ANNALES
POLONICI MATHEMATICI
XL (1981)

Certain closed flows on a 2-manifold

by RONALD A. KNIGHT (Kirksville, Mo., U.S.A.)

Abstract. The classification and characterization of certain closed flows in terms of the critical points and the space separating properties of their noncritical trajectories is the purpose of this paper.

1. Introduction. Closed flows on connected 2-manifolds were analyzed by Beck [3] and Wu [14] who gave topological characterizations of the set of critical points and by McCann [13] who classified such planar flows without critical points. Knight [8], [9] classified and characterized compact flows on certain 2-manifolds and closed planar flows in terms of the bilateral stability properties of the compact trajectories. In this paper we classify and characterize on particular 2-manifolds a class of closed flows satisfying a specific type of stability criterion. In the terminology of [10] these flows are said to be of characteristic 0.

The concept of a flow of characteristic 0^+ $(0^-, 0^\pm)$ was introduced by Ahmad in [1], where he classified such flows with planar phase spaces in terms of their critical points and characterized planar flows of characteristic $0^\pm$. In [2] Ahmad classified these flows on locally compact phase spaces. Furthermore, in [11] Knight characterized planar flows of characteristic 0^+ (0^-) in terms of the set of critical points. The bilateral version of these flows was introduced by Knight in [10], where planar flows of characteristic 0 were characterized in terms of the critical points, and in [12], where further characterizations were obtained for Hausdorff phase spaces.

The purpose of this paper is to generalize the results of [10] to surfaces, i.e. general 2-manifold phase spaces satisfying the Jordan curve separation property. In Section 2 flows of characteristic 0 on surfaces are shown to be closed and we classify such flows on connected surfaces in terms of the space separating properties of noncritical trajectories and in terms of the set of critical points. Section 3 is devoted to characterizations of flows of characteristic 0 on connected surfaces. The characterization

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AMS 1970 subject classifications: Primary 34C35; Secondary 54H20, 58F10.

Key words and phrases: Characteristic 0, closed flow, dynamical system, flow, 2-manifold, parallelizable, transversal, stability, surface.

theorem of [10] for such flows on the plane is generalized to surfaces. For paracompact connected surfaces the generalization is very close to the statement of Theorem 4.8 in [10].

A 2-manifold which does not satisfy the separation property need not possess the characteristics obtained herein. Indeed, the example flow of [12] violates most of these properties.

Many of the definitions and notations which follow are standard but are presented here for the convenience of the reader. A *dynamical system* (X, π) consists of a topological space X and a continuous mapping π: $X \times R \to X$ satisfying $x0 = x$ and $(xt)s = x(t+s)$, where $\pi(x, r)$ is denoted by xr. We denote the trajectory (orbit), orbit closure, limit, prolongation, and prolongational limit sets of a point x by $C(x)$, $K(x)$, $L(x)$, $D(x)$, and $J(x)$, respectively. The region of attraction for a set M is denoted by $A(M)$. The corresponding positive and negative versions of these concepts carry the appropriate superscript. A *transversal* on a 2-manifold is a section which is either an arc or a simple closed curve. (See [6] for a treatment of transversal theory.) By a *transversal* T *at* x we shall mean that T contains x as a non-end point. A flow (X, π) is said to be of *characteristic* 0 if and only if $D(x) = K(x)$ for each point x in X. A flow (X, π) is called a *closed flow* if and only if each orbit is a closed set.

A Hausdorff space X is called a *2-manifold* whenever each of its points has an open neighborhood homeomorphic to R^2. We say that X satisfies the Jordan curve separation property if every simple closed curve C in X decomposes X into two open sets with common boundary C. We refer to such a 2-manifold as a *surface*.

