

An iterative method of solving differential equations

by Z. KOWALSKI (Kraków)

§ 1. This paper is concerned with a modified method of successive approximations which arose in a natural way in connection with some problems often occurring in practice.

The solution of a system of differential equations

$$(1,1) \quad x'_j(t) = H_j(t, x_1(t), \dots, x_n(t)), \quad j = 1, \dots, n,$$

passing for example through the point $(0, \dots, 0)$ can be found with the aid of the classical method of successive approximations, connected with the construction of a sequence ${}_p x_j(t)$, $p = 0, 1, 2, \dots$, defined by

$$(1,2) \quad \begin{aligned} {}_{p+1}x'_j(t) &= H_j(t, {}_p x_1(t), \dots, {}_p x_n(t)), \quad j = 1, \dots, n, \\ {}_{p+1}x_j(0) &= {}_p x_j(0) = 0, \quad j = 1, \dots, n, \quad p = 0, 1, 2, \dots, \end{aligned}$$

the first approximation ${}_0 x_j(t)$, $j = 1, \dots, n$, being fixed.

In the first modification of this method occurring in practice the right-hand member H_j of (1,2) is of the form

$$(1,3) \quad H_j = L_j(x_1, \dots, x_n) + R_j(t, x_1, \dots, x_n), \quad j = 1, \dots, n,$$

where $L_j = a_j + \sum_{k=1}^n b_{jk} x_k$, $j = 1, \dots, n$, denotes the linear part of H_j , the derivatives of the perturbations R_j with respect to t and x_k are zero at the point $(t = 0, x_j = 0, j = 1, \dots, n)$, and the successive approximations are defined by

$$(1,4) \quad \begin{aligned} {}_{p+1}x'_j(t) &= L_j({}_{p+1}x_1(t), \dots, {}_{p+1}x_n(t)) + R_j(t, {}_p x_1(t), \dots, {}_p x_n(t)), \\ {}_{p+1}x_j(0) &= {}_p x_j(0) = 0, \quad j = 1, \dots, n, \quad p = 0, 1, 2, \dots \end{aligned}$$

The modified construction (1,4) of the sequence of successive approximations involves, at every step, the solution of the linear differential equations (1,4) for the functions ${}_{p+1}x_1(t), \dots, {}_{p+1}x_n(t)$ and the convergence is often more rapid than in case (1,2) (cf. Collatz [1], p. 183, Beispiel B).

Such a modification of the method leads in a natural way to more general systems than (1,4), namely to

$$(1,5) \quad x'_j(t) = Q_j(t, x_1(t), \dots, x_n(t), x_1(t), \dots, x_n(t)), \quad j = 1, \dots, n,$$

with successive approximations defined by

$$(1,6) \quad \begin{aligned} {}_{p+1}x'_j(t) &= Q_j(t, {}_px_1(t), \dots, {}_px_n(t), {}_px_1(t), \dots, {}_px_n(t)), \quad j = 1, \dots, n, \\ {}_{p+1}x_j(0) &= {}_px_j(0) = 0, \quad j = 1, \dots, n, \quad p = 0, 1, 2, \dots \end{aligned}$$

For the investigations of convergence in that case cf. T. Ważewski [4] and [5].

The second modification of the process of successive approximations arose in connection with some technical problems, where the right-hand as well as the left-hand members of differential equations

$$(1,7) \quad x'_j(t) = f_j(t, x_1(t), \dots, x_n(t), x'_1(t), \dots, x'_n(t)), \quad j = 1, \dots, n,$$

contained the derivatives of unknown functions.

One might expect that two steps, A and B, are needed for the solution of (1,7), namely:

Step A: solve the implicit equations

$$(1,8) \quad z_j = f_j(t, x_1, \dots, x_n, z_1, \dots, z_n), \quad j = 1, \dots, n,$$

with respect to z_1, \dots, z_n so as to obtain the equivalent (at least locally equivalent) system

$$(1,9) \quad z_j = F_j(t, x_1, \dots, x_n), \quad j = 1, \dots, n,$$

and consider the system of differential equations

$$(1,10) \quad x'_j(t) = F_j(t, x_1(t), \dots, x_n(t)), \quad j = 1, \dots, n,$$

instead of the equivalent system (1,7).

Step B: solve the system (1,10), for example with the aid of successive approximations defined by

$$(1,11) \quad \begin{aligned} {}_{p+1}x'_j(t) &= F_j(t, {}_px_1(t), \dots, {}_px_n(t)), \quad j = 1, \dots, n, \\ {}_{p+1}x_j(0) &= {}_px_j(0) = 0, \quad j = 1, \dots, n, \quad p = 0, 1, 2, \dots, \end{aligned}$$

where p denotes the index of successive approximation.

However, only one step is needed when the successive approximations are defined in a different way, namely by

$$(1,12) \quad \begin{aligned} {}_{p+1}x'_j(t) &= f_j(t, {}_px_1(t), \dots, {}_px_n(t), {}_px'_1(t), \dots, {}_px'_n(t)), \quad j = 1, \dots, n, \\ {}_{p+1}x_j(0) &= {}_px_j(0) = 0, \quad j = 1, \dots, n, \quad p = 0, 1, 2, \dots \end{aligned}$$

That form is especially convenient for numerical computation of successive approximations, since then the solving of (1,8) can be avoided, which reduces the required labour.

The third modification of the method of successive approximations deals with more general systems than (1,5), namely with

$$(1,13) \quad x'_j(t) = h_j(t, x_1(t), \dots, x_n(t), x_1(t), \dots, x_n(t), x'_1(t), \dots, x'_n(t)), \\ j = 1, \dots, n,$$

the successive approximations being defined analogously to (1.6), by

$$(1,14) \quad {}_{p+1}x'_j(t) = h_j(t, {}_px_1(t), \dots, {}_px_n(t), {}_px_1(t), \dots, {}_px_n(t), {}_px'_1(t), \dots, {}_px'_n(t)), \\ {}_{p+1}x_j(0) = {}_px_j(0) = 0, \quad j = 1, \dots, n, \quad p = 0, 1, 2, \dots$$

In that case the proof of convergence of successive approximations involves—as will be seen at the end of the paper—difficulties different in type from those encountered in connection with (1,2) or even (1,6).

