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Algebraic polynomially bounded operators

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Abstract. Let T be a polynomially bounded operator in the complex Hilbert space. We prove several theorems concerning the structure properties of such T which are algebraic in the sense that they satisfy the equality $u(T) = 0$ for some u which belongs to the unit disc algebra. There are used the Lebesgue type decompositions of polynomially bounded operators. One considers also polynomially bounded n-tuples of commuting operators which are algebraic in a quite general sense.

Let H be the complex Hilbert space and let $L(H)$ stand for the algebra of all linear bounded operators in H. The identity operator in H is denoted by I.

We say that the operator $T \in L(H)$ is *algebraic* if $p(T) = 0$ for some polynomial $p(z) = \sum_{i/0}^{n} a_i z^i$. The concept of an algebraic operator extends in a natural way to polynomially bounded operators, that is such ones that

$$\|p(T)\| \leqslant K \sup_{|z|=1} |p(z)|$$

for all polynomials and some finite K. If T is polynomially bounded, then we show easily that there is a unique representation $u \to u(T)$ of the unit disc algebra such that for $u_i(z) = z^i$ $(i = 0, 1)$ we have $u_i(T) = T^i$ and $\|u(T)\| \leqslant K \sup_{|z|=1} |u(z)|$. The generalization of the concept of an algebraic operator reduces now to the following definition: the polynomially bounded operator T is algebraic (in general sense) if $u(T) = 0$ for some u which belongs to the unit disc algebra.

In the circle of the concept of general algebraic operators Sz.–Nagy and Foiaş developed the theory of contractions of class C^0 (see [11], Chapter III) and a large part of what we discuss in the present paper is patterned after this theory. Also Arveson in [1] proved several theorems related to contractions which are algebraic in the general sense. The recent results of Furuta [3] and Wadhwa [13] who considered algebraic operators of class $\mathscr{C}_\varrho$ (see [11], Chapter I for basic properties of operators of this class) and the results of the above mentioned authors suggest the problem of characterization of quite general, as defined above, algebraic polynomially bounded operators. The present paper pays some tribute to several questions related to this problem.

1. Let $(T_1, \ldots, T_n)$ $(T_i \in L(H))$ be a n-tuple of commuting operators. We say that it is *polynomially bounded* if for some finite K

$$\|p(T_1, \ldots, T_n)\| \leqslant K \sup_{z \in C^n} |p(z)|$$

for all polynomials $p(z) = p(z_1, \ldots, z_n) = \sum_{k_i \geqslant 0} a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}$.

C^n stands here for the product $C \times C \times \ldots \times C$ (n times) of the unit circle $C = \{\lambda\colon |\lambda| = 1\}$. If $n = 1$, then the tuple reduces to the singleton $T = T_1$ and then we say that T is *polynomially bounded.*

If the n-tuple $(T_1, \ldots, T_n)$ is polynomially bounded, then there is a representation $u \to u(T_1, \ldots, T_n)$ of the polydisc algebra $A(C^n) =$ uniform closure on C^n of the algebra of polynomials. The algebra $A(C^n)$ may be identified with the algebra of functions which are analytic in the polydisc $D^n = D \times D \times \ldots \times D$ (n-times) $(D = \{\lambda\colon |\lambda| < 1\})$ and continuous on the closure $\bar{D}^n$ — see [9] for references. It is plain that $u_i(T_1, \ldots, T_n) = T_i$ for $u_i(z_1 \ldots z_n) = z_i$ and

$$\|u(T_1, \ldots, T_n)\| \leqslant K\|u\| \quad \text{for } u \in A(C^n),$$

where $\|u\| = \sup_{C^n} |u|$.

The Hahn–Banach theorem and the Riesz–Kakutani theorem yield that there are complex measures $\mu(f, g)$ $(f, g \in H)$ on Borel subsets of C^n such that:

$$\|\mu(f, g)\| \leqslant K\|f\|\|g\| \quad (f, g \in H); \tag{1}$$

$$\big(u(T_1, \ldots, T_n)f, g\big) = \int u\, d\mu(f, g) \quad (f, g \in H,\ u \in A(C^n)). \tag{2}$$

The measures $\mu(f, g)$ satisfying (1) and (2) are called *elementary measures* of the representation $u \to u(T_1, \ldots, T_n)$.

It follows from the Ando's theorem of [2] that if T_1, T_2 are commuting contractions, then the pair (T_1, T_2) is polynomially bounded with constant $K = 1$. Other examples of polynomially bounded tuples may be derived from the dilation theory — see [5], [11] for references. We mention yet the operators of class $\mathscr{C}^\varrho$. They are polynomially bounded because by Proposition 11.4 of [11], if $T \in \mathscr{C}^\varrho$, then

$$\|p(T)\| \leqslant \sup_{|z|=1} |p(z) + (1-\varrho)p(0)| \leqslant (\varrho + |1-\varrho|)\|p\|$$

for every polynomial p.