A trajectory is said to *separate* X if $X - C(x)$ is the union of two components having the common boundary $C(x)$. Whenever sets or points are in different components of $X - C(x)$ we shall say that they are *separated in* X *by* $C(x)$. If C is a simple closed curve in a subset V of X homeomorphic to R^2 we shall denote the bounded and unbounded components of $V-C$ by int C and ext C, respectively. Note that int C depends only on the existence of a set $V \approx R^2$ so that we shall feel free to use int C without referring to V.

The nonnegative and nonpositive real numbers are denoted by R^+ and R^-, respectively. By δM, M^0, and $\bar{M}$ we mean the boundary, interior, and closure of the set M, respectively.

For basic properties of dynamical system theory we refer the reader to [4], [5], and [6].

2. Classification of flows of characteristic 0 **on a surface.** Since each component of a surface is a surface we shall restrict our attention to connected surfaces. Throughout this section (X, π) denotes a flow of characteristic 0 on a connected surface X. We denote the sets of critical points and periodic points by S and P, respectively.

PROPOSITION 1. *A point x is critical or periodic provided $L^{+}(x) \neq \emptyset$ $(L^{-}(x) \neq \emptyset)$.*

Proof. By virtue of Corollary 3.1 of [12] we have $K(x) = L^{+}(x)$ whenever $L^{+}(x) \neq \emptyset$. Suppose that there is a regular point x $(x \notin P \cup S)$ for which $K(x) = L^{+}(x)$ and let T be a transversal arc at x. For any $\varepsilon > 0$, $T(-\varepsilon, \varepsilon)$ is a neighborhood of x. Since x is in $L^{+}(x)$, there is a $t > \varepsilon$ such that $xt \in [T(-\varepsilon, \varepsilon)]^{0}$, and hence, $C^{+}(x)$ meets T more than once. Let xt_1 and xt_2 be consecutive intersections of T and $C^{+}(x)$ with $t_1 < t_2$. The arcs C_1 in T and C_2 in $C^{+}(x)$ with end points xt_1 and xt_2 form a simple closed curve which separates X into two sets H and K with common boundary $C = C_1 \cup C_2$. The fact that a trajectory other than $C(x)$ intersecting C_1 does so only once follows from the proof of Lemma 4.5 page 173 of [6]. The key to the proof is the property $C_1(0, \delta) \subset H$ and $C_1(-\delta, 0) \subset K$ for sufficiently small $\delta > 0$ (and, of course, H and K properly labeled). Consequently, H and $\bar{H}$ are positively invariant whereas K and $\bar{K}$ are negatively invariant. Thus, we have $C^{+}(xt) \subset H$ for $t > t_2$ and $C^{-}(xt) \subset K$ for $t < t_1$. Since $L^{+}(x) = L^{+}(xt)$ for each t in R we have $C(x) \subset L^{+}(x) \subset \bar{H}$ which is clearly impossible.

COROLLARY 2. *A flow on a surface X is of characteristic 0 if and only if $D(x) = C(x)$ for each x in X.*

COROLLARY 3. *A flow of characteristic 0 on a surface is closed.*

PROPOSITION 4. *If $C(x)$ is a critical or periodic orbit, then $C(x)$ is bilaterally stable.*

Proof. The proposition is an immediate consequence of Theorem 7 of [12].

PROPOSITION 5. *Each of the sets P and $P \cup S$ is open.*

Proof. The proof is a direct result of Corollary 7.1 of [12].

For each regular boundary point x of the periodic regions in the flows given in Examples 2 and 3 of [10] we have $J(x) = C(x)$ so that the following proposition cannot be strengthened.