We shall deal from the beginning with the differential equation

$$(1,15) \quad y' = h(t, y, y, y'),$$

in a complete Banach space with a homogeneous norm, since then complicated calculations can be avoided, the system (1,13) being a special case of (1,15).

In that case we solve the problem of the location of successive approximations and of the solution, we evaluate the length of the interval in question, and derive some error estimates for successive approximations.

The example in § 7 gives an illustration of the fact that the results can be applied to numerical computations.

The method of proofs used here systematically, i.e. that of differential inequalities, supplies non-local theorems, the convergence of successive approximations being obtained without a series of comparison.

I should like to use this opportunity to thank prof. T. Ważewski for many valuable remarks during the preparation of this paper.

§ 2. We shall make use of the following well-known theorems:

THEOREM A. *Let us suppose that the right-hand member of the differential equation*

$$(2,1) \quad z' = F(t, z),$$

is a real-valued function of real variables t, z , continuous for $0 \leq t < a$, $-\infty < z < +\infty$.

Suppose that the real-valued and continuous function $\lambda(t)$, $\lambda(0) = 0$, possesses the right upper derivative $\bar{D}_+\lambda(t)$, and

$$(2,2) \quad \bar{D}_+\lambda(t) \leq F(t, \lambda(t)), \quad \text{for } 0 \leq t < a.$$

Let $z(t)$ be the greatest solution of the equation (2,1) through the point $(0, 0)$, defined in the interval $0 \leq t < a$.

Under these assumptions

$$(2,3) \quad \lambda(t) \leq z(t), \quad \text{for } 0 \leq t < a.$$

THEOREM B. *Let $\varphi(t)$, $\varphi(0) = 0$, be a function defined for real $t \in I$:*

$$I: \quad 0 \leq t < a,$$

with values in a complete Banach space with a homogeneous norm.

Suppose that there exists a derivative $\varphi'(t)$ for $t \in I$, and

$$\|\varphi'(t)\| \leq F(t, \|\varphi(t)\|),$$

where $F(t, z)$ is a real-valued function of real variables (t, z) , continuous for $0 \leq t < a$, $-\infty < z < +\infty$.

Finally, let $z(t)$ be the greatest solution of the equation (2,1), passing through the point $(0, 0)$, and defined in the interval $0 \leq t \leq a$.

Under these assumptions the function $\varphi(t)$ satisfies the inequality

$$\|\varphi(t)\| \leq z(t), \quad \text{for } t \in I.$$

Proof. It is sufficient to apply theorem A and the inequality

$$\bar{D}_+ \|\varphi(t)\| \leq \|\varphi'(t)\|, \quad \text{for } t \in I,$$

given by T. Ważewski [6].

THEOREM C (Dini). *Suppose that the real-valued functions $z_n(t)$, $n = 0, 1, 2, \dots$, continuous in the interval I*

$$(2,4) \quad I: \quad 0 \leq t < a \quad (a \leq +\infty),$$

satisfy the conditions

$$(2,5) \quad z(t) \leq z_{n+1}(t) \leq z_n(t), \quad \text{for } t \in I, \quad n = 0, 1, 2, \dots,$$

$$(2,6) \quad z_n(t) \rightarrow z(t), \quad \text{as } n \rightarrow +\infty, \quad t \in I,$$

where the function $z(t)$ is continuous in the interval I .

Then

$$z_n(t) \Rightarrow z(t), \quad \text{as } n \rightarrow +\infty, \quad t \in I,$$

where the sign \Rightarrow denotes the almost uniform convergence in the interval I , i.e. the uniform convergence on every closed interval, bounded and contained in I (cf. Sikorski [3], p. 143).

THEOREM D. *Consider the differential equation*

$$(2,7) \quad u' = g(t, u),$$

and denote by I the interval of the real t -axis:

$$I: \quad 0 \leq t < a.$$

Suppose that:

1° The function $g(t, u)$ is continuous on the set

$$\Omega: \quad 0 \leq t < a, \quad \|u\| < b,$$

where u is an element of a complete Banach space with a homogeneous norm and with the values of $g(t, u)$ in B :

$$g(t, u) \in B, \quad \text{for } (t, u) \in \Omega.$$

2° $g(t, u)$ satisfies the Lipschitz condition

$$\|g(t, \bar{u}) - g(t, u)\| \leq M \cdot \|\bar{u} - u\|,$$

for $(t, \bar{u}) \in \Omega, (t, u) \in \Omega$.

3° All terms $u_n(t)$ of the sequence of successive approximations

$$\begin{aligned} u_0(t) &\equiv 0, \quad t \in I, \\ (2,8) \quad u'_{n+1}(t) &= g(t, u_n(t)), \quad t \in I, \\ u_{n+1}(0) &= u_n(0) = 0, \quad n = 0, 1, 2, \dots, \end{aligned}$$

will not get out of the region

$$S: \quad 0 \leq t < a, \quad \|u\| \leq \sigma(t),$$

where the real-valued function $\sigma(t)$ of the class C^1 in the interval I , satisfies the conditions $0 < \sigma(t) < b$, for $0 < t < a$, $\sigma(0) = 0$.

