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On the characteristics for a system of partial differential equations of the first order in a special case

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Abstract. Basing ourselves on the characteristics of solutions of single first order partial differential equations, we give the definition and certain properties of the characteristics of the system

(1) $$z_x^{(i)} = f^{(i)}(x, Y, Z, z_Y^{(i)}) \quad (i = 1, 2, \ldots, m),$$

where $Y = (y_1, \ldots, y_n)$, $Z = (z^{(1)}, \ldots, z^{(m)})$, $z_Y^{(i)} = (z_{y_1}^{(i)}, \ldots, z_{y_n}^{(i)})$. We base ourselve on T. Ważewski's paper [8], where a metod is shown for the construction of the solution of system (1) as the limit of sequences of solutions of single partial differential equations.

We give here the following theorems:

1. *Assume that*:

(α) *The functions $f^{(i)}(x, Y, Z, Q)$ are of class C^1 in the convex domain Ω of the space x, Y, Z, Q and $|f_{q_j}^{(i)}| \leqslant M$ for $(x, Y, Z, Q) \in \Omega$.*

(β) *The solution $Z(x, Y)$ of system* (1) *is of class C^1 in $\Delta = \{(x, Y)\colon |x - x_0| < a, |y_j - \overset{0}{y}_j| \leqslant b_j - M|x - x_0|\}$ and the derivatives $z_Y^i(x, Y)$ fulfil the Lipschitz condition with respect to Y.*

(γ) $$f_{q_k}^{(1)}(x, Y, Z(x, Y), z_Y^{(1)}(x, Y)) = f_{q_k}^{(2)}(x, Y, Z(x, Y), z_Y^{(2)}(x, Y)) = \ldots = f_{q_k}^{(m)}(x, Y, Z(x, Y), z_Y^{(m)}(x, Y)) \quad (k = 1, 2, \ldots, n).$$

Under these assumptions the solution $Z(x, Y)$ is generated by characteristics in the set Δ.

2. *Assume that*

1) *functions $f^{(i)}$ fulfil* (α),

2) *the solution of the characteristic system is uniquely defined by the initial conditions,*

3) *the solutions $U(x, Y)$ and $V(x, Y)$ fulfil* (β) *and* (γ),

4) $U(\bar{x}, \bar{Y}) = V(\bar{x}, \bar{Y})$, $u_Y^{(i)}(\bar{x}, \bar{Y}) = v_Y^{(i)}(\bar{x}, Y)$, $(\bar{x}, \bar{Y}) \in \Delta$.

Under these assumptions the solutions $U(x, Y)$ and $V(x, Y)$ have a common characteristic issuing from the point $(\bar{x}, \bar{Y}, U(\bar{x}, \bar{Y}), u_Y^{(1)}(\bar{x}, \bar{Y}), \ldots, u_Y^{(m)}(\bar{x}, \bar{Y}))$.

The characteristics of partial differential equations play an important part in the investigations of properties of their solutions. In paper [1] the definition of characteristics has been extended to certain systems of partial differential equations of the first order of the form

(1) $$z_x^{(i)} = f^{(i)}(x, Y, Z, z_Y^{(i)}) \quad (i = 1, 2, \ldots, m),$$

where $Y = (y_1, \ldots, y_n)$, $Z = (z^{(1)}, \ldots, z^{(m)})$, $z_Y^{(i)} = (z_{y_1}^{(i)}, \ldots, z_{y_n}^{(i)})$. The same paper gives the basic properties of characteristics, constituting generalizations of known theorems on the characteristics of a single partial differential equation.

In the present paper we shall investigate the position of characteristics with respect to the solutions $Z(x, Y) = \big(z^{(1)}(x, Y), \ldots, z^{(m)}(x, Y)\big)$ of system (1). In the first part of the paper we shall discuss briefly, the basis of [1], the construction of the characteristics of solutions of system (1) fulfilling certain additional conditions. Theorem 3 forms the main part of the paper, giving sufficient conditions for a solution $Z(x, Y)$ of system (1) to be generated by characteristic. Theorem 3 is a generalization of Theorem 3 of [1], p. 66.

I. We shall now show briefly, on the basis of parer [1], the construction of the characteristic for the solution $Z(x, Y)$ of system (1) fulfilling certain additional conditions.

For this purpose we shall make the following

Assumption H.

(α) *Functions* $f^{(s)}(x, Y, Z, Q)$ $(s = 1, 2, \ldots, m)$ *are of class* C^2 *in the set*

$$|x-x_0| < a, \quad Y, Z, Q \text{ -- } \textit{arbitrary}, \tag{2}$$

where $a > 0$, $Q = (q_1, \ldots, q_n)$, *and*

$$\left.\begin{matrix} \left|\dfrac{\partial f^{(s)}}{\partial y_k}\right|, \left|\dfrac{\partial f^{(s)}}{\partial z^{(i)}}\right|, \left|\dfrac{\partial f^{(s)}}{\partial q_k}\right|, \left|\dfrac{\partial^2 f^{(s)}}{\partial y_k \partial y_l}\right| \\ \left|\dfrac{\partial^2 f^{(s)}}{\partial y_k \partial z^{(i)}}\right|, \left|\dfrac{\partial^2 f^{(s)}}{\partial y_k \partial q_l}\right| \\ \left|\dfrac{\partial^2 f^{(s)}}{\partial z^{(i)} \partial z^{(j)}}\right|, \left|\dfrac{\partial^2 f^{(s)}}{\partial z^{(i)} \partial q_k}\right|, \left|\dfrac{\partial^2 f^{(s)}}{\partial q_k \partial q_l}\right| \end{matrix}\right\} \leqslant M$$

$(i, j, s = 1, 2, \ldots, m,\ k, l, = 1, 2, \ldots, n)$, *where* $M > 0$.

