity set in Ω. For each fixed z, $P(z, \cdot)$ has at most k roots, where k = degree of P in w. So X lies at most k-sheeted over the disk $|z| < 1$. In 1909 F. Hartogs proved the following converse.

Hartogs' Theorem [7]. *Let X be a bounded singularity set in Ω such that, for some integer k, X lies at most k-sheeted over $|z| < 1$. Then X is an analytic subvariety of Ω. In particular, if $k = 1$, then X is the graph: $w = \Phi(z)$ of some analytic function on $|z| < 1$.*

In 1934 K. Oka gave in [9] a stronger version of Hartogs' theorem by assuming merely that X is a bounded singularity set in Ω which lies finite-sheeted over a subset e of $|z| < 1$ such that e has positive logarithmic capacity. He concluded, as before, that X is an analytic variety. There were no proofs in [9]. The proofs were later published by T. Nishino in [8], in 1962.

Meanwhile, and independently of the preceding, people had begun to study *maximal ideal spaces* of certain function algebras. For X a compact Hausdorff space, a *uniform algebra* A on X is a closed subalgebra of $C(X)$ such that A separates the points on X and contains the constants. Denote by M the maximal ideal space of A. Then X has a natural imbedding in M and we may regard the functions in A as defined and continuous on all of M. In 1963 Errett Bishop proved the following result:

BISHOP'S THEOREM [5]. *Let A be a uniform algebra on X and let M be its maximal ideal space. Fix $f \in A$. Choose a connected component W of $\mathbf{C} \setminus f(X)$. Assume that $\exists \Lambda \subset W$* such that

(1) *For each $\lambda \in \Lambda$ the set*

$$f^{-1}(\lambda) = \{y \in M | \, f(x) = \lambda\}$$

is a finite set.

(2) *Λ has positive area.*

Then

(3) *$\exists k < \infty$ such that the cardinality of $f^{-1}(\lambda) \leqslant k$ for each $\lambda \in W$, and*

(4) *$\exists$ a discrete set S in W such that $f^{-1}(W \setminus S)$ can be given the structure of a Riemann surface and every function in A is analytic with respect to this structure.*

Later it was shown, independently in [4] and [12] that hypothesis (2) in the theorem can be weakened to

(2′) *Λ has positive logarithmic capacity,*

without affecting the conclusion.

Indeed, (2′) cannot be weakened further, as is shown by an example of H. P. Lee given in [4] (as well as by an example in $\mathbf{C}^2$ by H. Alexander, unpublished).

Both this stronger version of Bishop's theorem and the Oka–Nishino generalization of Hartogs' theorem have as a hypothesis a finite sheeting over a set of positive capacity and as conclusion the existence of analytic structure. What explains the similarity of the two theorems? It turns out that a certain maximum principle is valid in both contexts, which extends the usual maximum principle for analytic functions.

Given an algebra $\mathfrak{A}$ of continuous complex-valued functions on a locally compact space Y, we say that $\mathfrak{A}$ is a *maximum modulus algebra on Y* if

(i) $\mathfrak{A}$ separates points on Y and contains the constants, and

(ii) For each compact set $K \subset Y$, with topological boundary ∂K,

$$|h(y)| \leqslant \max_{\partial K} |h|, \qquad y \in K, \ h \in A.$$

Fix $f \in \mathfrak{A}$ and choose a domain $W \subset \mathbf{C}$ such that f maps Y on W. If, for every compact set $K \subset W$, $f^{-1}(Y)$ is a compact subset of Y, we say that: *(Y, f) lies over W.*

PROPOSITION 1 [15]. *Let X be a bounded singularity set in $\Omega = \{|z| < 1\} \subset C^2$. Denote by $\mathfrak{A}$ the algebra of restrictions to X of all polynomials in z and w. Then $\mathfrak{A}$ is a maximum modulus algebra on X. Also (X, z) lies over $|z| < 1$.*

PROPOSITION 2 [10]. *Let A be a uniform algebra on X, and let M, f, W be as in Bishop's theorem. Let $\mathfrak{A}$ be the restriction of A to $M \backslash X$. Then $\mathfrak{A}$ is a maximum modulus algebra on $M \backslash X$. (It follows that the restriction of A to $f^{-1}(W)$ is a maximum modulus algebra on $f^{-1}(W)$. Also $(f^{-1}(W), f)$ lies over W.)*

