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A variant of Leja's approximations in Dirichlet's plane problem

by W. KLEINER (Kraków)

1. Let D be a bounded domain in the z-plane, D_∞ the set of finite points exterior to D, and C, their common boundary, a simple closed curve of continuous curvature ($C \in C^2$) and, for the sake of simplicity, of length 1. We use its natural representation

$$C\colon\ z = z(s) \equiv z(s+1) \in C^2, \qquad |z'(s)| \equiv 1\,. \tag{1}$$

When operating on an arc of C, or using two values of s, we always assume the values of s to be taken from an interval of the smallest possible length—in any case < 1.

The degenerate case with D empty is by no means excluded. C is then assumed to be a C^2-arc of length 1, and some obvious changes are to be introduced below.

Let $f(z)$ be a real-valued function defined for $z \in C$ and satisfying a Lipschitz condition

$$|f(z_1)-f(z_2)| \leqslant p|z_1-z_2| \qquad (z_1, z_2 \in C)\,.$$

Then Dirichlet's problem is certainly solvable in both D and D_∞ in the following sense: given any number q_0 there exists a function $u(z; f, q_0)$ continuous in $D \cup C \cup D_\infty$, harmonic in $D \cup D_\infty$, with $u(z; f, q_0) = f(z)$ $(z \in C)$, $u(z; f, q_0) - q_0 \log|z|$ bounded for $z \to \infty$.

2. Notations concerning measures. All measures (see [1]) that we use are supported by C. μ^+, μ^- denote the positive and the negative parts of the measure μ: $\mu = \mu^+ - \mu^-$, $|\mu| = \mu^+ + \mu^-$.

Let $h(z)$ range over the class of all continuous functions on C. If

$$\int h\,d\mu = \int_C h(z)g(z)|dz|\,,$$

g is called the *density* of μ, we write $g = \mu'$, $d\mu = g\,ds$. The *trace* of μ on a Borel set A is denoted by $\mu \mid A$, i.e.

$$\int h\,d(\mu \mid A) = \int h\varphi_A\,d\mu$$

(φ_Δ is the characteristic function of Δ). The last integral is sometimes denoted by $\int\limits_\Delta h d\mu$.

Let $\varepsilon(z)$ be the unit measure condensed at the point z, i.e.

$$\int h(\zeta)\, d\varepsilon(z) = h(z)\,.$$

Every measure of the form

$$\tau = \sum_{i=1}^{n} m_i \varepsilon(z_i) \qquad (z_i \in C) \tag{2}$$

is called a *discrete measure on C*. We have $\tau(\{z_i\}) = m_i$.

3. Notations concerning potentials and energies. For any measure μ on C its potential and its energy are defined by

$$U^\mu(z) = \int \log|z-\zeta|^{-1} d\mu, \qquad \|\mu\|^2 = \int U^\mu d\mu\,, \tag{3}$$

respectively. U^μ is harmonic in a domain Δ if and only if $|\mu|(\Delta) = 0$. Among the measures on C with $\mu(C) = 1$ (called unit measures) there is (see [3]) an $\eta = \eta^C$ of minimal energy. Its potential is constant on C, moreover

)4) $$U^\eta(z) = u(z, \|\eta\|^2, 1).$$

For discrete measures potentials (3) are well-defined but the energy is not. Thus we introduce two substitutes; as to the first one, see [4], [20]:

$$U_0^\tau(z) = \int \log_0|z-\zeta|^{-1} d\tau, \qquad U_n^\tau(z) = \int \log_n|z-\zeta|^{-1} d\tau\,, \tag{5}$$

where the mark $_0$ denotes that for $\zeta = z$ the integrand is to be replaced by 0, while

$$\log_n t = \min\{\log t,\ \log n^4\}\,. \tag{6}$$

Now we put

$$\begin{gathered} (\mu, \sigma) = \int U^\mu d\sigma; \quad (\mu, \sigma)_0 = \int U_0^\mu d\sigma; \quad (\mu, \sigma)_n = \int U_n^\mu d\sigma\,; \\ \|\mu\|^2 = (\mu, \mu); \quad \|\mu\|_0^2 = (\mu, \mu)_0; \quad \|\mu\|_n^2 = (\mu, \mu)_n\,. \end{gathered} \tag{7}$$

Let us notice some properties of these integrals.