Thus we have

$$\|u_n(t)\| \leq \sigma(t), \quad \text{for } t \in I, \quad n = 0, 1, 2, \dots$$

Under these assumptions:

a) the sequence of successive approximations converges to the function $\psi(t)$

$$u_n(t) \Rightarrow \psi(t), \quad \text{as } n \rightarrow +\infty, \quad t \in I,$$

where the sign \Rightarrow denotes the almost uniform convergence in the interval I , i.e. the uniform convergence on every closed interval, bounded and contained in I .

b) $\psi(t)$ is the unique solution of equation (2,7) for $t \in I$, satisfying the initial condition $\psi(0) = 0$;

c) the inequality

$$\|\psi(t)\| \leq \sigma(t),$$

holds for $t \in I$.

The proof of this theorem does not differ from that of the classical theorem on the convergence of successive approximations, cf. Lusternik, Sobolew [2].

§ 3. Throughout the rest of the paper we shall use the following assumptions H:

ASSUMPTIONS H. 1) Assume that the function $h(t, u, v, w)$ is defined and continuous for $(t, u, v, w) \in \omega$, where

$$(3,1) \quad \omega: \quad 0 \leq t < a, \quad \|u\| < b, \quad \|v\| < b, \quad \|w\| < c, \\ (a \leq +\infty, b \leq +\infty, c \leq +\infty).$$

2) The values of $h(t, u, v, w)$ are in the Banach space B :

$$h(t, u, v, w) \in B, \quad \text{for} \quad (t, u, v, w) \in \omega.$$

3) The Lipschitz condition

$$(3,2) \quad \|h(t, \bar{u}, \bar{v}, \bar{w}) - h(t, u, v, w)\| \leq M \cdot \|\bar{u} - u\| + N \cdot \|\bar{v} - v\| + L \cdot \|\bar{w} - w\|,$$

holds for $(t, \bar{u}, \bar{v}, \bar{w}) \in \omega$, $(t, u, v, w) \in \omega$, with arbitrary constants M and N , and the constant L satisfying

$$(3,3) \quad 0 \leq L < 1.$$

We suppose also that

$$\|h(0, 0, 0, 0)\| < c(1 - L).$$

Denote by $s(t)$ the solution of the non-homogeneous linear differential equation

$$(3,4) \quad s'(t) = \frac{M + N}{1 - L} \cdot s(t) + \frac{\|h(t, 0, 0, 0)\|}{1 - L},$$

satisfying the initial condition $s(0) = 0$, and let I' be the greatest interval contained in the interval I :

$$(3,5) \quad I': \quad 0 \leq t < \alpha \quad (\alpha \leq a),$$

such that

$$(3,6) \quad s(t) < b, \quad s'(t) < c, \quad \text{for} \quad t \in I'.$$

The existence of the interval I' follows from the theorem on continuations of the solutions of differential equations.

§ 4. The first lemma to be proved can be formulated as follows:

LEMMA 1. Suppose that the sequence of real functions $z_n(t)$, $t \in I'$, $z_n(0) = 0$, $n = 0, 1, 2, \dots$, satisfies the following assumptions:

1° The function $z_0(t)$ is given by the formula

$$(4,1) \quad z_0(t) = 2 \cdot s(t), \quad t \in I',$$

where $s(t)$ is the solution of the equation (3,4).

2° The function $z_{n+1}(t)$ is the solution $\zeta = z_{n+1}(t)$, $t \in I'$, of the non-homogeneous linear differential equation

$$(4,2) \quad \zeta'(t) = M \cdot \zeta(t) + N \cdot z_n(t) + L \cdot z_n'(t),$$

and $\zeta(0) = z_{n+1}(0) = 0$.

Thus (4,2) becomes

$$(4,3) \quad z_{n+1}'(t) = M \cdot z_{n+1}(t) + N \cdot z_n(t) + L \cdot z_n'(t), \quad t \in I',$$

identically for $t \in I'$.

Under these assumptions the sequence $z_n(t)$ satisfies the following conditions:

$$(4,4) \quad z_n(t) \geq 0, \quad z_n'(t) \geq 0, \quad t \in I', \quad n = 0, 1, 2, \dots,$$

$$(4,5) \quad z_n(t) \geq z_{n+1}(t) \geq 0, \quad z_n'(t) \geq z_{n+1}'(t) \geq 0,$$

for $t \in I'$, $n = 0, 1, 2, \dots$,

$$(4,6) \quad z_n'(t) \geq M \cdot z_n(t) + N \cdot z_n(t) + L \cdot z_n'(t),$$

for $t \in I'$, $n = 0, 1, 2, \dots$,

$$(4,7) \quad z_n(t) \Rightarrow 0, \quad z_n'(t) \Rightarrow 0, \quad \text{as } n \rightarrow +\infty, \quad t \in I',$$

where the sign \Rightarrow denotes the almost uniform convergence in the interval I' , i.e. the uniform convergence in every closed interval, bounded and contained in I' .

Proof. From the assumptions 1° and 2° it follows that the sequence $z_n(t)$, $n = 0, 1, 2, \dots$, is uniquely determined when the solution $s(t)$, $s(0) = 0$, $t \in I'$, of equation (3,4) is given.

First we shall prove that the function $z_0(t)$ satisfies the conditions

$$(A_0) \quad z_0(t) \geq 0, \quad z_0'(t) \geq 0, \quad t \in I',$$

and

$$(B_0) \quad z_0'(t) \geq M \cdot z_0(t) + N \cdot z_0(t) + L \cdot z_0'(t), \quad t \in I'.$$

In fact, condition (A_0) is fulfilled, since $s(t)$, $s(0) = 0$, is the solution of the non-homogeneous linear differential equation (3,4); solving (3,4) effectively, we show that $s(t) \geq 0$ for $t \in I'$.

Condition (B_0) is also fulfilled. In fact, from (3,4) follows

$$s'(t) \geq M \cdot s(t) + N \cdot s(t) + L \cdot s'(t);$$

multiplying both sides by 2 and comparing this result with (4,1) we obtain inequality (B_0) .