PROPOSITION 6. *$J(x) = \emptyset$ for any point x interior to the set of regular points.*

Proof. In view of Corollary 2, $J(x) \subset C(x)$ for each point x in X. Suppose that $J^{+}(x) = C(x)$ for a point x in $(X - P \cup S)^{0}$ and let T be a transversal arc of regular points at x. If $C(x)$ meets T more than once, then we can select consecutive intersections xt_1 and xt_2 of T and $C(x)$ with $t_1 < t_2$ as we did in the proof of Proposition 1. Like before we denote the arcs in T and $C(x)$ with end points xt_1 and xt_2 by C_1 and C_2, respectively. The simple closed curve $C = C_1 \cup C_2$ separates X into two sets H and K with $C^{+}(H) = H$ and $C^{-}(K) = K$. There is a subtransversal T_0 of T at xt_1 and an $\varepsilon > t_2 - t_1$ such that $T_0(-\varepsilon, \varepsilon)$ is a neighborhood of C_2 (2.9, p. 167, [6]). Thus, for any nets (x_i) converging to xt_1 and (t_i) converging to $+\infty$, there is an i_0 such that $x_i t_i \in H$ for each $i \geq i_0$. Hence, $J^+(x) \subset \bar{H}$. However, this means that $C(x) \subset \bar{H}$ which is absurd. The trajectory $C(x)$ meets T exactly once.

Next, let y be a point of T distinct from x with $C(y)$ meeting T at consecutive points yt_1 and yt_2. The situation is similar to that of the preceding paragraph for x so we adopt the notation given above replacing x by y. There are neighborhoods of C_1 and C_2 each of whose points are attracted to H since a trajectory through any point of K which comes sufficiently close to C meets the arc C_1 and is thereafter contained in H (2.5, p. 166, and 2.9, p. 167, [6]). Hence, $\bar{H}$ is an attractor with $x \notin A^+(\bar{H})$. Since $A^+(\bar{H})$ is an open invariant set, there is a net (z_i) in $A^+(\bar{H})$ converging to a point z in $\delta(A^+(\bar{H}))$ and a subnet of $\{w_i\colon w_i \in C^+(z_i) \cap C_1\}$ which converges in C_1 implying that $D^+(z) \cap C_1 \neq \emptyset$. This contradicts $D(z) = C(z) \subset X - A^+(\bar{H})$. Hence, $C(y) \cap T$ is a singleton.

Let T_0 denote T less its end points. Then T_0 is a section of $T_0 R$. The flow $(T_0 R, \pi | T_0 R)$ is parallelizable, and hence, is dispersive (5.10 and 5.11, I, p. 83, [5]). Since $T_0 R$ is a neighborhood of $C(x)$ and $J(x) \subset C(x)$, we have $J(x) = \emptyset$ which of course contradicts our assumption that $J^+(x) = C(x)$. Similarly, $J^-(x) = C(x)$ leads to a contradiction. Hence, $J(x) = \emptyset$.

COROLLARY 7. *The flow is locally parallelizable on* $(X - P \cup S)^0$.

COROLLARY 8. *Each component of* $X - P \cup S$ *is a locally parallelizable subflow of* X.

Proof. The boundary points of $X - P \cup S$ have not yet been shown dispersive in $X - P \cup S$. Let x be a point of $\delta(X - P \cup S)$ and let T be a transversal arc at x. Using the notation of Proposition 6, if $C(x)$ meets T more than once, then $C_1 \cap P = \emptyset$ since H is positively invariant. There are periodic points arbitrarily near xt_1 in the component of $T - \{xt_1\}$ that does not contain C_1. Thus, as in the proof of Proposition 6, periodic points sufficiently close to xt_1 in K eventually meet C_1 which is impossible. Consequently, $C(x)$ is isolated in T from periodic points making it interior to $X - P \cup S$. Since this is impossible we have $C(x) \cap T = \{x\}$. Each component T_0 of $T \cap (X - P \cup S)$ is a section of $T_0 R$, and hence, $(T_0 R, \pi | T_0 R)$ is a parallelizable subflow of (X, π).

COROLLARY 9. *If T is a transversal arc at a regular or periodic point x, then $C(x)$ meets T only once.*

Proof. The proof for a transversal arc at a regular point is complete. If T is a transversal arc at a periodic point x, then the sets H and K from the proof of Proposition 6 are constructable whenever $C(x)$ meets T more than once. But this would mean that $C(x) \subset H \cap K = \emptyset$.