2. Suppose we are given the polynomially bounded operator T. Let m be the normalized linear Lebesgue measure on C. We say that T is *m-continuous* (*m-singular* respectively) if the representation $u \to u(T)$ of the algebra $A(C)$ generated by T has a system of elementary measures which are absolutely continuous (singular respectively) with regard to m.

PROPOSITION 1. *Suppose $T \in L(H)$ is polynomially bounded. Then T has a decomposition $T = T_a + T_s$ (direct sum), where T_a is m-continuous and T_s is m-singular. Both T_a, T_s are polynomially bounded with the same constant K as that of T. Moreover, T_s is similar to a singular unitary operator.*

Proof. The first parts of the assertion follow easily from Theorem 2.1 of [6]. Only the last part requires a proof. This part may be derived from Theorem 10.5 of [12]. We will give the direct proof.

Suppose just that T_s is the singular part of T. Then there are m-singular measures $\mu(f, g)$ with f, g varying over the space corresponding to T_s in the decomposition $T = T_a + T_s$ and such that

$$(u(T_s)f, g) = \int u \, d\mu(f, g) \quad \text{for } u \in A(C). \tag{3}$$

Since by the M. and F. Riesz theorem (see [4], Chapter 4) the only m-singular measures which are orthogonal to $A(C)$ are zero measures, the measures $\mu(f, g)$ are determined uniquely by condition (3). Since the left-hand side of (3) is a bilinear form in f and g, the values of $\mu(f, g)$ on a fixed but arbitrary Borel set in C are bilinear in f and g. On the other hand $|\mu(f, g)(\sigma)| \leqslant \|\mu(f, g)\| \leqslant K\|f\|\|g\|$ for Borel subset σ of C. It follows that for every such σ there is the unique operator $E(\sigma)$ such that $\mu(f, g)(\sigma) = (E(\sigma)f, g)$ for all $f, g \in H$. Also $\|E(\sigma)\| \leqslant K$. Consequently, $E(\cdot)$ is a weakly countably additive operator measure. Hence $E(\cdot)$ is strongly countably additive and

$$u(T_s) = \int u \, dE \quad \text{for } u \in A(C).$$

We next show that E is a spectral measure i.e. $E(\sigma \cap \gamma) = E(\sigma)E(\gamma)$ for Borel subsets of C. The proof is patterned after the proof of the spectral theorem for normal operators. To begin with we consider the measure

$$v_u(\sigma) = \int_\sigma u \, d\mu(f, g)$$

for some $u \in A(C)$ and fixed f, g. Then for every $v \in A(C)$

$$\int v \, dv_u = \int uv \, d\mu(f, g) = (v(T_s)f, u(T_s)^* g) = \int v \, d\mu(f, u(T_s)^* g).$$

Since both measures v_u, $\mu(f, u(T_s)^* g)$ are singular and v is varying over $A(C)$, the Riesz Brothers theorem yields that $v_u = \mu(f, u(T_s)^* g)$.

It follows that for a fixed Borel set σ

$$\int u\chi_\sigma d\mu(f,g) = \mu(f, u(T_s)^* g)(\sigma) = (u(T_s)E(\sigma)f, g) = \int u d\mu(E(\sigma)f, g).$$

Since the measures in the above equality are m-singular and u is an arbitrary element of $A(C)$, the Riesz Brothers theorem used once again implies that the measures $\chi_\sigma\mu(f,g)$ and $\mu(E(\sigma)f, g)$ are equal.

We conclude therefore that for every Borel set γ

$$\chi_\sigma\mu(f,g)(\gamma) = (E(\sigma\cap\gamma)f, g) = \mu(E(\sigma)f, g)(\gamma) = (E(\gamma)E(\sigma)f, g).$$

Since f and g are arbitrary vectors we have $E(\sigma\cap\gamma) = E(\sigma)E(\gamma)$ q.e.d.

We conclude now that the mapping $h\to\int h dE$ is a representation of the algebra of all continuous functions on C. It follows that T_s is strictly invertible and $T_s^n = \int z^n dE$ for $n = 0, \pm 1, \pm 2, \ldots$ Hence $\|T_s^n\| \leqslant 4K$ for all n, K being the constant corresponding to T. Applying now the theorem of Sz.-Nagy of [10] we conclude that T_s is similar to a unitary operator, which is obviously singular.