Now we proceed by induction. We assume that for some integer n the function $z_n(t)$, $t \in I'$, satisfies the inequalities

$$(A_n) \quad z_n(t) \geq 0, \quad z_n'(t) \geq 0, \quad t \in I',$$

$$(B_n) \quad z_n'(t) \geq M \cdot z_n(t) + N \cdot z_n(t) + L \cdot z_n'(t).$$

We shall prove that the function $z_{n+1}(t)$, obtained from the non-homogeneous linear equation (4,2) and the initial condition $z_{n+1}(0) = 0$, satisfies the inequalities

$$(A_{n+1}) \quad z_n(t) \geq z_{n+1}(t) \geq 0, \quad z'_n(t) \geq z'_{n+1}(t) \geq 0,$$

$$(B_{n+1}) \quad z'_{n+1}(t) \geq M \cdot z_{n+1}(t) + N \cdot z_n(t) + L \cdot z'_n(t).$$

We prove successively (A_{n+1}) and (B_{n+1}) .

a) First we show that

$$(4,8) \quad z_n(t) \geq z_{n+1}(t).$$

In fact, the function $z_n(t)$ satisfies the differential inequality (B_n) (because of the induction assumption), the function $z_{n+1}(t)$ satisfies the differential equation (4,6) (because of the definition), and both functions satisfy the same initial condition $z_n(0) = z_{n+1}(0) = 0$; hence the theorem on differential inequalities implies that $z_n(t) \geq z_{n+1}(t)$. This completes the proof of part a).

b) We show that $z_{n+1}(t) \geq 0$.

In fact, let us substitute the function $\gamma(t) \equiv 0$, $t \in I'$, in the differential equation (4,2) in place of $\zeta(t)$; then we obtain the differential inequality

$$\gamma'(t) \equiv 0 \leq M \cdot 0 + N \cdot z_n(t) + L \cdot z'_n(t),$$

because inequalities (A_n) are fulfilled.

So the function $\gamma(t) \equiv 0$ decreases with respect to the differential equation (4,2), the function $z_{n+1}(t)$ satisfies equation (4,2), and the initial values are equal: $\gamma(0) = z_{n+1}(0) = 0$, whence

$$(4,9) \quad z_{n+1}(t) \geq \gamma(t) \equiv 0.$$

This completes the proof of part b).

c) We show that

$$(4,10) \quad z'_n(t) \geq z'_{n+1}(t).$$

In fact, (4,3), (4,8) and (B_n) imply

$$\begin{aligned} z'_{n+1}(t) &= M \cdot z_{n+1}(t) + N \cdot z_n(t) + L \cdot z'_n(t) \\ &\leq M \cdot z_n(t) + N \cdot z_n(t) + L \cdot z'_n(t) \leq z'_n(t). \end{aligned}$$

This completes the proof of part c).

d) Now we shall prove that $z'_{n+1}(t) \geq 0$.

In fact, because of (4,3), (4,9) and (A_n) we have

$$z'_{n+1}(t) = M \cdot z_{n+1}(t) + N \cdot z_n(t) + L \cdot z'_n(t) \geq 0.$$

This completes the proof of part d).

e) We show that $z_{n+1}(t)$ satisfies inequality (B_{n+1}) .

In fact, (4,3), (4,8) and (4,10) implies that

$$\begin{aligned} z'_{n+1}(t) &= M \cdot z_{n+1}(t) + N \cdot z_n(t) + L \cdot z'_n(t) \\ &\geq M \cdot z_{n+1}(t) + N \cdot z_{n+1}(t) + L \cdot z'_{n+1}(t). \end{aligned}$$

This completes the proof of part e).

Thus we have obtained the desired inequalities (A_{n+1}) and (B_{n+1}) . These inequalities are true for all $n = 0, 1, 2, \dots$, because of the principle of finite induction, and this completes the proof of (4,4), (4,5) and (4,6).

What remains to be shown is that (4,7) holds.

First we observe that the sequences $z_n(t)$ and $z'_n(t)$ converge because of (4,5):

$$(4,11) \quad z_n(t) \rightarrow z(t) \geq 0, \quad z'_n(t) \rightarrow r(t) \geq 0, \quad \text{as } n \rightarrow +\infty, \quad t \in I'.$$

Hence the sequence of derivatives $z'_n(t)$, $n = 0, 1, 2, \dots$, is uniformly bounded on every closed interval bounded and contained in the interval I' (since inequalities (A_{n+1}) hold). This implies that the functions $z_n(t)$, $n = 0, 1, 2, \dots$, are equicontinuous in every closed interval bounded and contained in the interval I' . But the convergence and equicontinuity of functions $z_n(t)$, $n = 0, 1, 2, \dots$, implies almost uniform convergence on I' :

$$(4,12) \quad z_n(t) \Rightarrow z(t), \quad \text{as } n \rightarrow +\infty, \quad t \in I',$$

therefore the limit function $z(t)$ is continuous on I' .

We shall show that the function $r(t)$ is also continuous on I' .

In fact, $z_{n+1}(t)$ satisfies identity (4,3), whence we have, as $n \rightarrow +\infty$,

$$r(t) = (M + N) \cdot z(t) + L \cdot r(t),$$

and

$$r(t) = \frac{M + N}{1 - L} \cdot z(t).$$

Consequently, the continuity of $z(t)$ implies the continuity of $r(t)$ in the interval I' .