(β) *Functions* $\big(\omega^{(1)}(Y), \ldots, \omega^{(m)}(Y)\big) = \Omega(Y)$ *are of class* C^2 *for any arbitrary* Y *and*

$$\left|\frac{\partial \omega^{(i)}}{\partial y_k}\right| \leqslant M, \quad \left|\frac{\partial^2 \omega^{(i)}}{\partial y_k \partial y_l}\right| \leqslant M,$$
$$(i = 1, 2, \ldots, m,\ k, l = 1, 2, \ldots, n).$$

Remark 1. If assumption H is satisfied, then it follows from Theorem 1 of [8], p. 113, that there exists a solution $Z(x, Y)$ of system (1) fulfilling the initial condition

$$Z(x_0, Y) = \Omega(Y) \tag{3}$$

for any arbitrary Y. The solution $Z(x, Y)$ is of class C^1 in the set

(4) $$|x-x_0| < b, \qquad Y \text{ — arbitrary},$$

where $b = \min(a, p, c)$, and p is the root of the equation with respect to x

$$N(1+Nn+N)\int_0^x e^{\int_0^x \lambda(t)dt} dx = \frac{1}{n},$$

whereas

$$N = M(1+3mr+m^2r^2),$$

$$r = 2M + \frac{1}{2(m+n)},$$

$$\lambda(t) = N + 2nN(1+N)e^{Nt},$$

$$c = \{4M(m+n)[1+2M(m+n)]\}^{-1}.$$

This solution can be obtained as the limit of sequences of functions $\big(z_\lambda^{(1)}(x, Y), \ldots, z_\lambda^{(m)}(x, Y)\big) = Z_\lambda(x, Y)$ $(\lambda = 0, 1, 2, \ldots)$ defined in the following way:

$$Z_0(x, Y) = \Omega(Y),$$

whereas functions $z_\lambda^{(i)}(x, Y)$, where $\lambda = 1, 2, \ldots$ are solutions of equations

(5) $$\frac{\partial z^{(i)}}{\partial x} = f^{(i)}\left(x, Y, Z_{\lambda-1}(x, Y), \frac{\partial z^{(i)}}{\partial y_1}, \ldots, \frac{\partial z^{(i)}}{\partial y_n}\right)$$

fulfilling the initial conditions

(6) $$z_\lambda^{(i)}(x_0, Y) = \omega^{(i)}(Y).$$

Functions $z_\lambda^{(i)}(x, Y)$ are defined in (4) and are of class C^2 in (4) and in an arbitrary closed and bounded domain enclosed in (4) the sequence $\{z^{(i)}(x, Y)\}$ and the sequences of partial derivatives

$$\left\{\frac{\partial z_\lambda^{(i)}(x, Y)}{\partial x}\right\}, \quad \left\{\frac{\partial z_\lambda^{(i)}(x, Y)}{\partial y_k}\right\}$$

are uniformly convergent to the function $z^{(i)}(x, Y)$ and its derivatives

$$\frac{\partial z^{(i)}(x, Y)}{\partial x}, \quad \frac{\partial z^{(i)}(x, Y)}{\partial y_k}$$

respectively.

We shall now define sequences of characteristics corresponding to single partial differential equations obtained from (5). With the help of these sequences we shall define the characteristics corresponding to the integral of system (1).

For this purpose we shall choose in each of the spaces x, Y, $z^{(i)}$, $Q^{(i)}$ for $i = 1, 2, \ldots, m$, where $Q^{(i)} = (q_1^{(i)}, \ldots, q_n^{(i)})$, the point $P_i(x_0, \overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)})$ $\big(\overset{0}{Y} = (\overset{0}{y}_1, \ldots, \overset{0}{y}_n)$, $\overset{0}{Q}{}^{(i)} = (\overset{0}{q}{}_1^{(i)}, \ldots, \overset{0}{q}{}_n^{(i)})\big)$ and assume that the functions $\Omega(Y)$ satisfy, apart from assumption H, also the following conditions:

$$(7) \qquad \omega^{(i)}(\overset{0}{Y}) = \overset{0}{z}{}^{(i)}, \quad \omega_Y^{(i)}(\overset{0}{Y}) = \overset{0}{Q}{}^{(i)} \quad (i = 1, 2, \ldots, m).$$

Let us now fix an index $\lambda \geqslant 1$ and assume that the functions

$$(8) \qquad z_l^{(1)}(x, Y), \ldots, z_l^{(m)}(x, Y),$$

where $l = 1, 2, \ldots, \lambda-1$, are known. In this case each of equations (5), for $i = 1, 2, \ldots, m$, with the initial condition (6) can be solved independently of the remaining ones.

