ANNALES
POLONICI MATHEMATICI
L (1989)

# Multistep methods for ordinary differential equations with parameters

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**Abstract.** Multistep methods combined with iterative ones are used for finding a numerical solution of ordinary differential equations with parameters. Consistency and convergence of our methods are considered.

**1. Introduction.** Let $R^q$ be some $q$-dimensional real linear space of elements $x = (x_1, x_2, \ldots, x_q)^T$ and $I = [\alpha, \beta]$, $\alpha < \beta$. We denote by $C(I, R^q)$ the class of all continuous functions defined in $I$ with a range in $R^q$.

We consider the differential equation

$$y'(t) = f(t, y(t), \lambda), \quad t \in I \tag{1}$$

with boundary conditions

$$y(\alpha) = y_0, \tag{2}$$

$$\tilde{M}\lambda + \tilde{N}y(\beta) = \tilde{S}. \tag{3}$$

Here $f\colon I \times R^q \times R^p \to R^q$, $y_0 \in R^q$, $\tilde{S} \in R^p$ and matrices $\tilde{M}_{p \times p}$, $\tilde{N}_{p \times q}$ are given. We seek a parameter $\lambda \in R^p$ and a function $\varphi \in C(I, R^q)$ such that (1)–(3) to be satisfied. It is a solution of (1)–(3).

Existence and uniqueness theorems were obtained by many authors (for example, see [8], [10], [12], [13]). The task of this paper is a numerical solution of BVP (1)–(3). It will be assumed that (1)–(3) has the solution.

We choose a positive number $N$ and select the mesh point $t_{h0}, t_{h1}, \ldots, t_{hN}$, where $t_{hi} = \alpha + ih$, $i = 0, 1, \ldots, N$. Here $h = (\beta - \alpha)/N$ is the common distance between our points. To determine the numerical solutions $(y_h, \lambda_{hj})$ we apply a multistep method for $y_h$ combined with an iterative method for $\lambda_{hj}$.

The iterative method $\lambda_{hj}$ is obtained from condition (3) and it has the form

$$\begin{aligned} &\lambda_{h0} = \lambda_0, \\ &\lambda_{h,j+1} = \lambda_{hj} - B^{-1}[\tilde{M}\lambda_{hj} + \tilde{N}y_h(\beta; \lambda_{hj}) - \tilde{S}], \quad j = 0, 1, \ldots, \end{aligned} \tag{4}$$

where $\lambda_0$ is given. The nonsingular matrix $B_{p\times p}$ will be defined later. Now we may introduce the stationary multistep method for $y_h$ defined as

$$(5)\quad \sum_{i=0}^{k} a_i(t,h)\,y_h(t+ih;\lambda_{hj}) = hF(t,\ldots,t+kh-h,h,y_h(t;\lambda_{hj}),\ldots,y_h(t+kh-h;\lambda_{hj}),\lambda_{hj}) \equiv h\mathscr{F}(t,h,y_h,\lambda_{hj}),\qquad n=0,1,\ldots,j=0,1,\ldots$$

The $a_i$ and $F$ are certain functions. This formula needs an approximate solution $y_h$ for the starting values $t_{h0},\ldots,t_{h,k-1}$. Generally we assume that those values are generated by some one-step procedures. Now, knowing the approximate solution $y_h$ at the last point $t_{hN}=\beta$, we are able to determine the new value $\lambda_{h,j+1}$ and the corresponding numerical solution $y_h$ on the mesh points for this new value $\lambda_{h,j+1}$. Consequently, it is the multistep iterative method.

For finding a numerical solution of our problem (1)–(3) one-step methods may be applied too. A detailed treatment of this method for the case $p=q$ is given in [7]. There are some numerical examples too.

The task of this paper is to obtain reasonable sufficient conditions for the convergence of the method (4)–(5). Of course, the method (4)–(5) must be consistent. A Lipschitz condition on $F$ with suitable constants is assumed too.