PROPOSITION 10. *Either* $S = \emptyset$, $S = X$, *or S consists of Poincaré centers.*

Proof. Let s_0 be a boundary point of S and U be a neighborhood of s_0 homeomorphic to R^2. Then s_0 has an open connected invariant neighborhood V in U with compact closure since s_0 is bilaterally stable and $P \cup S$ is open. Furthermore, we can select V to be simply connected in U because any simple closed curve in V encloses compact orbits. Since V is homeomorphic to R^2, s_0 is a Poincaré center (4.8, [10]).

A parallel flow on an open tube with circular sections is not separated by any trajectory. The following proposition indicates when such a separation can be effected.

PROPOSITION 11. *Each noncritical trajectory separates X if at least one orbit separates X.*

Proof. No critical orbit separates X and each periodic orbit separates X. Let $C(y)$ be a trajectory which separates X. If $C(y)$ is regular and $X-C(y) = P \cup S$, then we are done. Let $C(x)$ be a regular trajectory in $X-C(y)$ and let A_y be the component of $X-C(y)$ which excludes x. Denote $\{z\colon C(z)$ separates x from $A_y\}$ by M and let the component of $X-C(z)$ which contains A_y be denoted by A_z for each z in M. Note that $C(y) \subset M$ so that $M \neq \emptyset$. The set $\{A_z\colon z \in M\}$ is linearly ordered by set inclusion and $\bar{A}_z = A_z \cup C(z)$ for each z in M. Let $A = \bigcup \{A_z\colon z \in M\}$. Then A is an open connected invariant set. For distinct points p and q in δA let V_1 and V_2 be disjoint neighborhoods of p and q, respectively, homeomorphic to R^2. Let C_1 and C_2 be simple closed curves surrounding p and q in V_1 and V_2, respectively. There is a point z_1 in $M \cap \operatorname{int} C_1$ since $p \in \delta A$. There is a point z_2 in $M \cap (X - A_{z_1}) \cap (\operatorname{int} C_2) \cap A$ since $q \notin A_{z_1}$ and $q \in \delta A$. Thus, $A_{z_1} \subset A_{z_2}$, and hence, $C(z_2) \cap \operatorname{int} C_1 \neq \emptyset$. Consequently, we can find nets (x_i) and $(x_i t_i)$ converging to q and p, respectively; therefore, $q \in D(p) = C(p)$ and $\delta A = C(p)$.

We now show that p is a regular point. Suppose that p is critical. Then by Proposition 10, p is a Poincaré center and there is an orbit $C(z)$ in A surrounding p in a neighborhood homeomorphic to R^2. But no orbit in $\overline{\operatorname{int} C(z)}$ separates the regular point x from A_y implying that $p \notin \delta A$. Hence, p is not critical. Next, if p is periodic, then since $C(p)$ is compact and P is open there is a compact neighborhood V of $C(p)$ in P homeomorphic to an annular region in R^2 with simple closed curve boundary components C_1 and C_2 separated by $C(p)$. There is a connected invariant neighborhood W of $C(p)$ in V because $C(p)$ is bilaterally stable. If $C(z)$ is an orbit in W which does not separate C_1 from C_2, then $X-C(z)$ has a component U in V homeomorphic to R^2, and so, V contains a critical point. Since $V \subset P$, each orbit of W separates C_1 from C_2 implying that W is an annular neighborhood of $C(p)$ which separates C_1 from C_2. Thus, there is a periodic orbit $C(z)$ in $W-A$ which separates A from x. The component A_z of $X-C(z)$ contains $C(p)$ placing $C(p)$ interior to A which is impossible. Hence, p is a regular point.