Let us consider the characteristic system of ordinary differential equations corresponding to equation (5)

$$(9) \qquad \begin{aligned} \frac{dy_k}{dx} &= -f_{q_k}^{(i)}(S_\lambda^{(i)}) \quad (k = 1, 2, \ldots, n), \\ \frac{dz^{(i)}}{dx} &= f^{(i)}(S_\lambda^{(i)}) - \sum_{j=1}^{n} q_j^{(i)} f_{q_j}^{(i)}(S_\lambda^{(i)}), \\ \frac{dq_k^{(i)}}{dx} &= f_{y_k}^{(i)}(S_\lambda^{(i)}) - \sum_{l=1}^{m} f_{z^{(l)}}^{(i)}(S_\lambda^{(i)}) \frac{\partial z_{\lambda-1}^{(l)}(x, Y)}{\partial y_k} \\ & \qquad\qquad\qquad\qquad (k = 1, 2, \ldots, n), \end{aligned}$$

where $S_\lambda^{(i)} = \big(x, Y, Z_{\lambda-1}(x, Y), Q^{(i)}\big)$.

Let

$$(10) \qquad Y = Y_\lambda^{(i)}(x), \quad z^{(i)} = z_\lambda^{(i)}(x), \quad Q^{(i)} = Q_\lambda^{(i)}(x),$$

where

$$Y_\lambda^{(i)}(x) = \big(\overset{1}{y}{}_\lambda^{(i)}(x), \ldots, \overset{n}{y}{}_\lambda^{(i)}(x)\big), \quad Q_\lambda^{(i)}(x) = \big(\overset{1}{q}{}_\lambda^{(i)}(x), \ldots, \overset{n}{q}{}_\lambda^{(i)}(x)\big),$$

denote the solution of system (9) satisfying the initial conditions

$$Y_\lambda^{(i)}(x_0) = \overset{0}{Y}, \quad z_\lambda^{(i)}(x_0) = \overset{0}{z}{}^{(i)}, \quad Q_\lambda^{(i)}(x_0) = \overset{0}{Q}{}^{(i)}.$$

Assumption H is satisfied and functions (8) are of class C^2 in (4); therefore exactly one solution (10) of system (9) will pass through any point $P_i(x_0, \overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)})$. As the integral

$$(11) \qquad z^{(i)} = z_\lambda^{(i)}(x, Y)$$

of equation (5) fulfilling the initial condition (6) is of class C^2 in (4), it follows from Theorem 1.4 of [6], p. 22, that the characteristic (10), which in the sequel will be denoted by $C_\lambda^{(i)}$, is situated on surface (11), which means that

$$(12)\qquad z_\lambda^{(i)}\big(x, Y_\lambda^{(i)}(x)\big) = z_\lambda^{(i)}(x), \qquad \frac{\partial z_\lambda^{(i)}\big(x, Y_\lambda^{(i)}(x)\big)}{\partial y_k} = \overset{k}{q}{}_\lambda^{(i)}(x)$$

$$(k = 1, 2, \ldots, n).$$

Now taking for λ the successive values $1, 2, \ldots$, we shall obtain for any $i = 1, 2, \ldots, m$ a sequence of differential equations (5) and a sequence of characteristic systems (9) connected with it and also a sequence of characteristics $C_\lambda^{(i)}$ issuing from the point $P_i(x_0, \overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)})$.

Let

$$(13)\qquad Y = Y^{(i)}(x), \quad z^{(i)} = z^{(i)}(x), \quad Q^{(i)} = Q^{(i)}(x),$$

where $Y^{(i)}(x) = \big(\overset{1}{y}{}^{(i)}(x), \ldots, \overset{n}{y}{}^{(i)}(x)\big)$, $Q^{(i)}(x) = \big(\overset{1}{q}{}^{(i)}(x), \ldots, \overset{n}{q}{}^{(i)}(x)\big)$, denote the solution of a system of ordinary equations

$$(14)\qquad \begin{aligned} \frac{dy_k}{dx} &= -f_{q_k}^{(i)}(S^{(i)}) \quad (k = 1, 2, \ldots, n), \\ \frac{dz^{(i)}}{dx} &= f^{(i)}(S^{(i)}) - \sum_{j=1}^{n} q_j^{(i)} f_{q_j}^{(i)}(S^{(i)}), \\ \frac{dq_k^{(i)}}{dx} &= f_{y_k}^{(i)}(S^{(i)}) + \sum_{l=1}^{m} f_{z^{(l)}}^{(i)}(S^{(i)}) \frac{\partial z^{(l)}(x, Y)}{\partial y_k} \\ & \qquad\qquad (k = 1, 2, \ldots, n), \end{aligned}$$

where $S^{(i)} = \big(x, Y, Z(x, Y), Q^{(i)}\big)$, fulfilling the initial conditions

$$(15)\qquad Y^{(i)}(x_0) = \overset{0}{Y}, \quad z^{(i)}(x_0) = \overset{0}{z}{}^{(i)}, \quad Q^{(i)}(x_0) = \overset{0}{Q}{}^{(i)}.$$

The properties of the curves $C_\lambda^{(i)}$ and of the solutions (13) of system (14) are given in Theorem 1 of [1], p. 55, which we shall quote here without proof as

THEOREM 1. *If assumption* H *holds, then the curve* (13), *which in the sequel will be denoted by* $C^{(i)}$, *is the unique solution of the system of differential equations* (14) *fulfilling the initial conditions* (15). *Moreover, the sequence* $C_\lambda^{(i)}$ *is for* $\lambda \to \infty$ *uniformly convergent to* $C^{(i)}$ *in an arbitrary closed interval*

$$(16)\qquad |x - x_0| \leqslant b',$$

where $0 < b' < b$.

DEFINITION 1. The set of curves $(C^{(1)}, \ldots, C^{(m)})$, any of which is the limit of a sequence of characteristics $C_\lambda^{(i)}$ almost uniformly convergent in the interval $(x_0 - b, x_0 + b)$, is said to be the *quasi-characteristic* corresponding to the system of points $P_i(x_0, \overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)})$ and the solution $Z(x, Y)$ of system (1).