**2. Convergence and consistency.** We take the following basic definitions.

DEFINITION 1. We say that the method (4)–(5) is *convergent to the solution* $(\varphi,\lambda)$ of (1)–(3) if

$$\lim_{\substack{N\to\infty\\ j\to\infty}}\ \max_{i=0,1,\ldots,N}\|\varphi(t_{hi};\lambda)-y_h(t_{hi};\lambda_{hj})\|=0,\qquad \lim_{\substack{N\to\infty\\ j\to\infty}}\|\lambda_{hj}-\lambda\|=0.$$

DEFINITION 2. We say that the method (4)–(5) is *consistent with the problem* (1)–(3) on the solution $(\varphi,\lambda)$ if there exists a function $\varepsilon\colon J_j\times H\to R_+=[0,\infty)$, $J_h=[\alpha,\beta-kh]$ such that

$$\text{(i)}\qquad \Big\|\sum_{i=0}^{k} a_i(t,h)\,\varphi(t+ih;\lambda)-h\mathscr{F}(t,h,\varphi,\lambda)\Big\|\leqslant\varepsilon(t,h),\qquad t\in J_j,$$

$$\text{(ii)}\qquad \lim_{N\to\infty}\sum_{i=0}^{N-k}\varepsilon(t_{hi},h)=0.$$

Remark 1. Since $\varphi$ is a solution of (1)–(3), condition (i) may be written also in the following way

$$\Big\|\varphi(t)\sum_{i=0}^{k} a_i(t,h)+\sum_{i=1}^{k} a_i(t,h)\int_{t}^{t+ih} f(s,\varphi(s),\lambda)\,ds-$$

$$-h\mathscr{F}(t,h,\varphi,\lambda)\Big\| \leqslant \varepsilon(t,h)$$

with the extra condition

$$\tilde{M}\lambda+\tilde{N}\varphi(t)+\tilde{N}\int_{t}^{\beta} f(s,\varphi(s),\lambda)\,ds = \tilde{S} \quad \text{for } t\in J_h.$$

The following theorem we can prove similarly as Theorem 1, [6]. It deals with the consistency of the method (4)–(5).

THEOREM 1. *If*

(I) $f\colon I\times R^q\times R^p\to R^q$, $F\colon I^k\times H\times R^{qk}\times R^p\to R^q$, $a_j\colon I\times H\to R$, $j=0,1,\ldots,k-1$, $a_k(t,h)\equiv 1$, $H=[0,h_0]$, $h_0>0$ *and* $f$ *and all* $a_j$ *are bounded, and* $f$ *is continuous,*

(II) *there exists the solution* $(\varphi,\lambda)$, $\varphi\neq 0$ *of* (4)–(5), *where* $0$ *is the zero vector in* $R^q$,

*then the method* (4)–(5) *is consistent with problem* (1)–(3) *on* $(\varphi,\lambda)$ *if*

$$\lim_{N\to\infty}\sum_{i=0}^{N-k}\Big|\sum_{j=0}^{k} a_j(t_{hi},h)\Big| = 0,$$

$$\lim_{N\to\infty} h\sum_{i=0}^{N-k}\Big\|\sum_{j=1}^{k} ja_j(t_{hi},h)f\big(t_{hi},\varphi(t_{hi};\lambda),\lambda\big)-\mathscr{F}(t_{hi},h,\varphi,\lambda)\Big\| = 0.$$

**3. Convergence of the method** (4)–(5). We introduce the following assumption:

ASSUMPTION A. Suppose that

(a) $F\colon I^k\times H\times R^{qk}\times R^p\to R^q$, $f\colon I\times R^q\times R^p\to R^q$;