(a) (μ, σ), $(\mu, \sigma)_0$ and $(\mu, \sigma)_n$ are bilinear forms. The first one is positive definite, this means that $\|\mu\|^2 > 0$ for any signed measure with the only exception of $\mu \equiv 0$ ([3], [17]). In $\mathcal{K}_n$ (section 8) $(\mu, \sigma)_n$ is also positive definite; this was, among others, the reason of taking $\log n^4$ in (6). We omit the simple proof since the property mentioned is of no importance for our present purposes. $\|\mu\|_0^2$ may happen to be negative, and so may $\|\mu\|_n^2$ for μ not in $\mathcal{K}_n$.

(b) $(\mu, \sigma) = (\sigma, \mu)$ at least for σ, μ of finite energy (3) by Fubini's theorem.

(c) $(\mu, \sigma)_0 = (\sigma, \mu)_0$ at least if 1° μ and σ have finite energy (since then $(\mu, \sigma)_0 = (\mu, \sigma)$), or 2° U^{μ} is continuous and σ discrete, or 3° both μ and σ are discrete.

(d) $(\mu, \sigma)_n = (\sigma, \mu)_n$ for any measures, since $\log_n$ is continuous.

4. Leja's method (see [14]). Let f be merely continuous, and C the common boundary of D and D_∞, not necessarily smooth. Denote by $\mathfrak{T}_n$ the class of measures (2) with fixed n and $m_i = 1/n$ $(i = 1, \ldots, n)$ and arbitrary $z_i \neq z_k$ $(i \neq k)$. There exists an $\mu_{n\lambda} \in \mathfrak{T}_n$ with

(8) $$I_0(\mu_{n\lambda}) \leqslant I_0(\tau) \overset{\mathrm{df}}{=} \frac{n}{n-1}\|\tau\|_0^2 + 2\lambda \int f\, d\tau \qquad (\lambda > 0).$$

There exist a function $b(\lambda)$ of λ alone and a set of positive capacity (transfinite diameter) $C_{\lambda f} \subset C$, depending on λf only, such that the following limits exist:

(9) $$\begin{gathered} U^{-\mu_{n\lambda}}(z) \to u_\lambda(z) \quad (z \in D \cup D_\infty), \qquad u_\lambda(z) - b(\lambda) \to f(\zeta) \quad (z \to \zeta \in C_{\lambda f}), \\ \lambda^{-1}\big(u_\lambda(z) - b(\lambda)\big) \to u(z; f, q_0) \quad (\lambda \downarrow 0,\ z \in D) \quad (\text{for any } q_0). \end{gathered}$$

5. Suppose now that f is continuous and solvable in the sense of Siciak ([19]), i.e. $C_f = C$, or, equivalently, for $q_0 = 1$ (§ 1) $u(z; f, 1)$ is continuous subharmonic in the open plane. Then a measure ψ_f on C exists with

(10) $$u(z; f, 1) = U^{-\psi_f}(z) + H(z), \qquad H(z) = b_f = \text{const} \qquad (z \text{ arbitrary}).$$

This formula, with H harmonic in the whole plane, results from Riesz's theorem ([18]); $U^{-\psi_f}$ is harmonic in D and D_∞, and so ψ_f is supported by C; $H(z)$ is easily seen to be bounded from either above or below, and thus constant ([13], p. 282).

In particular, let us return to the assumptions of § 1. Then, as shown by Górski ([5]), $\lambda_0 f$ is solvable for some $\lambda_0 > 0$. Thus

(11) $$u(z; f, q_0) = \lambda_0^{-1} u(z, \lambda_0 f, \lambda_0 q_0) = U^{-\lambda_0^{-1}\psi_{\lambda_0 f}}(z) + \lambda_0^{-1} b_{\lambda_0 f} = U^{-\varphi}(z)$$

with $\varphi = \lambda_0^{-1}\psi_{\lambda_0 f} - \lambda_0^{-1} b_{\lambda_0 f}\|\eta\|^{-2}\eta$ (see (4)) and $q_0 = \lambda_0^{-1}$. In particular

(12) $$f(z) = U^{-\varphi}(z) \qquad (z \in C).$$

Moreover, by (11) and by [2], p. 213, U^{φ} satisfies a Lipschitz condition, and this implies that φ is of bounded density:

(13) $$|\varphi(E)| \leqslant \varrho_\varphi |E|$$

for any Borel set $E \subset C$ (the bars denote length). A proof of this implication (written for the space) will be found in [12].

6. We now propose a variant of Leja's method, in which the supporting points z_i in (2) are fixed, while the masses m_i are to be determined from a minimum condition. The main advantages are:

1° the parameter λ and the corresponding limit process (9) are eliminated (¹),

2° minimizing the polynomial (16) is rather easy, particularly for computer programming.