Now, we can apply theorem C and we obtain the almost uniform convergence:

$$(4,13) \quad z'_n(t) \Rightarrow r(t), \quad \text{as } n \rightarrow +\infty, \quad t \in I'.$$

Furthermore, from (4,12), (4,13) and the classical theorem follows $r(t) = z'(t)$, $t \in I'$, whence

$$z_n(t) \Rightarrow z(t), \quad z'_n(t) \Rightarrow z'(t), \quad \text{as } n \rightarrow +\infty, \quad t \in I'.$$

It will be shown that $z(t) \equiv 0$. In fact, it may be seen from (4,2) that, if $\zeta = z_{n+1}(t)$ and $n \rightarrow +\infty$, then

$$z'(t) = (M + N) \cdot z(t) + L \cdot z'(t).$$

This differential equation for $z(t)$ and the initial condition $z(0) = 0$ imply $z(t) \equiv 0$ for $t \in I'$.

Hence (4,7) holds. This concludes the proof of lemma 1.

§ 5. Now we shall prove a theorem connected with the existence and location of successive approximations for the equation

$$y' = h(t, y, y, y').$$

THEOREM 1. *Suppose that the function $h(t, u, v, w)$ satisfies assumptions H, and the first approximation $y_0(t)$ is of the form*

$$(5,1) \quad y_0(t) \equiv 0, \quad t \in I'.$$

Under these assumptions

1° the sequence of successive approximations is determined uniquely by the first approximation $y_0(t) \equiv 0$, $t \in I'$, and the conditions

$$(5,2) \quad y'_{n+1}(t) = h(t, y_{n+1}(t), y_n(t), y'_n(t)),$$

where $y_{n+1}(0) = y_n(0) = 0$, $n = 0, 1, 2, \dots$, $t \in I'$.

2° Successive approximations satisfy the inequalities

$$(5,3) \quad \|y_n(t)\| \leq s(t), \quad \|y'_n(t)\| \leq s'(t),$$

for $t \in I'$: $n = 0, 1, 2, \dots$, where $s(t)$ is the solution of the non-homogeneous linear differential equation (3,4) and $s(0) = 0$.

Proof. Let us consider the differential equation for $\tilde{y}(t)$:

$$(5,4) \quad \tilde{y}'(t) = h(t, \tilde{y}(t), y(t), y'(t)),$$

where $y(t)$, $y'(t) \in B$, $y(0) = 0$, is a given function of class C^1 on the interval I' , and

$$(5,5) \quad \|y(t)\| \leq s(t), \quad \|y'(t)\| \leq s'(t), \quad t \in I'.$$

It will be shown that then the solution $\tilde{y}(t)$, $\tilde{y}(t) \in B$, of equation (5,4), satisfying the condition $\tilde{y}(0) = 0$, exists on I' and fulfils the inequalities

$$(5,6) \quad \|\tilde{y}(t)\| \leq s(t), \quad \|\tilde{y}'(t)\| \leq s'(t), \quad t \in I',$$

in the interval I' : $0 \leq t < a$.

For that purpose we prove first that the inequalities

$$(5,7) \quad \|\tilde{y}_\mu(t)\| \leq s(t), \quad \|\tilde{y}'_\mu(t)\| \leq s'(t),$$

hold for $t \in I'$, $\mu = 0, 1, 2, \dots$, where $\tilde{y}_\mu(t)$, $\mu = 0, 1, 2, \dots$, denotes the sequence of successive approximations for differential equation (5,4):

$$(5,8) \quad \begin{aligned} \tilde{y}_0(t) &\equiv 0, \quad t \in I', \\ \tilde{y}'_{\mu+1}(t) &= h(t, \tilde{y}_\mu(t), y(t), y'(t)), \quad t \in I', \\ \tilde{y}_{\mu+1}(0) &= y_\mu(0) = 0, \quad \mu = 0, 1, 2, \dots \end{aligned}$$

We observe that the first approximation $\tilde{y}_0(t) \equiv 0$, $t \in I'$, satisfies the inequalities

$$\|\tilde{y}_0(t)\| \leq s(t), \quad \|\tilde{y}'_0(t)\| \leq s'(t), \quad t \in I',$$

where $s(t)$ is the solution of the differential equation (3,4) and $s(0) = 0$.

Proceeding by induction, assume that the approximation $\tilde{y}_\mu(t)$, $t \in I'$, satisfies the inequalities

$$(5,9) \quad \|\tilde{y}_\mu(t)\| \leq s(t), \quad \|\tilde{y}'_\mu(t)\| \leq s'(t), \quad t \in I'.$$

We prove that the next approximation $\tilde{y}_{\mu+1}(t)$ satisfies

$$(5,10) \quad \|\tilde{y}_{\mu+1}(t)\| \leq s(t), \quad \|\tilde{y}'_{\mu+1}(t)\| \leq s'(t), \quad t \in I'.$$

In fact, from (5,5), (5,9) and (3,4) follows

$$\begin{aligned} \|\tilde{y}'_{\mu+1}(t)\| &= \|h(t, \tilde{y}_\mu(t), y(t), y'(t))\| \\ &\leq \|h(t, \tilde{y}_\mu(t), y(t), y'(t)) - h(t, 0, 0, 0)\| + \|h(t, 0, 0, 0)\| \\ &\leq M \cdot \|\tilde{y}_\mu(t)\| + N \cdot \|y(t)\| + L \cdot \|y'(t)\| + \|h(t, 0, 0, 0)\| \\ &\leq M \cdot s(t) + N \cdot s(t) + L \cdot s'(t) + \|h(t, 0, 0, 0)\| = s'(t), \quad t \in I', \end{aligned}$$

i.e.

$$(5,11) \quad \|\tilde{y}'_{\mu+1}(t)\| \leq s'(t), \quad t \in I'.$$

Consider the differential equation for the function $\delta(t)$:

$$(5,12) \quad \delta'(t) = s'(t).$$

The function $s(t)$ is the only (and also the greatest) solution of equation (5,12), satisfying the condition $\delta(0) = 0$, whence from (5,11), (5,12) and theorem B follows

$$(5,13) \quad \|\tilde{y}_{\mu+1}(t)\| \leq s(t), \quad t \in I',$$

and inequalities (5,10) hold.