Remark 2. It follows from Theorem 1 that if assumption H holds and the solution $Z(x, Y)$ of system (1) is known, we can obtain the quasi-characteristic $(C^{(1)}, \ldots, C^{(m)})$ corresponding to the set of points $P_i(x_0, \overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)})$ $(i = 1, 2, \ldots, m)$ by solving m systems of ordinary differential equations (14) with initial conditions (15), where $\overset{0}{z}{}^{(i)}$ and $\overset{0}{Q}{}^{(i)}$ satisfy equalities (7).

For quasi-characteristics we can give the following geometrical interpretation:

For each $i = 1, 2, \ldots, m$ the first $n+1$ equations of system (13) form a characteristic curve situated in the space $x, Y, z^{(i)}$, whereas the remaining n equations form at each point of this curve a set of directional coefficients of the tangent plane.

Remark 3. If assumption H holds, then the quasi-characteristic $(C^{(1)}, \ldots, C^{(m)})$ corresponding to the integral $Z(x, Y)$ of system (1) is situated on this integral, which means that

$$\begin{aligned} z^{(i)}\big(x, Y^{(i)}(x)\big) &= z^{(i)}(x), \\ z_Y^{(i)}\big(x, Y^{(i)}(x)\big) &= Q^{(i)}(x) \qquad (i = 1, 2, \ldots, m) \end{aligned}$$

for $x \in (x_0 - b, x_0 + b)$. This follows from (12) and from Theorem 1 of [8], p. 113.

Remark 4. To a fixed set of points $P_i(x_0, \overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)})$ and two different solutions $Z(x, Y)$ and $\bar{Z}(x, Y)$ of system (1) different quasi-characteristics can correspond, as can be seen from the following example.

Example 1. Given a system

$$\begin{aligned} z_x^{(1)} &= 1 + 2x - x^2 - \sin y - \cos y + z^{(2)} - z_y^{(1)}, \\ z_x^{(2)} &= 2x. \end{aligned} \tag{17}$$

The functions

$$\begin{aligned} z^{(1)}(x, y) &= 1 + x + x^2 + \cos y - \sin y + \sin(y - x) - \cos(y - x), \\ z^{(2)}(x, y) &= x^2 \end{aligned} \tag{18}$$

are solutions of the given system and satisfy the following conditions:

$$z^{(1)}(0, y) = 1, \quad z^{(2)}(0, y) = 0.$$

The functions

$$\begin{aligned} \bar{z}^{(1)}(x, y) &= \cos y + x^2, \\ \bar{z}^{(2)}(x, y) &= x^2 + \cos y - 1 \end{aligned} \tag{19}$$

are also solutions of the same system and satisfy the initial conditions

$$\bar{z}^{(1)}(0, y) = \cos y, \quad \bar{z}^{(2)}(0, y) = \cos y - 1.$$

It follows that the curve $C^{(1)}$ issuing from the point $P_1(0, 0, 1, 0)$ corresponding to solution (18) has the form

$$y = x, \quad \begin{aligned} z^{(1)} &= x + x^2 + \cos x - \sin x, \\ q^{(1)} &= 1 - \sin x - \cos x, \end{aligned}$$

whereas the curve $\bar{C}^{(1)}$ issuing from the same point but corresponding to solution (19) has the form

$$y = x, \quad z^{(1)} = x^2 + \cos x, \quad q^{(1)} = -\sin x.$$

Thus two different curves $C^{(1)}$ and $\bar{C}^{(1)}$ can correspond to two different integral surfaces having a common point $P_1(0, 0, 1, 0)$.

Therefore it is not sufficient for the definition of a quasi-characteristic $(C^{(1)}, \ldots, C^{(m)})$ to give a set of points $P_i(x_0, \overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)})$ $(i = 1, 2, \ldots, m)$ through which the curves $C^{(i)}$ should pass. We have to establish also a solution of system (1) fulfilling conditions (3) and (7) on which the quasi-characteristic should be situated.

The above example shows also that two solutions $Z(x, Y)$ and $\bar{Z}(x, Y)$ of system (1) which are of class C^2 for any x, Y and fulfil at some point $P(\bar{x}, \bar{Y})$ the equalities

$$Z(\bar{x}, \bar{Y}) = \bar{Z}(\bar{x}, \bar{Y}), \quad z_Y^{(i)}(\bar{x}, \bar{Y}) = \bar{z}_Y^{(i)}(\bar{x}, \bar{Y}) \quad (i = 1, 2, \ldots, m)$$

do not necessarily have a common quasi-characteristic corresponding to the same set of points $P_i\big(\bar{x}, \bar{Y}, z^{(i)}(\bar{x}, \bar{Y}), z_Y^{(i)}(\bar{x}, \bar{Y})\big)$ $(i = 1, 2, \ldots, m)$.

Further properties of quasi-characteristics are given in the following theorem.

THEOREM 2. *Assume that the functions $f^{(i)}(x, Y, Z, Q)$ $(i = 1, 2, \ldots, m)$ and $\Omega(Y)$ fulfil assumption* H *and that $Z(x, Y)$ is an integral of system* (1) *fulfilling the initial conditions* (3).