(b) there exist constants $L_i\geqslant 0$, $i=0,1,\ldots,k$ and a function $\varepsilon_F\colon I\times H\to R_+$ such that for $(s_0,\ldots,s_{k-1},h)\in I^k\times H$ and $z_i,\bar{z}_i\in R^q$, $i=0,1,\ldots,k-1$, $\mu,\bar{\mu}\in R^p$ we have

$$\|F(s_0,\ldots,s_{k-1},h,z_0,\ldots,z_{k-1},\mu)-F(s_0,\ldots,s_{k-1},h,\bar{z}_0,\ldots,\bar{z}_{k-1},\mu)\|$$

$$\leqslant\sum_{i=0}^{k-1} L_i\|z_i-\bar{z}_i\|+L_k\|\mu-\bar{\mu}\|+\varepsilon_F(s_0,h),$$

and

$$\lim_{N\to\infty} h\sum_{i=0}^{N-k}\varepsilon_F(t_{hi},h) = 0;$$

(c) for matrices $\tilde{M}_{p\times p}$ and $\tilde{N}_{p\times q}$ there exist a nonsingular matrix $B_{p\times p}$ and constants $m_1$ and $m_2$ such that

$$\|B^{-1}(B-M)\| \leqslant m_1 < 1, \quad \|B^{-1}N\| \leqslant m_2.$$

We consider a family of recurrent equations of order $k$

$$\sum_{i=0}^{k} a_i(t_{hn}, h) z_{n+i}^h = c_n^h, \quad n = 0, 1, \ldots, N-k,$$

where $a_i\colon I\times H \to R$, $i = 0, 1, \ldots, k$, $a_k(t, h) \equiv 1$. It may be written by

$$U_{n+1}^h = A_n^h U_n^h + W_n^h, \quad n = 0, 1, \ldots, N-k, \tag{6}$$

where

$$U_n^h = [z_n^h, \ldots, z_{n+k-1}^h]^T, \quad W_n^h = [\theta, \ldots, \theta, c_n^h]^T, \quad \theta \in R^q.$$

$$A_n^h = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{0n}^h & a_{1n}^h & a_{2n}^h & \ldots & a_{k-1,n}^h \end{bmatrix},$$

$a_{in}^h = -a_i(t_{hn}, h)$, $i = 0, 1, \ldots, k-1$.

Now we can prove the following main theorem:

THEOREM 2. *If Assumption* A *is satisfied and if*:

(A) *there exists the solution* $(\varphi, \lambda)$ *of* (1)–(3),

(B) *there exists a nonnegative constant* $\tilde{R}$ *such that for* $n = 0, 1, \ldots, N-k$ *and* $h \in H$ *we have*

$$\|A_n^h\|_\infty \leqslant 1 + \tilde{R}h \quad (\textit{maximum norm}),$$

(C) $d = m_1 + m_2 A < 1$, *where* $A = \dfrac{L_k}{L}(D-1)$ *and* $D = \exp(L(\beta-\alpha))$ *and* $L = \tilde{R} + \sum_{i=0}^{k-1} L_i$,

(D) *there exists a function* $\eta\colon H \to R_+$, $\lim_{h\to 0} \eta(h) = 0$ *such that*

$$\max_j \max_{s=0,\ldots,k-1} \|y_h(t_{hs}; \lambda_{hj}) - \varphi(t_{hs}; \lambda)\| \leqslant \eta(h),$$

(E) *the method* (4)–(5) *is consistent with* (1)–(3) *on the solution* $(\varphi, \lambda)$,

*then the method* (4)–(5) *is convergent to the solution* $(\varphi, \lambda)$ *of BVP* (1)–(3) *and the estimations*

$$\|\lambda_{hj} - \lambda\| \leqslant u_j(h), \quad j = 0, 1, \ldots, \tag{7}$$

$$\max_{n=0,\ldots,N} \|y_h(t_{hn};\lambda_{hj})-\varphi(t_{hn};\lambda)\| \leqslant Au_j(h)+Dw(h), \qquad j=0,1,\ldots, \tag{8}$$