By finite induction all approximations $\tilde{y}_\mu(t)$, $\mu = 0, 1, 2, \dots$, for equation (5,4) satisfy conditions (5,7), whence assumption 3° of theorem D is fulfilled.

Assumptions 1° and 2° of theorem D are also fulfilled, since the function $h(t, \tilde{y}, y(t), y'(t))$ is continuous with respect to (t, \tilde{y}) on the set

$$t \in I', \quad \|\tilde{y}\| < b,$$

and satisfies in it the Lipschitz condition with respect to \tilde{y} (the Lipschitz constant being M).

Hence theorem D implies the existence of one and only one solution $\tilde{y}(t)$ of equation (5,4), satisfying the condition $\tilde{y}(0) = 0$. In addition, from theorem D and inequality (5,11) one obtains, as $\mu \rightarrow +\infty$:

$$\|\tilde{y}(t)\| \leq s(t), \quad \|\tilde{y}'(t)\| \leq s'(t), \quad t \in I'.$$

This concludes the proof of (5,6).

We shall now deal with the sequence of successive approximations (5,2). The first approximation $y_0(t)$, $t \in I'$ satisfies the inequalities

$$\|y_0(t)\| \leq s(t), \quad \|y'_0(t)\| \leq s'(t), \quad t \in I',$$

because of definition (5,1).

Hence the next approximation $y_1(t)$ is determined uniquely on the interval I' , and fulfil the inequalities

$$\|y_1(t)\| \leq s(t), \quad \|y'_1(t)\| \leq s'(t), \quad t \in I'.$$

By induction we easily verify that all approximations $y_n(t)$, $n = 0, 1, 2, \dots$, satisfy inequalities (5,3).

This completes the proof of theorem 1.

§ 6. We shall now prove the convergence of successive approximations and the existence of the solution.

THEOREM 2. *Suppose that the right-hand member of the differential equation*

$$(6,1) \quad y' = h(t, y, y, y'),$$

satisfies assumptions H.

Under these assumptions

1° *The sequence of successive approximations defined by means of the formula*

$$(6,2) \quad \begin{aligned} y_0(t) &\equiv 0, \quad t \in I', \\ y'_{n+1}(t) &= h(t, y_{n+1}(t), y_n(t), y'_n(t)), \quad t \in I', \\ y_{n+1}(0) &= y_n(0) = 0, \quad n = 0, 1, 2, \dots, \end{aligned}$$

converges almost uniformly on the interval I' to the solution $y = \varphi(t)$, $t \in I'$, of equation (6,1), satisfying the initial condition $\varphi(0) = 0$.

2° *The error estimates of the form*

$$(6,3) \quad \|y_{n+1}(t) - \varphi(t)\| \leq z_{n+1}(t),$$

$$(6,4) \quad \|y'_{n+1}(t) - \varphi'(t)\| \leq z'_{n+1}(t), \quad t \in I', \quad n = 0, 1, 2, \dots,$$

hold for the approximation $y_{n+1}(t)$ and its derivative $y'_{n+1}(t)$.

Here the function $z_0(t)$ is defined by means of the formula (4,1), where $s(t)$, $s(0) = 0$, denotes the solution of the differential equation (3,4); furthermore, if the function $z_n(t)$, $z_n(0) = 0$, is defined, then the function $z_{n+1}(t)$ is the solution $\zeta = z_{n+1}(t)$ of the non-homogeneous linear differential equation (4,2) satisfying the initial condition $\zeta(0) = z_{n+1}(0) = 0$.

Hence the identity (4,3) holds for $t \in I'$.

Proof. a) First it will be shown that the sequence $y_n(t)$ of successive approximations and the sequence of derivatives $y'_n(t)$, $n = 0, 1, 2, \dots$, satisfy Cauchy criterion for uniform convergence.

To this end, let $z_n(t)$, $z_n(0) = 0$, $t \in I'$, $n = 0, 1, 2, \dots$, be a sequence of real-valued functions, as in lemma 1.

We shall verify first that for arbitrary integers p, q satisfying $p \geq 0$, $q \geq 0$, the inequalities

$$(C_0) \quad \|y_p(t) - y_q(t)\| \leq z_0(t), \quad \|y'_p(t) - y'_q(t)\| \leq z'_0(t),$$

hold for $t \in I'$.

In fact, inequalities (C_0) are satisfied according to (4,1) and part 2° of theorem 1 (that connected with the location of the sequence of successive approximations).

Proceeding by induction, assume that for some positive integer n and arbitrary integers $p \geq n$, $q \geq n$ the inequalities

$$(C_n) \quad \|y_p(t) - y_q(t)\| \leq z_n(t), \quad \|y'_p(t) - y'_q(t)\| \leq z'_n(t),$$

hold for $t \in I'$.

We shall prove that for arbitrary integers $p \geq n+1$, $q \geq n+1$, the inequalities

$$(C_{n+1}) \quad \|y_p(t) - y_q(t)\| \leq z_{n+1}(t), \quad \|y'_p(t) - y'_q(t)\| \leq z'_{n+1}(t),$$

are satisfied for $t \in I'$.

In fact, let us consider arbitrary $p \geq n+1$, $q \geq n+1$; then $p-1 \geq n$, $q-1 \geq n$, whence according to the Lipschitz condition and assumption (C_n) we obtain successively

$$\begin{aligned} & \|y'_p(t) - y'_q(t)\| \\ & \leq M \cdot \|y_p(t) - y_q(t)\| + N \cdot \|y_{p-1}(t) - y_{q-1}(t)\| + L \cdot \|y'_{p-1}(t) - y'_{q-1}(t)\|, \end{aligned}$$

and

$$(6,5) \quad \|y'_p(t) - y'_q(t)\| \leq M \cdot \|y_p(t) - y_q(t)\| + N \cdot z_n(t) + L \cdot z'_n(t).$$

Thus the difference $\|y_p(t) - y_q(t)\|$ satisfies the differential inequality (6,5), the function $z_{n+1}(t)$ satisfies the differential equation (4,2), and the initial values are equal: $\|y_p(0) - y_q(0)\| = z_{n+1}(0) = 0$, whence

$$(6,6) \quad \|y_p(t) - y_q(t)\| \leq z_{n+1}(t), \quad t \in I',$$

for arbitrary $p \geq n+1$, $q \geq n+1$.