Next, we show that if $C(p) \neq C(x)$, then there is a transversal arc T at p such that $T \cap (X-A) \cap P = \emptyset$. Suppose the contrary. Then, given a transversal arc T at p, there is a net (x_i) of periodic points in $T \cap (X-A)$ converging to p where $x_i < x_{i+1} < p$ in T with T properly ordered. Let $B = \bigcup B_i$ where B_i is the component of $X - C(x_i)$ contained in $X-A$. Now $\{B_i\}$ is linearly ordered by set inclusion and B is an open connected invariant set having p as a boundary point. An argument similar to the one demonstrating that $\delta A = C(p)$ yields $\delta B = C(p)$, and hence, $B = X-A$. For some subscript j, x is in B_j implying that the orbit $C(x_j)$ separates x from A. In fact $C(x_j)$ separates x from $C(p)$ so that $C(p)$ is interior to A which is a contradiction.

Finally, we show that $C(p) = C(x)$ by assuming the opposite. Let T be a transversal arc at p with endpoints a and b satisfying $T \cap (X-A) \cap P = \emptyset$ and $b \in A$. The set TR is a neighborhood of $C(p)$ so that $W = TR \cup A$ is an invariant neighborhood of A. Obviously, $C(a) \subset \delta W$. Let z be an element of δW. There is a sequence (x_i) in $T \cap (W-A)$ converging to a with $x_i \leqslant x_{i+1} < a$ in T properly ordered. Let C be a simple closed curve surrounding z in a neighborhood V of z homeomorphic to R^2 where $a \notin V$ and $V \cap A = \emptyset$. For q in $(W-A) \cap \operatorname{int} C$ there is a t in R and a subscript j such that $x_j \leqslant qt < x_{j+1}$ in T where $x_0 = p$. The subarc T_j of T from x_j to x_{j+1} generates a neighborhood T_jR of $C(q)$ which meets $\operatorname{int} C$ and $X - \operatorname{int} C$. If $\operatorname{int} C \subset T_jR$, then z is interior to W which is impossible since $z \in \delta W$. Hence, one of the trajectories $C(x_j)$ and $C(x_{j+1})$ meets $\operatorname{int} C$. Thus, there is a subsequence (x_{n_i}) of (x_i) and a sequence (t_i) in R such that $(x_{n_i} t_i)$ converges to q. Thus, $q \in D(a) = C(a)$ and $\delta W = C(a)$. The sets W and $X - \bar{W}$ are separated in $X - C(a)$, and so, $C(a)$ separates x from A_y placing $C(p)$ interior to A which is absurd. We conclude that $C(p) = C(x)$.

Corollary 12. *If $P \neq \emptyset$, then every orbit in $X-S$ separates X.*

Corollary 13. *If no trajectory separates X, then X is separated by each pair of trajectories.*

Proof. If no trajectory separates X, then $P \cup S = \emptyset$. Let $Y = X - C(x)$ be connected and T be a transversal arc with end points x and y. The region $T_0 R$, where $T_0 = T - \{x, y\}$ is an open invariant connected subset of Y. As we have seen before $\delta(T_0 R)$ in Y is $C(y)$ so that $T_0 R$ and $Y - T_0 R \cup C(y)$ are separated in $Y - C(y)$. According to Proposition 11, the space Y is separated by each of its trajectories.

Corollary 14. *If $S \neq X$, then S consists of at most two Poincaré centers.*

Proof. Let s_1, s_2, and s_3 be Poincaré centers with s_1 distinct from s_2 and s_3. Denote the set $\{x\colon C(x)$ separates s_1 from s_2 and $s_3\}$ by M. Since s_1 is a Poincaré center, $M \neq \emptyset$. For each point x in M denote by A_x the component of $X - C(x)$ containing s_1. We can show that $A = \bigcup \{A_x\colon x \in M\}$ is an open invariant connected set whose boundary is a single trajectory $C(p)$ by arguing as we did for the set A in the first paragraph of the proof for Proposition 11. Moreover, we can show that $C(p)$ is neither periodic nor regular by the techniques used in the proof for Proposition 11. Hence, $C(p)$ is a Poincaré center. Each orbit near p separates s_1 from p, s_2, and s_3 since $p \in \delta A$. Thus, $p = s_2 = s_3$.