Let us denote by $(C^{(1)}, \ldots, C^{(m)})$ the quasi-characteristic situated on this integral and corresponding to the system of points $P_i(x_0, \overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)})$, where $\overset{0}{Y}, \overset{0}{z}{}^{(i)}, \overset{0}{Q}{}^{(i)}$ fulfil equalities (7). *Furthermore, let us assume that for any (x, Y) of the set* (4) *the following equalities are satisfied*:

$$(20) \quad \begin{aligned} &f_{q_k}^{(1)}\big(x, Y, Z(x, Y), z_Y^{(1)}(x, Y)\big) \\ &\qquad = f_{q_k}^{(2)}\big(x, Y, Z(x, Y), z_Y^{(2)}(x, Y)\big) \\ &\qquad = \ldots = f_{q_k}^{(m)}\big(x, Y, Z(x, Y), z_Y^{(m)}(x, Y)\big) \\ &\qquad\qquad (k = 1, 2, \ldots, n). \end{aligned}$$

Under these conditions the quasi-characteristic $(C^{(1)}, \ldots, C^{(m)})$ *possesses the following properties:*

1° *The projections of the curves* $C^{(i)}$ *onto the space* x, Y *are identical for all* $C^{(i)}$ $(i = 1, 2, \ldots, m)$ *(to define a quasi-characteristic it is therefore sufficient to quote the equation* $Y = Y(x)$, *where* $Y(x) = (y_1(x), \ldots, y_n(x))$, *the common projection of the curves* $(C^{(1)}, \ldots, C^{(m)})$ *onto the space* x, Y *and the functions* $z^{(i)} = z^{(i)}x$, $Q^{(i)} = Q^{(i)}(x)$, $i = 1, 2, \ldots, m$).

2° *The functions*

$$Y = Y(x), \quad Z = Z(x), \quad Q^{(1)} = Q^{(1)}(x), \ldots, Q^{(m)} = Q^{(m)}(x), \tag{21}$$

where $Z(x) = (z^{(1)}(x), \ldots, z^{(m)}(x))$ *form the solution of the system of ordinary differential equations*

$$\begin{aligned} \frac{dy_k}{dx} &= -f^{(1)}_{q_k}(x, Y, Z, Q^{(1)}) \quad (k = 1, 2, \ldots, n), \\ \frac{dz^{(i)}}{dx} &= f^{(i)}(x, Y, Z, Q^{(i)}) - \sum_{j=1}^{n} q_j^{(i)} f^{(i)}_{q_j}(x, Y, Z, Q^{(i)}) \quad (i = 1, 2, \ldots, m), \\ \frac{dq_k^{(i)}}{dx} &= f^{(i)}_{y_k}(x, Y, Z, Q^{(i)}) - \sum_{l=1}^{m} q_k^{(l)} f^{(i)}_{z^{(l)}}(x, Y, Z, Q^{(i)}) \\ &\qquad (i = 1, 2, \ldots, m, \ k = 1, 2, \ldots, n) \end{aligned} \tag{22}$$

fulfilling the initial conditions

$$Y(x_0) = \overset{0}{Y}, \quad Z(x_0) = \overset{0}{Z}, \quad Q^{(1)}(x_0) = \overset{0}{Q}{}^{(1)}, \ldots, Q^{(m)}(x_0) = \overset{0}{Q}{}^{(m)},$$

where $\overset{0}{Z} = (\overset{0}{z}{}^{(1)}, \ldots, \overset{0}{z}{}^{(m)})$.

Basing ourselves on the above theorem we can define the characteristics for the solution of the system of partial differential equations (1).

DEFINITION 2. Assume that $Z(x, Y)$ is the solution of the system of differential equations (1) fulfilling conditions (20) in the domain D of the space x, Y and that $(\bar{x}, \bar{Y})$ is a point belonging to D. Any solution (21) of the system of ordinary differential equations (22) fulfilling the initial conditions

$$Y(\bar{x}) = \bar{Y}, \quad Z(\bar{x}) = \bar{Z}, \quad Q^{(1)}(\bar{x}) = \bar{Q}^{(1)}, \ldots, Q^{(m)}(\bar{x}) = \bar{Q}^{(m)},$$

where $\bar{Z} = Z(\bar{x}, \bar{Y})$, $\bar{Q}^{(i)} = z^{(i)}_Y(\bar{x}, \bar{Y})$ for $i = 1, 2, \ldots, m$, will be referred to as the characteristic corresponding to the solution $Z(x, Y)$ of system (1).

We shall quote an example of a family of characteristics corresponding to the solution of a system of partial differential equations.

EXAMPLE 2. Characteristics situated on the solution

$$z^{(1)}(x, y) = y^2 + 1, \quad z^{(2)}(x, y) = x^2 + y^2 \tag{23}$$

of the system of differential equations

$$(24)\qquad \begin{aligned} z_x^{(1)} &= (z^{(2)}-x^2-y^2)(z_y^{(1)})^2, \\ z_x^{(2)} &= 2x+(z^{(1)}-2z^{(2)}+2x^2+y^2-1)[(z_y^{(2)})^2-z_y^{(2)}] \end{aligned}$$

are solutions of the system of ordinary differential equations

$$(25)\qquad \begin{aligned} \frac{dy}{dx} &= -2q^{(1)}(z^{(2)}-x^2-y^2), \\ \frac{dz^{(1)}}{dx} &= -(q^{(1)})^2(z^{(2)}-x^2-y^2), \\ \frac{dz^{(2)}}{dx} &= 2x-(q^{(2)})^2(z^{(1)}-2z^{(2)}+2x^2+y^2-1), \\ \frac{dq^{(1)}}{dx} &= (q^{(1)})^2(-2y+q^{(2)}), \\ \frac{dq^{(2)}}{dx} &= [(q^{(2)})^2+q^{(2)}](2y+q^{(1)}-2q^{(2)}), \end{aligned}$$

because in this case

$$(26)\qquad f_q^{(1)}\big(x, y, z^{(1)}(x, y), z^{(2)}(x, y), z_y^{(1)}(x, y)\big) = f_q^{(2)}\big(x, y, z^{(1)}(x, y), z^{(2)}(x, y), z_y^{(2)}(x, y)\big).$$