Furthermore, (6,5), (6,6) and (4,3) imply

$$\begin{aligned} \|y'_p(t) - y'_q(t)\| & \leq M \cdot \|y_p(t) - y_q(t)\| + N \cdot z_n(t) + L \cdot z'_n(t) \\ & \leq M \cdot z_{n+1}(t) + N \cdot z_n(t) + L \cdot z'_n(t) = z'_{n+1}(t). \end{aligned}$$

This concludes the proof of inequalities (C_{n+1}) .

By induction, inequalities (C_n) hold for all $n = 0, 1, 2, \dots$ (as well as $p \geq n$, $q \geq n$).

Consequently, the condition (4,7), obtained in lemma 1, implies that the sequence of successive approximations $y_n(t)$ and the sequence of derivatives $y'_n(t)$ satisfy the Cauchy criterion for almost uniform convergence.

b) Denote by $\varphi(t)$ the limit function of the almost uniformly convergent sequence $y_n(t)$. The convergence of derivatives $y'_n(t)$ is also almost uniform, whence

$$y_n(t) \Rightarrow \varphi(t), \quad y'_n(t) \Rightarrow \varphi'(t), \quad \text{as } n \rightarrow +\infty, \quad t \in I'.$$

In view of

$$y'_{n+1}(t) = h(t, y_{n+1}(t), y_n(t), y'_n(t)), \quad t \in I', \quad n = 0, 1, 2, \dots,$$

the last two lines of the formula imply that

$$\varphi'(t) = h(t, \varphi(t), \varphi(t), \varphi'(t)), \quad t \in I',$$

and $\varphi(0) = 0$, since $y_n(0) = 0$, $n = 0, 1, 2, \dots$

This proves part 1° of the theorem.

c) The proof of the second part follows from (C_{n+1}) , as $q \rightarrow +\infty$, since then

$$(6,7) \quad \|y_p(t) - \varphi(t)\| \leq z_{n+1}(t),$$

$$(6,8) \quad \|y'_p(t) - \varphi'(t)\| \leq z'_{n+1}(t),$$

for $t \in I'$, $p \geq n+1$.

In those lines of the formula $\varphi(t)$ denotes the solution of the differential equation (6,1), and $z_{n+1}(t)$ — the solution of the linear equation (4,2) (cf. lemma 1).

This completes the proof of the theorem.

Remark 1. There is no loss of generality in assuming

$$y_0(t) \equiv 0, \quad t \in I',$$

for the first approximation. In fact, if the first approximation $\eta(t)$ satisfies the conditions

$$\eta(t) \not\equiv 0, \quad \eta(0) = 0, \quad t \in I',$$

then the sequence of successive approximations and the sequence of derivatives converge almost uniformly to the solution $\varphi(t)$ and the derivative $\varphi'(t)$ respectively.

In order to see this it is sufficient to apply the transformation

$$(6,9) \quad y - \eta(t) = Y,$$

since then (6,9) transforms equation (6,1) into a new differential equation with the right-hand member satisfying the Lipschitz condition with the same constants M, N, L , and the first approximation is identically zero.

§ 7. EXAMPLE. It is not easy to solve the differential equation

$$(7,1) \quad y' = t + (y')^5,$$

with respect to the derivative y' . We shall use the same method as that in theorem 2 to deal with equation (7,1), assuming that

$$h(t, u, v, w) = t + w^5,$$

and the Lipschitz constant L satisfies the condition

$$\left| \frac{\partial h}{\partial w} \right| = |5 \cdot w^4| \leq L \leq 1,$$

which implies

$$(7,2) \quad |w| \leq \sqrt[4]{\frac{L}{5}} < 1.$$

A simple calculation shows that the fifth approximation $y_5(t)$, $y_5(0) = 0$, fulfils the condition

$$y'_5(t) = t + \{t + [t + (t + t^5)^5]^5\}^5.$$

That approximation is bounded by the solution $s(t)$, $s(0) = 0$, of equation (3,4), which reduces in this case to

$$(7,3) \quad s'(t) = \frac{t}{1-L};$$

hence

$$(7,4) \quad s(t) = \frac{1}{1-L} \cdot \frac{t^2}{2}.$$

We shall evaluate the length of the greatest interval for which the solution exists (this length depends obviously on L). To this end it is sufficient to apply theorem 2 and solve the inequality

$$|s'(t)| = \frac{1}{1-L} \cdot t \leq \sqrt[4]{\frac{L}{5}}.$$

So from the last line of the formula it follows that the length of the interval is

$$(7,5) \quad 0 \leq t \leq (1-L) \cdot \sqrt[4]{\frac{L}{5}}.$$

It is easy to verify that the greatest length can be obtained if $L = \frac{1}{5}$, and then

$$(7,6) \quad 0 \leq t \leq \frac{4}{5} \cdot \frac{1}{\sqrt[4]{5}} = 0,358.$$

Now, as regards the error estimate for $y_5(t)$ in the interval (7,6), let us observe that (4,1) and (4,3) imply

$$z_0(t) = 2 \cdot s(t) = \frac{1}{1-L} \cdot t^2,$$

and

$$(7,7) \quad z_n(t) = \frac{L^n}{1-L} \cdot t^2, \quad n = 0, 1, 2, \dots$$

From (7,6) and (7,7) it follows that

$$z_5(t) = \frac{L^5}{1-L} \cdot t^2 \leq \frac{5}{4} \left(\frac{1}{5}\right)^5 \cdot 0,358^2 = 0,000051,$$

whence the approximation $y_5(t)$ to the unknown solution $\varphi(t)$ (as well as all higher approximations $y_p(t)$, $p \geq 5$) satisfies the inequality

$$|y_5(t) - \varphi(t)| \leq 0,000051,$$

for $0 \leq t \leq 0,358$.