COROLLARY 15. *If at least one trajectory separates X, then for any three distinct noncritical trajectories one separates the other two.*

COROLLARY 16. *If at least one trajectory separates X, then for any two distinct noncritical trajectories there is a trajectory which separates them.*

COROLLARY 17. *If X is a subspace of R^2, then there are at most two nested sequences of annular periodic regions.*

We now summarize the results of this section letting $T = X - P \cup S$.

Case 1. $S = \emptyset$.

I. $P = \emptyset$ and each of the following holds:
  (a) π is a locally parallel flow.
  (b) If an orbit separates X, then
    (i) each orbit separates X;
    (ii) each pair of orbits is separated by a third; and
    (iii) one of each three orbits separates the other two.
  (c) If no orbit separates X, then each pair of orbits separates X.

II. $P \neq \emptyset$ and each of the following holds:
  (a) P is open.
  (b) Each orbit in P is bilaterally stable.
  (c) $\pi|T$ is a locally parallel subflow of π.
  (d) Each orbit separates X.
  (e) Each pair of orbits is separated by a third.
  (f) One of each triple of orbits separates the other two.

Case 2. $S \neq \emptyset$.

I. $S \neq X$ and each of the following holds:
  (a) $P \cup S$ is open.
  (b) Each orbit in $P \cup S$ is bilaterally stable.
  (c) S consists of at most two Poincaré centers.
  (d) $\pi|T$ is a locally parallel subflow of π.
  (e) Each orbit in $T \cup P$ separates X.
  (f) Each pair of orbits in $T \cup P$ is separated by a third orbit in $T \cup P$.
  (g) One of each three orbits in $T \cup P$ separates the other two.

II. $S = X$.

3. Characterizations of flows of characteristic 0 on a surface. The purpose of this section is to present characterizations of flows of characteristic 0 on connected surfaces. We thus have characterizations of such flows on a surface componentwise.

First, we recall the somewhat useful characterization of Corollary 2, namely, a flow on a surface X is of characteristic 0 if and only if $D(x) = C(x)$ for each x in X.

The condition given in the following proposition is not necessary as can be seen by the simplistic critical flow on a surface. However, in view of Corollaries 13 and 16 the condition is both necessary and sufficient for noncritical connected surfaces as we have indicated in Theorem 19.

PROPOSITION 18. *A flow (X, π) on a Hausdorff space X is of characteristic 0 if each pair of distinct trajectories is separated by a third trajectory.*

Proof. Let $C(z)$ separate $C(x)$ from $C(y)$ and let A be the component of $X - C(z)$ containing $C(x)$. Then $D(x) = \bigcap \{\overline{VR}: V$ is a neighborhood of $x\} \subset \overline{AR} = \bar{A} = A \cup C(z)$. Since $y \in X - \bar{A}$, we have $y \notin D(x)$. Thus, $D(x) = C(x)$ for each x in X.

THEOREM 19. *A flow (X, π) on a connected surface is of characteristic 0 if and only if*

(1) *$S = X$, or*

(2) *each pair of distinct trajectories is separated by a noncritical trajectory either in X or in the connected subsurface $X - C(x)$ for any x in X.*

The following theorem is a generalization of Theorem 4.8 of [10] which characterizes planar flows in terms of the set of critical points. Even though the statement of Theorem 19 is succinct and more asthetically pleasing, Theorem 20 furnishes more insight into the structure of a flow of characteristic 0 on a surface.

THEOREM 20. *A flow (X, π) on a connected surface X is of characteristic 0 if and only if one of the following holds:*

(1) *(X, π) is locally parallelizable ($S \cup P = \emptyset$) and each pair of trajectories is separated by a third trajectory in either X or the connected subsurface $X - C(x)$ for any x in X.*

(2) *S consists of at most two Poincaré centers. Each noncritical boundary component of P is a single regular trajectory. The restriction of π to $X - P \cup S$ is locally parallelizable and each pair of regular trajectories is separated by a third trajectory.*

(3) *$S = X$.*

Proof. The proof is complete except for showing that case (2) is sufficient. To complete the proof we need only show that each periodic orbit is separated from any other noncritical trajectory.