Each of the characteristics given by the equations

$$(27)\qquad \begin{gathered} y = \eta, \quad z^{(1)} = \eta^2+1, \quad z^{(2)} = x^2+\eta^2, \\ q^{(1)} = 2\eta, \quad q^{(2)} = 2\eta, \end{gathered}$$

where η is an arbitrary constant, is situated on the solution (23) of system (24).

II. We shall now give a sufficient condition for the existence o a common characteristic for the integrals $U(x, Y) = \big(u^{(1)}(x, Y), \ldots, u^{(m)}(x, Y)\big)$ and $V(x, Y) = \big(v^{(1)}(x, Y), \ldots, v^{(m)}(x, Y)\big)$ of the system of partial differential equations (1).

First of all we shall adopt the following definition of generating the solutions of system (1) by characteristics.

DEFINITION 3. We shall say that the solution $Z(x, Y)$ of system (1) defined in the set Δ given by the inequalities

$$28)\qquad |x-x_0| < a, \quad |y_j-\overset{0}{y}_j| \leqslant b_j-M|x-x_0| \quad (j = 1, 2, \ldots, n),$$

where $a > 0$, $b_j > 0$, $a < b_j/M$, is generated by the characteristics if for any point $P(\bar{x}, \bar{Y}, \bar{Z}, \bar{Q}^{(1)}, \ldots, \bar{Q}^{(m)})$, where $(\bar{x}, \bar{Y}) \in \Delta$ and $\bar{Z} = Z(\bar{x}, \bar{Y})$, $\bar{Q}^{(i)} = z_Y^{(i)}(\bar{x}, \bar{Y})$, there exists a solution (21) of system (22) such that

1) $Y(\bar{x}) = \bar{Y}$,

2) the functions $Y(x)$ are defined on the interval $\langle x_0, \bar{x}\rangle$ or $\langle \bar{x}, x_0\rangle$ (depending on whether $\bar{x} \geqslant x_0$ or $\bar{x} < x_0$),

3) $Z(x, Y(x)) = Z(x)$, $z_Y^{(i)}(x, Y(x)) = Q^{(i)}(x)$ $(i = 1, 2, \ldots, m)$ for $x \in \langle x_0, \bar{x}\rangle$ (or $x \in \langle \bar{x}, x_0\rangle$).

Remark 5. Definition 3 is a generalization of the definition of generating by characteristics the solution of a single partial differential equation given in [7], p. 3, to the case of a system of partial differential equations.

We shall now give a theorem containing a sufficient condition for the solutions of system (1) to be generated by characteristics. It is a generalization of Theorem 3 of [1]. An analogous theorem for a single equation can be found in [6], p. 38.

THEOREM 3. *Assume that*

(α) *The functions $f^{(i)}(x, Y, Z, Q)$ $(i = 1, 2, \ldots, m)$ are of class C^1 in the convex domain Ω of the space x, Y, Z, Q, the projection of which onto the space x, Y contains domain E. The set Δ defined by inequalities* (28) *is contained in E. Moreover,*

$$|f_{q_j}^{(i)}| < M \quad (i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n). \tag{29}$$

(β) *The solution $Z(x, Y)$ of system* (1) *defined in E is of class C^1 in E and the points $(x, Y, Z(x, Y), z_Y^{(i)}(x, Y))$ belong to Ω for arbitrary $(x, Y) \in E$ and $i = 1, 2, \ldots, m$.*

(γ) *The derivatives $z_Y^{(i)}(x, Y)$ $(i = 1, 2, \ldots, m)$ fulfil in E the Lipschitz condition with respect to Y.*

(δ) *The solution $Z(x, Y)$ fulfil conditions* (20) *for $(x, Y) \in E$.*

Under these assumptions the solution $Z(x, Y)$ of system (1) *is generated by characteristics in the set Δ.*

Proof. Let us denote by $Y = Y(x)$ the solution of the system of ordinary differential equations

$$\frac{dy_i}{dx} = -f_{q_i}^{(1)}(x, Y, Z(x, Y), z_Y^{(1)}(x, Y)) \quad (i = 1, 2, \ldots, n) \tag{30}$$

fulfilling the initial condition

$$Y(\bar{x}) = \bar{Y}, \tag{31}$$

where $(\bar{x}, \bar{Y})$ is an arbitrary point of Δ. It follows from assumptions (α) and (β) that the curve $Y = Y(x)$ is defined in the interval $\langle x_0, \bar{x}\rangle$ (we assume $\bar{x} \geqslant x_0$; the remaining case is analogous).

Let us denote by $Z(x)$ and $Q^{(i)}(x)$ the functions

$$Z(x) = Z(x, Y(x)), \quad Q^{(i)}(x) = z_Y^{(i)}(x, Y(x)) \quad (i = 1, 2, \ldots, m). \tag{32}$$

We shall demonstrate that the functions $Y = Y(x)$ and the functions defined by formulas (32) are solutions of system (22).