Remark 2. It is worth noting that the method of proofs used here, i.e. that of differential inequalities, can be applied also to prove theorems with more general assumptions than the Lipschitz condition. The proofs of those theorems are connected with some further difficulties, and will be published later.

§ 8. Now we shall derive another error estimate for the approximation $y_{n+1}(t)$ to the unknown solution $\varphi(t)$ of the differential equation (6,1). In this connection we shall evaluate the difference $\|y_{n+1}(t) - \varphi(t)\|$ with the aid of differences $\|y_{n+1}(t) - y_n(t)\|$ and $\|y'_{n+1}(t) - y'_n(t)\|$ of two successive approximations $y_n(t)$ and $y_{n+1}(t)$.

This derivation leads to a form especially convenient for numerical computation, supplying successively $y_0(t), y_1(t), \dots, y_n(t), y_{n+1}(t)$, since there is no need of constructing the real-valued functions $z_0(t), z_1(t), \dots, z_{n+1}(t)$ in (6,3) and (6,4).

The theorem can be formulated as follows:

THEOREM 3. Suppose that the right-hand member of the equation (6,1) satisfies the assumptions H.

Let $\varepsilon(t)$ and $\delta(t)$ denote the given functions, such that the estimates

$$(8,1) \quad \|y_{n+1}(t) - y_n(t)\| \leq \varepsilon(t), \quad \|y'_{n+1}(t) - y'_n(t)\| \leq \delta(t), \quad t \in I',$$

hold for some n , the two successive approximations $y_n(t), y_{n+1}(t)$ being defined by formula (6,2).

Under these assumptions the error estimates are provided by

$$(8,2) \quad \|y_{n+1}(t) - \varphi(t)\| \leq x(t), \quad \|y'_{n+1}(t) - \varphi'(t)\| \leq x'(t), \quad t \in I',$$

where $x(t)$ denotes the solution of the linear equation

$$(8,3) \quad x'(t) = \frac{M+N}{1-L} \cdot x(t) + \frac{N \cdot \varepsilon(t) + L \cdot \delta(t)}{1-L},$$

issuing from the point $(0, 0)$, and $\varphi(t)$ the unknown solution of equation (6,1), satisfying the condition $\varphi(0) = 0$.

Proof. By assumptions we obtain successively the identities

$$(8,4) \quad y'_{n+1}(t) \equiv h(t, y_{n+1}(t), y_n(t), y'_n(t)), \quad \varphi'(t) \equiv h(t, \varphi(t), \varphi(t), \varphi'(t)), \quad t \in I',$$

and

$$(8,5) \quad y'_{n+1}(t) - \varphi'(t) \\ = h(t, y_{n+1}(t), y_n(t), y'_n(t)) - h(t, y_{n+1}(t), y_{n+1}(t), y'_{n+1}(t)) + \\ + h(t, y_{n+1}(t), y_{n+1}(t), y'_{n+1}(t)) - h(t, \varphi(t), \varphi(t), \varphi'(t)).$$

(8,5), (8,1) and the Lipschitz condition implies that

$$\|y'_{n+1}(t) - \varphi'(t)\| \leq N \cdot \|y_{n+1}(t) - y_n(t)\| + L \cdot \|y'_{n+1}(t) - y'_n(t)\| + \\ + M \cdot \|y_{n+1}(t) - \varphi(t)\| + N \cdot \|y_{n+1}(t) - \varphi(t)\| + L \cdot \|y'_{n+1}(t) - \varphi'(t)\|;$$

hence

$$(8,6) \quad (1-L) \cdot \|y'_{n+1}(t) - \varphi'(t)\| \leq (M+N) \cdot \|y_{n+1}(t) - \varphi(t)\| + N \cdot \varepsilon(t) + L \cdot \delta(t),$$

and

$$(8,7) \quad \bar{D}_+ \|y_{n+1}(t) - \varphi(t)\| \leq \frac{M+N}{1-L} \cdot \|y_{n+1}(t) - \varphi(t)\| + \frac{N \cdot \varepsilon(t) + L \cdot \delta(t)}{1-L}.$$

Consider the non-homogeneous linear differential equation

$$(8,8) \quad \xi'(t) = \frac{M+N}{1-L} \cdot \xi(t) + \frac{N \cdot \varepsilon(t) + L \cdot \delta(t)}{1-L},$$

for the real-valued function $\xi(t)$.

There exists exactly one solution (i.e. also the greatest solution) $\xi = x(t)$ of equation (8,8), passing through the point $(0, 0)$; hence from (8,8), (8,7) and theorem B we obtain

$$(8,9) \quad \|y_{n+1}(t) - \varphi(t)\| \leq x(t), \quad \text{for } t \in I'.$$

Since (8,6) and (8,9) imply

$$\|y'_{n+1}(t) - \varphi'(t)\| \leq \frac{M+N}{1-L} \cdot \|y_{n+1}(t) - \varphi(t)\| + \frac{N \cdot \varepsilon(t) + L \cdot \delta(t)}{1-L} \\ \leq \frac{M+N}{1-L} \cdot x(t) + \frac{N \cdot \varepsilon(t) + L \cdot \delta(t)}{1-L} = x'(t),$$

the proof of the last theorem is complete.

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