On the other hand, the advantage of Leja's original method is in its far greater generality and in giving the existence proof.

7. Conventions. In what follows, C, f and φ have a fixed meaning and the properties stated in § 1 and (12). σ denotes any measure with bounded density: $|\sigma'| \leqslant \varrho$. The letters c and N denote constants, not the same ones at each occurrence but always positive and depending only on C, ϑ, θ and ϱ (in particular on ϱ_φ (13)).

n is fixed but arbitrary, till a limit process is mentioned; then, n is fixed anew. The range of i, j, k is always $\{1, \ldots, n\}$.

For any measure μ, its trace (§ 2) on C_{in} (§ 8) is denoted by μ_i. The only exception: ψ_n is a measure on the whole C, not on C_{nn}.

$O(a_n)$ is the uniform Landau symbol: $b_n = O(a_n)$ means $|b_n| \leqslant ca_n$ where c depends only on C and ϱ (ev. ϱ_φ). We abbreviate: $O_n = O(n^{-1}\log n)$.

Conventional symbols are also introduced in § 8 and in the first lines of § 9.

8. Result. Divide C into n almost equal arcs C_{in} and take their centres for z_i in (2). To be precise, take two positive constants $0 < \vartheta < 1 < \theta$ (independent of n). Then, if n is large enough, we can choose $0 = t_1 < t_2 < \ldots < t_n = 1 = t_0$ with

$$(14)\qquad \begin{gathered} \vartheta/n \leqslant |z_{in} - z_{kn}| \quad (i \neq k), \qquad |C_{in}| \leqslant \theta/n, \\ z_{in} = z(s_i), \quad s_i = \tfrac{1}{2}(t_i + t_{i-1}), \quad C_{in} = z(\langle t_{i-1}, t_i \rangle), \quad |C_{in}| = t_i - t_{i-1}. \end{gathered}$$

The errors of approximations below increase with $\theta - \vartheta$ but comparatively slowly.

Denote by $\mathcal{K}_n$ the class of discrete "bounded" measures on $\bigcup_i \{z_{in}\}$:

$$(15)\qquad \mathcal{K}_n = \Big\{\tau\colon\ \tau = \sum_{i=1}^{n} m_i \varepsilon(z_{in}),\ |m_i| \leqslant 1\Big\}$$

(¹) Observe that even with our assumptions (§ 1), we have no effective formula for λ_0 from § 5. Giving it is equivalent to solving our problem stated in § 8.

and put

(16) $$I(\tau) = \|\tau\|_n^2 - 2(\varphi, \tau) = \sum_{i,k=1}^{n} m_i m_k \log_n |z_{in} - z_{kn}|^{-1} + 2\sum_{i=1}^{n} m_i f(z_{in}).$$

$I(\tau)$ attains its minimum in $\mathcal{K}_n$. Denote by $\psi_n^{\cdot} \in \mathcal{K}_n$ the minimal measure:

(17) $$I(\psi_n^{\cdot}) \leqslant I(\tau) \qquad (\tau \in \mathcal{K}_n).$$

THEOREM. *On the conditions of* § 1, $\psi_n^{\cdot} \to \varphi$. *Moreover,*

(18) $$[\psi_n^{\cdot} - \varphi] = O(n^{-1/2}\log n),$$

whence

(19) $$|U^{\psi_n^{\cdot}}(z) - U^{\varphi}(z)| \leqslant V(z)\, O(n^{-1/2}\log n) \qquad (z \in D \cup D_\infty),$$

where φ *is defined by* (12) *and consequently*

$$-U^{\varphi}(z) = u\big(z; f, \varphi(C)\big)$$

(see § 1), *while*

(20) $$[\mu] \stackrel{\mathrm{df}}{=} \sup\{|\mu(L)|: L \text{ a Jordan arc} \subset C\}$$

(*for any measure* μ *on* C) *and*

$$V(z) = \mathrm{Var}_{\zeta \in C} \log |z - \zeta|^{-1}.$$

It seems possible to eliminate the condition $|m_i| \leqslant 1$ from the definition of $\mathcal{K}_n$. But the computer programmist will welcome this limitation—he would even be glad to have a condition like $|m_i| \leqslant c/n$. Our proof shows this is available, but with $c = \varrho_\varphi = \max|\varphi'|$ which is effectively unknown. So we raise a problem:

Give an effective estimate for $|\varphi'|$ *in terms of* C *and* f *alone.*

For the use of the computer we suggest without proof the following procedure: Fix an n and find $\psi_p^{\cdot}$ for $p \approx n^{1/2}$. Take $p\psi_p^{\cdot}(\{z_{ip}\}) = c_i$ as the approximative density of φ on C_{ip}. Define

$$\mathcal{K}_n^{\sim} = \big\{\tau:\ \tau = \sum m_j\, \varepsilon(z_{jn}),\ 0 \leqslant m_j/c_i \leqslant 2/n \text{ if } z_{jn} \in C_{ip}\big\}$$

(if a $c_i = 0$, replace the corresponding condition on m_j by, say, $|m_j| \leqslant |\psi_p^{\cdot}|(C)/p$) and calculate the minimal measure $\psi_n^{\sim} \in \mathcal{K}_n^{\sim}$ analogous to (17). The interval for m_j may be reduced, but if the computation gives an m_j at an end of the jth interval, this interval must be enlarged and the computation repeated with the new $\mathcal{K}_n^{\sim}$. Then our theorem is valid with $\psi_n^{\sim}$ instead of $\psi_n^{\cdot}$. Now, allowing the errors of, say, $1/2n^2$ in the m_i's (see § 13), the time required to find $\psi_n^{\sim}$ will equal that needed for $\psi_q^{\cdot}$ with $q \ll n$.

Moreover, a concrete computing machine which can reach, say, $\psi_{100}^{\sim}$, will be able to calculate only $\psi_{10}^{\cdot}$. Thus, the twofold computation described above seems to be much more economical.

9. To prove our theorem we need some condensation lemmas. With the notation of the preceding section, denote by

$$C_{in}^{*} = z(\langle s_i - |C_{in}|/n,\ s_i + |C_{in}|/n \rangle) \tag{21}$$

the arc of C of centre z_{in} and length $|C_{in}^{*}| = |C_{in}|/n$. For any measure σ on C, let σ^{*} and $\sigma^{\cdot}$ denote its condensations to $\bigcup_i C_{in}^{*}$ and $\bigcup_i \{z_{in}\}$ respectively:

$$\begin{gathered} d\sigma^{*}\big(z(s_i + t/n)\big) = d\sigma\big(z(s_i + t)\big), \qquad |t| \leqslant \tfrac{1}{2}|C_{in}|, \\ \sigma^{\cdot}(\{z_{in}\}) = \sigma(C_{in}) = \sigma^{*}(C_{in}^{*}). \end{gathered} \tag{22}$$

We suppose once for all lemmas that σ denotes any measure on C of bounded density: $|\sigma'(z)| \leqslant \varrho$.

LEMMA 1.

$$\big|\|\sigma^{*}\|^2 - \|\sigma^{\cdot}\|_n^2\big| = O_n.$$

Proof. We have

$$\|\sigma^{*}\|^2 - \|\sigma^{\cdot}\|_n^2 = \sum_{i,k} [(\sigma_i^{*}, \sigma_k^{*}) - (\sigma_i^{\cdot}, \sigma_k^{\cdot})_n]$$

(see Conventions). By Lagrange's mean value theorem,

$$|\log a^{-1} - \log b^{-1}| \leqslant |a-b| \cdot 1/r$$

with some $r > \min\{a, b\}$ $(a, b > 0)$.

Let $i \neq k$. Put $a = |z-\zeta|$ $(z \in C_{in}^{*}, \zeta \in C_{kn}^{*})$, $b = |z_{in} - z_{kn}|$. By (8), the corresponding $r > \vartheta n^{-1} - \theta n^{-2}$ and $|a-b| \leqslant \theta n^{-2}$. On the other hand, $|\sigma_j^{*}(C_{jn}^{*})| \leqslant \theta n^{-1}\varrho$. So for $n \geqslant N$

$$|(\sigma_i^{*}, \sigma_k^{*}) - (\sigma_i^{\cdot}, \sigma_k^{\cdot})_n| \leqslant \int_{C_{in}^{*}} \int_{C_{kn}^{*}} \big|\log|z-\zeta|^{-1} - \log|z_{in} - z_{kn}|^{-1}\big|\, d|\sigma_i^{*}|\, d|\sigma_k^{*}| \leqslant O(n^{-3}),$$

while

$$\|\sigma_i^{*}\|^2 \leqslant \int_{C^{*}} \int_{C^{*}} \log|s-t|^{-1} (n\varrho)^2 ds\, dt \leqslant n^{-1} O_n$$

(both integrations with respect to the natural parameter); thus

$$\sum_i \|\sigma_i^{*}\|^2 = O_n \quad \text{and} \quad \sum_i \|\sigma_i^{\cdot}\|_n^2 = O_n$$

and our lemma follows.