It is not difficult to show that if $T\in\mathscr{C}^\varrho$, then elementary measures $\mu(f,f)$ may be chosen real. We get then

COROLLARY 1. *If $T\in\mathscr{C}^\varrho$, then the decomposition $T = T_a + T_s$ is an orthogonal one i.e. $T = T_a\oplus T_s$ and T_s is unitary.*

Let H^∞ be the algebra of all analytic bounded functions in D and identify it in a well-known way with the subalgebra of $L^\infty(m)$. Theorem 3.1 of [6] yields the following proposition:

PROPOSITION 2. *Let T_a be the m-continuous part of the polynomially bounded operator T with the corresponding constant K. Then there is the unique representation $u\to u(T_a)$ of H^∞ such that the following conditions hold true:*

(4) $$u_i(T_a) = T_a^i \quad \text{for } u_i(z) = z^i,\ i = 0, 1;$$

(5) $$\|u(T_a)\| \leqslant K\|u\|_\infty \quad \text{for } u\in H^\infty,$$

(6) *If $u_n\to u$ a.e. with respect to m on C, $\sup\|u_n\|_\infty < +\infty$, then $u_n(T_a)\to u(T_a)$ weakly.*

3. In what follows we refer to [4], [11] for terminology as well for the basic properties of Hardy spaces in the unit disc. The lemma below is the first part of Proposition 3.1 ([11], p. 118), proved for completely non-unitary contractions. The proof applies to m-continuous polynomially bounded operators.

LEMMA. *Suppose that T is a polynomially bounded and m-continuous operator. If $u\in H^\infty$ is an outer function, then $u(T)$ is an invertible operator.*

If $T \in L(H)$ is polynomially bounded operator such that $u(T) = 0$ for some $u \in A(C)$, then $u(T_a) = 0$ and $u(T_s) = 0$. Let $u = u_i u_e$, where u_i is the inner factor of u and u_e the outer one. Lemma implies that $u_i(T_a) = 0$. On the other hand T_s is similar to a unitary operator U_s such that $u(U_s) = 0$. Consequently, the spectrum of T_s is included in $N_u = \{z \in C: u(z) = 0\}$. Summing up we get the following theorem:

THEOREM 1. *Let $T \in L(H)$ be a polynomially bounded operator such that $u(T) = 0$ for some $u = u_i u_e \in A(C)$. Then $u_i(T_a) = 0$ and T_s is similar to a unitary operator and the spectrum of T_s is included in N_u.*

COROLLARY 2. *If u in the above theorem is outer, then $T = T_s$. In particular, if u is a polynomial having all it's zeros on C, then $T = T_s$.*

COROLLARY 3. *If u is a polynomial (a minimal one) such that $u(T) = 0$, then the spectrum of T_a is in the open unit disc. Then by theorem of G. C. Rota [8] T_a is similar to a contraction. Consequently, T itself is similar to a contraction.*

The precise description of the m-singular part T_s in terms of the *whole* ideal of all v such that $v(T_s) = 0$ may be given by applying theorems of Arveson [1] Lemma 3.6.10 and Proposition 3.6.11.

Applying Theorem 1 and Corollary 1, we derive easily the following theorem:

THEOREM 2. *Suppose $T \in \mathscr{C}^{\varrho}$ for some $\varrho > 0$. If $u(T) = 0$ for some $u \in A(C)$, then T is an orthogonal sum $T = T_a \oplus T_s$, where T_s is unitary with spectrum in N_u and $u_i(T_a) = 0$, u_i being the inner factor of u.*

Since every operator in $\mathscr{C}^{\varrho}$ is similar to a contraction ([11], Chapter II, 8) the study of T_a in the above theorem may be reduced to the study of contractions of class C^0.

Applying Theorem 2 in the case when u is a polynomial (a minimal one) we get the Furuta–Wadhwa theorem [3], [13]:

If $T \in \mathscr{C}^{\varrho}$ and $u(T) = 0$, then $T = (\oplus z_k I_k) \oplus T_a$, where z_k are all zeros of u which are on C and $u_i(T_a) = 0$, where u_i is the factor of u including only, but all zeros of u which belong to the open unit disc.

4. Let $(T_1, \ldots, T_n)$ be a n-tuple of commuting operators and assume that it is polynomially bounded. Let $u \to u(T_1, \ldots, T_n)$ be the representation of $A(C^n)$ generated by this tuple. Assume now that the set $\sigma \subset C^n$ is a peak set for $A(C^n)$. It follows from the results of [7] that the representation $u \to u(T_1, \ldots, T_n)$ is a direct sum of two representations $u \to u(T_1, \ldots, T_n)_\sigma$, $u \to u(T_1, \ldots, T_n)_{C^n - \sigma}$ such that the first one has a system of elementary measures with closed carriers included in σ and the other one has a system of elementary measures vanishing off σ.