*hold true with*

$$w(h)=\eta(t)+\sum_{i=0}^{N-k}[\varepsilon(t_{hi},h)+h\varepsilon_F(t_{hi},h)],$$

$$u_j(h)=d^j\|\lambda_0-\lambda\|+m_2Dw(h)\frac{1-d^j}{1-d}.$$

Proof. First of all we have

$$\sum_{i=0}^{k} a_i(t,h)[y_h(t+ih;\lambda_{hj})-\varphi(t+ih;\lambda)] = h\mathscr{F}(t,h,y_h,\lambda_{hj})-$$

$$-h\mathscr{F}(t,h,\varphi,\lambda)+h\mathscr{F}(t,h,\varphi,\lambda)-\sum_{i=0}^{k} a_i(t,h)\varphi(t+ih;\lambda).$$

Put

$$z_{hn}^j=\|y_h(t_{hn};\lambda_{hj})-\varphi(t_{hn};\lambda)\|, \qquad n=0,1,\ldots,N,\ j=0,1,\ldots,$$

$$\tilde{\varepsilon}(t,h)=\varepsilon(t,h)+h\varepsilon_F(t,h).$$

By (6) and assumptions we have

$$\|U_{h,n+1}^j\| \leqslant \|A_n^h\|\,\|U_{hn}^j\|+\|W_{hn}^j\|,$$

and

$$e_{h,n+1}^j=\max_{s=0,\ldots,k-1} z_{h,n+1+s}^j \leqslant (1+h\tilde{R})e_{hn}^j+h\sum_{i=0}^{k-1}L_i e_{hn}^j+hL_k\|\lambda_{hj}-\lambda\|+\tilde{\varepsilon}(t_{hn},h).$$

Hence

$$\begin{aligned} e_{hn}^j &\leqslant (1+hL)^n e_{h0}^j+\sum_{i=0}^{n-1}[hL_k\|\lambda_{hj}-\lambda\|+\tilde{\varepsilon}(t_{hi},h)](1+hL)^{n-i-1}\\ &\leqslant (1+hL)^n e_{h0}^j+\frac{L_k}{L}\|\lambda_{hj}-\lambda\|[(1+hL)^n-1]+\\ &\qquad\qquad +\sum_{i=0}^{n-1}\tilde{\varepsilon}(t_{hi},h)(1+hL)^{n-i-1}\end{aligned}$$

or

$$e_{hn}^j \leqslant A\|\lambda_{hj}-\lambda\|+D\Big[e_{h0}^j+\sum_{i=0}^{n-1}\tilde{\varepsilon}(t_{hi},h)\Big],$$

$$n=0,1,\ldots,N-k+1,\ j=0,1,\ldots$$

Further, according to the definition of $\lambda_{hj}$, we see that

$$\begin{aligned}\|\lambda_{h,j+1}-\lambda\| &\leqslant \|\lambda_{hj}-\lambda-B^{-1}[\tilde{M}\lambda_{hj}+\tilde{N}y_h(\beta;\lambda_{hj})-\tilde{M}\lambda-\tilde{N}\varphi(\beta;\lambda)]\| \\ &\leqslant \|B^{-1}\{B(\lambda_{hj}-\lambda)-\tilde{M}(\lambda_{hj}-\lambda)-\tilde{N}[y_h(\beta;\lambda_{hj})-\varphi(\beta;\lambda)]\}\| \\ &\leqslant m_1\|\lambda_{hj}-\lambda\|+m_2 z_{hN}^j\end{aligned}$$

and

$$\|\lambda_{h,j+1}-\lambda\| \leqslant d\|\lambda_{hj}-\lambda\|+m_2 Dw(h), \qquad j = 0, 1, \ldots$$

Now it is easy to get estimations (7)–(8) and the convergence of the method (4)–(5) is obvious.

Remark 2. We know that if condition (b) is satisfied, then our method is stable. The similar result will take another norm of $A$ in (b).

Remark 3. If $p = q$ we can put $B = \tilde{M}+\tilde{N}$ if the matrix $\tilde{M}+\tilde{N}$ is nonsingular. In this case $m_1 = m_2$. Such result for one-step methods is given in [7] together with some numerical examples.

Remark 4. There is no problem to consider Lipschitz functions $L_i(t, h)$ instead of constants $L_i$.

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*Reçu par la Rédaction le 16.02.1987*