The general result on maximum modulus algebras which is required is

THEOREM [16]. *Let $\mathfrak{A}$ be a maximum modulus algebra on a space X and let $f \in \mathfrak{A}$. Let W be a domain in C such that (X, f) lies over W. Assume $\exists$ a subset E of W such that $f^{-1}(\lambda)$ is a finite set for each $\lambda \in E$, such that E has positive capacity. Then $\exists$ a discrete set $S \subset W$ (which may be empty, finite, or countably infinite) such that $f^{-1}(W \backslash S)$ can be made into a Riemann surface and each $h \in \mathfrak{A}$ is analytic on that Riemann surface. Also $\exists k$ such that $f^{-1}(\lambda)$ has at most k points, $\lambda \in W$.*

Using the Theorem in conjunction with Proposition 1 we get Hartogs' theorem, and similarly using Proposition 2, we get Bishop's theorem.

As another application, we can take X to be a domain in the complex z-plane. Let $\mathfrak{A}$ be a maximum modulus algebra on X such that $\mathfrak{A}$ contains a non-constant holomorphic function f. Fix $a \in X$ with $f'(a) \neq 0$. Then $\exists r > 0$ such that, restricted to the disk Δ: $|z-a| < r$, $\mathfrak{A}$ is a maximum modulus algebra and f maps Δ one-to-one onto the domain $f(\Delta)$. It follows by the theorem that $\Delta \backslash S$ has analytic structure and since $f \in \mathfrak{A}$, this is the usual analytic structure. Hence every $g \in \mathfrak{A}$ is analytic in $\Delta \backslash S$ and hence analytic in Δ. Every a with $f'(a) \neq 0$, $a \in X$, is then a removable singularity for each such g. We thus have RUDIN'S THEOREM [11] that *each maximum modulus algebra on a plane domain which contains one non-constant holomorphic function consists entirely of holomorphic functions.*

It was discovered by Bernard Aupetit that, associated with each analytic map from a plane domain into a Banach algebra, there are certain phenomena related to Bishop's and Hartogs' theorems [3], and later a theory was developed in [13] by Zbigniew Słodkowski, concerning a class of *set-valued functions*, which unifies these different situations.

Finally, one wishes to explain the maximum principle which is known to hold for maximal ideal spaces away from the Silov boundary and for singularity sets (Propositions 1 and 2) by means of the maximum principle for analytic functions on analytic spaces.

It was tempting to try to do this by introducing analytic structure on suitable subsets of the space, relative to which to given functions are analytic. However, a series of examples ([6], [14], [17]) beginning with an example of Stolzenberg in 1963 shows that this cannot always be done. There remains the hope that in some sense one can *approximate* the given space X by analytic spaces. A particular example of this question arises when X is the polynomially convex hull in C^2 of the intersection $X \cap (|z| = 1)$.

PROBLEM. Let X be a compact set in C^2 such that X meets $(|z| < 1)$ and X is the polynomially convex hull of $X \cap (|z| = 1)$. To exhibit analytic subvarieties of $|z| < 1$ arbitrarily near X.

Herbert Alexander and I have found some results in this direction. Let $r > 0$ and let T be the tube

$$T = \{(z, w) |\ |w - a(z)| < r,\ |z| < 1\},$$

where $z \to a(z)$ is a continuous map of $|z| \leqslant 1$ into the unit disk. Assume that X (as above) is contained in T. Then $\exists$ an analytic variety Σ, where Σ is the graph of an analytic function on the unit disk, with Σ contained in the tube

$$T' = \{(z, w) |\ |w - a(z)| \leqslant 4r,\ |z| \leqslant 1\}. \quad ([1])$$

We also have a generalization of this result to the case when X is contained in a tube whose intersection with each complex line: $z = z_0$ is contained in the union of n disks of radius r, with r, n fixed [2].

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DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RHODE ISLAND, USA

Reçu par la Rédaction le 4.01.1984