LEMMA 2. *U^{σ} satisfies a Dini-Lipschitz condition on C:*

$$\big|U^{\sigma}\big(z(s_1)\big) - U^{\sigma}\big(z(s_2)\big)\big| \leqslant c\,|s_1 - s_2| \log|s_1 - s_2|^{-1}.$$

A proof is given in [7]; another one can be extracted from [2], p. 47-50.

LEMMA 3.

$$|U^\sigma(z) - U^\sigma_n(z)| \leqslant 17\varrho n^{-4}\log n,$$

$$|\|\sigma\|^2 - \|\sigma\|^2_n| \leqslant 17\varrho^2 n^{-4}\log n \qquad (n \geqslant N).$$

Proof. If the distance of z to C is $\geqslant n^{-4}$, the first difference is 0. In the opposite case, let

$$A = C \cap \{\zeta: |\zeta - z| < n^{-4}\} \quad \text{and} \quad z_0 = z(s_0) \in A.$$

Then

$$A \subset B \overset{\mathrm{df}}{=} C \cap \{\zeta: |\zeta - z_0| < 2n^{-4}\}.$$

Choose an $r \in (1, 17/16)$, then $|\zeta - z_0| > r^{-1}|s - s_0|$ $(\zeta = z(s))$ if $\zeta \in B$, provided $n \geqslant N$. So

$$|U^\sigma(z) - U^\sigma_n(z)| = \Big|\int_C \ldots\Big| = \Big|\int_B \ldots\Big|$$
$$\leqslant \varrho \int_{s_0 - 2rn^{-4}}^{s_0 + 2rn^{-4}} \log r|s - s_0|^{-1} ds \leqslant 17\varrho n^{-4}\log n$$

if n is sufficiently large. Thus the first inequality is established, and the second one follows by integration.

LEMMA 4.

$$\|\sigma^*\|^2 - \|\sigma\|^2 = \|\sigma^* - \sigma\|^2 + a_n, \qquad |a_n| = O_n.$$

Proof. Using Lemma 2, we estimate (max and min taken for $z \in C_{in}$, $\tau = \sigma^* - \sigma$):

$$|a_n| = \Big|2\int U^\sigma(z)\,d\tau\Big| \leqslant 2\sum_i \int_{C_{in}} \{\max U^\sigma d\tau^+ - \min U^\sigma d\tau^-\}$$
$$\leqslant 2\sum_i \int_{C_{in}} c|C_{in}|\log|C_{in}|^{-1} d|\tau| \leqslant 4\varrho c\theta n^{-1}\log n\theta^{-1}.$$

COROLLARY. *If, moreover,* $\sigma \geqslant 0$, *then* $\|\sigma^* - \sigma\|^2 = O_n$.

Indeed, $\|\sigma^*\|^2 - O_n \leqslant \|\sigma^\cdot\|^2_0 \leqslant \|\sigma\|^2 + O_n$, the first inequality resulting from Lemma 1, the second one from Lemma 1 in [7]. So $\|\sigma^*\|^2 - \|\sigma\|^2 \leqslant O_n$ and by Lemma 4 $\|\sigma^* - \sigma\|^2 \leqslant O_n - a_n$. The energy being positive, our assertion follows.

Now, the condition in this corollary may be droped—this in the matter of our chief lemma:

LEMMA 5. $\|\sigma - \sigma^*\|^2 = O_n$.

Proof. Take an auxiliary measure $\tau = \sigma + \alpha$, where $d\alpha = \varrho ds$. Then $\alpha \geqslant 0$, $\tau \geqslant 0$, and so by the corollary above and by the triangle inequality (valid by § 3 (a))

$$0 \leqslant \|\sigma - \sigma^*\|^2 = \|(\tau - \tau^*) - (\alpha - \alpha^*)\|^2 \leqslant (\|\tau - \tau^*\| + \|\alpha - \alpha^*\|)^2 = O_n .$$

LEMMA 6. $\|\varphi - \varphi^{\cdot}\|_n^2 = O_n$ (see §§ 1, 3).