It is known — see [9], Theorem 6.1.2, that every peak set γ of $A(C^n)$ has the following property:

If the measure μ defined on Borel subsets of C^n is orthogonal to $A(C^n)$, then $|\mu|(\gamma) = 0$.

Using arguments similar to those used in the proof of Proposition 1 one proves that the following proposition holds true:

PROPOSITION 3. *Let $(T_1, \ldots, T_n)$ be a n-tuple of commuting operators. Suppose that it is polynomially bounded. If, for some peak set $\sigma \subset C^n$ of $A(C^n)$, $u(T_1, \ldots, T_n) = u(T_1, \ldots, T_n)_\sigma$ for all $u \in A(C^n)$, then there is a similarity S (i.e. an one to one linear bounded operator in H with range equal to H) and a n-tuple of unitary operators $(U_1, \ldots, U_n)$ such that $T_i = SU_iS^{-1}$ for $i = 1, \ldots, n$. If the representation $u \to u(T_1, \ldots, T_n)$ is contractive i.e. if $\|u(T_1, \ldots, T_n)\| \leqslant \|u\|$ for $u \in A(C^n)$, then T_i are themselves unitary.*

In what follows we need the following property of some closed ideals in $A(C^n)$ — see [9], p. 78:

(7) *Suppose that u $A(C^n)$ and $\operatorname{Re} u \geqslant 0$ in $\overline{D^n}$. If k is a positive integer, then the closed ideal generated by u^k consists of all v in $A(C^n)$ that vanish on the zero set N_u of u (relative to $\overline{D^n}$).*

Suppose now that the zero set N_u of $u \in A(C^n)$ (relative to $\overline{D^n}$) is included in C^n. Then by Theorem 6.1.2 of [9] N_u is a peak set of $A(C^n)$. We derive now from (7) that the following condition holds true:

(8) *Suppose $u \in A(C^n)$ and $\operatorname{Re} u \geqslant 0$ in $\overline{D^n}$. Assume that $N_u \subset C^n$. Then for every positive integer k and every function $v \in A(C^n)$ which peaks on N_u there is a sequence $u_p \in A(C^n)$ such that $1 - v = \lim_p u^k u_p$ uniformly on C^n.*

We are now able to prove the following theorem:

THEOREM 3. *Let $u \to u(T_1, \ldots, T_n)$ be the representation generated by the polynomially bounded n-tuple $(T_1, \ldots, T_n)$ of commuting operators. Suppose that $u \in A(C^n)$, $\operatorname{Re} u \geqslant 0$ in $\overline{D^n}$ and $N_u \subset C^n$. Then, if for some positive integer k holds the equality $u(T_1, \ldots, T_n)^k = 0$, then there exists a similarity S and a n-tuple $(U_1, \ldots, U_n)$ of unitary operators with joint spectrum included in N_u, such that $T_i = SU_iS^{-1}$ for $i = 1, \ldots, n$. If the representation $u \to u(T_1, \ldots, T_n)$ is contractive, then T_i are themselves unitary and the joint spectrum of $(T_1, \ldots, T_n)$ is included in N_u.*

Proof. It follows from the assumptions and from (8) that if v peaks on N_u, then $1 - v = \lim_p u^k u_p$ uniformly on C^n. Since $u(T_1, \ldots, T_n)^k = 0$ we have $u_p(T_1, \ldots, T_n) u(T_1, \ldots, T_n)^k = 0$. Hence $v(T_1, \ldots, T_n) = I$. Let $\mu(f, g)$ be an elementary measure of considered representation. It follows that for every $h \in A(C^n)$ and $p = 1, 2, 3, \ldots$

$$\big(h(T_1, \ldots, T_n)f, g\big) = \big(h(T_1, \ldots, T_n)v(T_1, \ldots, T_n)^p f, g\big)$$
$$= \int h v^p d\mu(f, g) \underset{p\to\infty}{\to} \int_{N_u} h d\mu(f, g),$$

We just proved that $h(T_1, \ldots, T_n) = h(T_1, \ldots, T_n)_{N_u}$ for all $h \in A(C^n)$. The assertion follows now from Proposition 3 and well-known properties of representations of algebras of continuous functions.

COROLLARY 4. *Since* $\log|u^*| \in L^1(m_n)$ *for* $u \in N(D^n)$ (m_n = *the product Lebesgue measure on* C^n) — *see* [9], *Theorem* 3.3.5, N_u *of Theorem* 3 *has the measure* m_n *equal to zero. Consequently, the spectral measure of* $(U_1, \ldots, U_n)$ *is singular with respect to* m_n.

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