Let $x \in P$, $z \in X - S$, and A denote the component of $X - C(x)$ containing z. Since P is open and $C(x)$ is compact, $C(x)$ is contained in the union of finitely many open periodic sets homeomorphic to R^2. Hence, there is a simple closed curve C in P separating $C(x)$ from $C(z)$ and forming a closed annular region M in P with boundary curves C and $C(x)$.

If $C(y)$ is a periodic orbit interior to M, then either $X-C(y)$ has a component in M or $C(y)$ separates C and $C(x)$. If there were a component of $X-C(y)$ in M, then it would be homeomorphic to R^2, and hence, contain a point of S which is impossible. Thus, each orbit in M separates C and $C(x)$. The set $B = X-A$ is closed with compact boundary $C(x)$ and $M \cup B$ is a neighborhood of B. According to Ura's alternatives (4.10, p. 49, [10]) either $A^+(B)$ is open, $A^-(B)$ is open, $L(y) \cap B \neq \emptyset$ for some $y \in A$, or M contains an orbit $C(y)$. Since $B = A^+(B) = A^-(B)$ and $L(y) \cap B = \emptyset$ for each y in A, there is a periodic orbit $C(y)$ in M separating C from $C(x)$. Hence, $C(y)$ separates $C(x)$ and $C(z)$.

The following corollary is a simplification of Theorem 20 using [7]. Note that it is essentially the same as Theorem 4.8 of [10] for planar flows.

Corollary 21. *A flow (X, π) on a connected paracompact surface is of characteristic 0 if and only if one of the following holds*:

(1) *(X, π) is parallelizable $(S \cup P = \emptyset)$.*

(2) *S consist of at most two Poincaré centers. Each noncritical boundary component of P is a single regular trajectory which is separated from any other boundary component by a periodic orbit. The restriction of π to $X-P \cup S$ is parallelizable.*

(3) $S = X$.

One important class of surfaces to which our results directly apply are open subspaces of the plane. Concentric open annular regions of concentric periodic orbits form such a flow. If a closed disc or a point is deleted from the closed annular region or simple closed curve boundary between two open annular regions so that the set remaining is homeomorphic to the cartesian product of a closed interval and an open interval or is an open arc, then a parallelizable flow can be defined in that region in such a way that its union with the two open annular regions forms a flow of characteristic 0. Of course infinitely many such parallelizable regions can exist. For example, let A be the Cantor set and B be its complement in $[0, 1]$. In the closed unit disc D the set $\{(r, \theta)\colon r \in B,\ 0 \leqslant \theta < 2\pi\}$ consists of concentric open annular regions and the space $X = D - \{(r, 0)\colon r \in A\}$ is of the type described above.

In [10] the following classes of flows of characteristic 0 were identified:

(i) parallelizable flows,

(ii) global Poincaré centers,

(iii) one nonglobal Poincaré center with a connected parallelizable regular region,

(iv) two Poincaré centers with a connected parallelizable regular region separating the centers, and

(v) critical flows.

We identify the following categories of flows of characteristic 0 on open connected subspaces of R^2 or its one point compactification R^{2*}:

(1) parallelizable flows on R^2 or on an annular region X of R^{2*} having a simple closed curve section separating the boundary components of X in R^{2*};

(2) subspaces of planar type flows (ii), (iii), and (iv) as well as flows formed by replacing the nonregular orbits of such flows by regions similar to those described in the examples above where closed discs or points were deleted; and

(3) critical flows.

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NORTHEAST MISSOURI STATE UNIVERSITY

Reçu par la Rédaction le 17.5.1978