It follows from (30) and (32) that the first n equations of system (22) are satisfied by these functions.

It follows from (1), (30), (32) and from assumption (δ) that

$$\begin{aligned}\frac{dz^{(i)}}{dx} &= z_x^{(i)}(x, Y(x)) + \sum_{j=1}^{n} z_{y_j}^{(i)}(x, Y(x)) \frac{dy_j(x)}{dx}\\ &= f^{(i)}(x, Y(x), Z(x), Q^{(i)}(x)) + \sum_{j=1}^{n} q_j^{(i)}(x) f_{q_j}^{(i)}(x, Y(x), Z(x), Q^{(i)}(x)).\end{aligned}$$

Thus the next m equations of system (22) are also satisfied. We shall now demonstrate that the functions $Y(x), Z(x), Q^{(i)}(x)$ $(i = 1, 2, \ldots, m)$ satisfy also the remaining equations of the system.

Let us denote by $Q^{(i)}(x, h) = (q_1^{(i)}(x, h), \ldots, q_n^{(i)}(x, h))$ the functions

$$q_k^{(i)}(x, h) = \frac{z^{(i)}(x, Y(x) + H_k) - z^{(i)}(x, Y(x))}{h}, \tag{33}$$

where $h \neq 0$, $H_k = (0, \ldots, 0, h, 0, \ldots, 0)$ and h is the k-th coordinate of the vector H_k. Then

$$\lim_{h \to 0} q_k^{(i)}(x, h) = z_{y_k}^{(i)}(x, Y(x)) = q_k^{(i)}(x) \tag{34}$$

and the convergence is uniform with respect to x for $x \in \langle x_0, \bar{x} \rangle$.

Differentiating functions (33) with respect to x, we obtain

$$\begin{aligned}\frac{dq_k^{(i)}(x, h)}{dx} &= \frac{1}{n}\Big\{[z_x^{(i)}(x, Y(x) + H_k) - z_x^{(i)}(x, Y(x))] +\\ &\qquad + \sum_{j=1}^{n} [z_{y_j}^{(i)}(x, Y(x) + H_k) - z_{y_j}^{(i)}(x, Y(x))]\Big\} \frac{dy_j(x)}{dx}\\ &= \frac{1}{h}\big[f^{(i)}(x, Y(x) + H_k, Z(x, Y(x) + H_k), z_Y^{(i)}(x, Y(x) + H_k)) -\\ &\qquad - f^{(i)}(x, Y(x), Z(x, Y(x)), z_Y^{(i)}(x, Y(x)))\big] -\\ &\qquad - \frac{1}{h} \sum_{j=1}^{n} [z_{y_j}^{(i)}(x, Y(x) + H_k) -\\ &\qquad - z_{y_j}^{(i)}(x, Y(x))] f_{q_j}^{(1)}(x, Y(x), Z(x, Y(x)), z_Y^{(1)}(x, Y(x))).\end{aligned} \tag{35}$$

We put

$$P_k(x, Y, t, h) = \big(x, Y + tH_k, Z(x, Y) + t[Z(x, Y + H_k) - Z(x, Y)],$$
$$z_Y^{(i)}(x, Y) + t[z_Y^{(i)}(x, Y + H_k) - z_Y^{(i)}(x, Y)]\big),$$

where $(x, Y) \in \Delta$, $0 \leqslant t \leqslant 1$.

Then

$$\lim_{h \to 0} P_k\big(x, Y(x), t, h\big) = \Big(x, Y(x), Z\big(x, Y(x)\big), z_Y^{(i)}\big(x, Y(x)\big)\Big) \tag{36}$$
$$= \big(x, Y(x), Z(x), Q^{(i)}(x)\big),$$

and the convergence is uniform with respect to x.

Applying Hadamard's mean value theorem to the extreme right-hand member of (35), we obtain

$$\frac{dq_k^{(i)}(x, h)}{dx} = \frac{1}{h}\Bigg\{\Big(\int_0^1 f_{y_k}^{(i)}\big(P_k(x, Y(x), t, h)\big) h\,dt +$$
$$+ \sum_{l=1}^{m}\Big[\Big(\int_0^1 f_{z^{(l)}}^{(i)}\big(P_k(x, Y(x), t, h)\big)dt\Big(z^{(l)}\big(x, Y(x) + H_k\big) - z^{(l)}\big(x, Y(x)\big)\Big)\Big] +$$
$$+ \sum_{j=1}^{n}\int_0^1 f_{q_j}^{(i)}\big(P_k(x, Y(x), t, h)\big)dt\big[z_{y_j}^{(i)}\big(x, Y(x) + H_k\big) - z_{y_j}^{(i)}\big(x, Y(x)\big)\big]\Bigg\} -$$
$$- \frac{1}{h}\sum_{j=1}^{n} f_{q_j}^{(i)}\Big(x, Y(x), Z\big(x, Y(x)\big), z_Y^{(i)}\big(x, Y(x)\big)\Big)\big[z_{y_j}^{(i)}\big(x, Y(x) + H_k\big) -$$
$$- z_{y_j}^{(i)}\big(x, Y(x)\big)\big] = \int_0^1 f_{y_k}^{(i)}\big(P_k(x, Y(x), t, h)\big)dt +$$
$$+ \sum_{l=1}^{m}\int_0^1 f_{z^{(l)}}^{(i)}\big(P_k(x, Y(x), t, h)\big)dt\big[z^{(l)}\big(x, Y(x) + H_k\big) - z^{(l)}\big(x, Y(x)\big)\big]\frac{1}{h} +$$
$$+ \sum_{j=1}^{n}\frac{1}{h}\big[z_{y_j}^{(i)}\big(x, Y(x) + H_k\big) - z_{y_j}^{(i)}\big(x, Y(x)\big)\big]\cdot$$
$$\cdot\Big[\int_0^1 f_{q_j}^{(i)}\big(P_k(x, Y(x), t, h)\big)dt - f_{q_j}^{(i)}\Big(x, Y(x), Z\big(x, Y(x)\big), z_Y^{(i)}\big(x, Y(x)\big)\Big)\Big].$$

It follows from assumption (γ) that the fractions

$$\frac{z_{y_j}^{(i)}\big(x, Y(x) + H_k\big) - z_{y_j}^{(i)}\big(x, Y(x)\big)}{h}$$

are bounded. Hence from assumption (δ) and from (34) and (36) we obtain in the limit for $h \to 0$

$$\frac{dq_k^{(i)}(x)}{dx} = f_{y_k}^{(i)}\big(x, Y(x), Z(x), Q^{(i)}(x)\big) + \\ + \sum_{l=1}^{m} q_k^{(l)}(x) f_{z^{(l)}}^{(i)}\big(x, Y(x), Z(x), Q^{(i)}(x)\big),$$

which means that the functions $Y(x), Z(x), Q^{(i)}(x)$ $(i = 1, 2, \ldots, m)$ satisfy the last mn equations of system (22).

Hence all the conditions of Definition 3 are satisfied and the proof is complete.

THEOREM 4. *Assume that the functions $f^{(i)}(x, Y, Z, Q)$ $(i = 1, 2, \ldots, m)$ fulfil assumption* (α) *of Theorem* 3 *and that the integrals $U(x, Y) = \big(u^{(1)}(x, Y), \ldots, u^{(m)}(x, Y)\big)$ and $V(x, Y) = \big(v^{(1)}(x, Y), \ldots, v^{(m)}(x, Y)\big)$ fulfil assumptions* (β), (γ), (δ) *of that theorem.*

Assume further that the solution of system (22) *is uniquely defined by the initial conditions.*

Assume furthermore that

$$U(\bar{x}, \bar{Y}) = V(\bar{x}, \bar{Y}), \quad u_Y^{(i)}(\bar{x}, \bar{Y}) = v_Y^{(i)}(\bar{x}, \bar{Y}) \quad (i = 1, 2, \ldots, m), \tag{37}$$

where $(\bar{x}, \bar{Y}) \in \Delta$.

Let

$$Y = Y(x), \quad Z = Z(x), \quad Q^{(i)} = Q^{(i)}(x) \quad (i = 1, 2, \ldots, m) \tag{38}$$

denote the solution of system (22) *satisfying the initial conditions*

$$Y(\bar{x}) = \bar{Y}, \quad Z(\bar{x}) = \bar{Z}, \quad Q^{(i)}(\bar{x}) = \bar{Q}^{(i)} \quad (i = 1, 2, \ldots, m), \tag{39}$$

where

$$\bar{Z} = U(\bar{x}, \bar{Y}), \quad \bar{Q}^{(i)} = u_Y^{(i)}(\bar{x}, \bar{Y}) \quad (i = 1, 2, \ldots, m). \tag{40}$$

Under these assumptions

$$U\big(x, Y(x)\big) = V\big(x, Y(x)\big) \tag{41}$$

and

$$u_Y^{(i)}\big(x, Y(x)\big) = v_Y^{(i)}\big(x, Y(x)\big) \quad (i = 1, 2, \ldots, m) \tag{42}$$

for the values of x for which $\big(x, Y(x)\big) \in \Delta$.

Proof. It follows from Theorem 3 that the solutions $U(x, Y)$ and $V(x, Y)$ of system (1) are generated in Δ by characteristics. As the solution of system (22) is defined uniquely by the initial conditions, characteristic (38) satisfying conditions (39) is situated by (37) and (40) simultaneously on the integral $U(x, Y)$ and on $V(x, Y)$. Hence follow statements (41) and (42).

EXAMPLE 3. The solution

$$\bar{z}^{(1)}(x, y) = 1, \quad \bar{z}^{(2)}(x, y) = x^2 + y^2/2$$

of system (24) and solution (23) of the same system satisfy the conditions of Theorem 3, and furthermore

$$z^{(1)}(0, 0) = \bar{z}^{(1)}(0, 0) = 1, \quad z^{(2)}(0, 0) = \bar{z}^{(2)}(0, 0) = 0,$$
$$z_y^{(1)}(0, 0) = \bar{z}_y^{(1)}(0, 0) = 0, \quad z_y^{(2)}(0, 0) = \bar{z}_y^{(2)}(0, 0) = 0.$$

Therefore these solutions possess a common characteristic which is a solution of system (25) passing through the point $P(0, 0, 1, 0, 0, 0)$. This characteristic is defined by the equations

$$y = 0, \quad z^{(1)} = 1, \quad z^{(2)} = x^2, \quad q^{(1)} = 0, \quad q^{(2)} = 0.$$

The properties of characteristics formulated in this paper for systems of partial differential equations, especially Theorems 3 and 4, will enable us to formulate more general sufficient conditions that those of paper [2], where the initial inequalities between solutions of system (1) propagate onto sets formed by the characteristics. Similar problems for the solutions of a single equation have been discussed in papers [3], [4], [5].

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