Proof. By Lemma 3,

$$(\varphi, \varphi^{\cdot})_n - (\varphi, \varphi^{\cdot}) = \int (U_n^{\varphi} - U^{\varphi}) d\varphi^{\cdot} = O(n^{-4} \log n),$$

and by the argument used when proving Lemma 4

$$\begin{aligned} (23) \qquad |(\varphi, \varphi^{\cdot}) - (\varphi, \varphi^*)| &= \left| \int U^{\varphi} d(\varphi^{\cdot} - \varphi^*) \right| \\ &\leqslant \sum_i |C_{in}^*| \max |df/ds| 2\varrho_{\varphi} |C_{in}| = O(n^{-2}), \qquad \varrho_{\varphi} = \max|\varphi'| . \end{aligned}$$

By Lemmas 3 and 1 the differences between $\|\varphi\|_n^2$, $\|\varphi^{\cdot}\|_n^2$ and $\|\varphi\|^2$, $\|\varphi^*\|^2$ respectively are O_n; thus

$$\begin{aligned} \|\varphi - \varphi^{\cdot}\|_n^2 &= \|\varphi\|_n^2 - 2(\varphi, \varphi^{\cdot})_n + \|\varphi^{\cdot}\|_n^2 \\ &= \|\varphi\|^2 - 2(\varphi, \varphi^*) + \|\varphi^*\|^2 + O_n = \|\varphi - \varphi^*\|^2 + O_n = O_n \end{aligned}$$

by Lemma 5.

10. Proof of the theorem. We estimate at first $\|\psi_n^{\cdot} - \varphi\|_n^2$. Let us write simply $\psi^{\cdot}$ instead of $\psi_n^{\cdot}$. The first inequality resulting from Lemma 3 and the second one from (17), we have:

$$\begin{aligned} \|\psi^{\cdot} - \varphi\|_n^2 &= \|\psi^{\cdot}\|_n^2 - 2(\varphi, \psi^{\cdot})_n + \|\varphi\|_n^2 \leqslant \|\psi^{\cdot}\|_n^2 - 2(\varphi, \psi^{\cdot}) + \|\varphi\|^2 + O_n \\ &\leqslant \|\varphi^{\cdot}\|_n^2 - 2(\varphi, \varphi^{\cdot}) + \|\varphi\|^2 + O_n \qquad (n \geqslant N) \end{aligned}$$

(N being so large as to ensure $|\varphi(C_{in})| \leqslant 1$ for $n \geqslant N$). In the last two integrals we replace log by $\log_n$; by Lemma 3, this produces a change of O_n, and thus

$$(24) \qquad \|\psi^{\cdot} - \varphi\|_n^2 \leqslant \|\varphi^{\cdot} - \varphi\|_n^2 + O_n = O_n$$

by Lemma 6.

We must now pass to a continuous measure. Fix an i, put $C_r = z(\langle s_i - r, s_i + r \rangle)$ (see (14)) and let $\mu = \mu_{ir}$ be the uniform unit measure on C_r: $d\mu = ds/2r$. There is a $\delta > 0$ such that $\frac{1}{2}|s-t| \leqslant |z(s) - z(t)| \leqslant |s-t|$ provided $|s-t| < 2\delta$, and so with $0 < r \leqslant \delta$

$$J(r) \leqslant \|\mu_{ir}\|^2 \leqslant J(r) + \log 2 ,$$

where

$$J(r) = (2r)^{-2} \int_{-r}^{r} \int_{-r}^{r} \log|s-t|^{-1} ds\, dt = 1 + \log(2r)^{-1} .$$

Thus there exists an $r = r(i) \in (e/2n^4, e/n^4)$ such that $\mu_i = \mu_{i,r(i)}$ has the energy $\|\mu_i\|^2 = \log n^4$. Put $\psi_i = \psi^{\cdot}(\{z_{in}\})\mu_i$, then $\|\psi_i\|^2 = \|\psi_i^{\cdot}\|_n^2$. This being done for $i = 1, \dots, n$, put $\psi = \sum_i \psi_i$. As in the proof of Lemma 1, we obtain—owing to $r(i)$ being $O(n^{-4})$ and to the inequality in (14)—

$$|(\psi_i, \psi_k) - (\psi_i^{\cdot}, \psi_k^{\cdot})| = O(n^{-3}), \quad i \neq k,$$

and by the construction of ψ the left side equals 0 for $i = k$. Thus

$$\big|\|\psi\|^2 - \|\psi^{\cdot}\|_n^2\big| = O(n^{-1}). \tag{25}$$

Following the example of (23) in the proof of Lemma 6, we estimate the first term required, and by Lemma 3 the second one:

$$\begin{aligned}|(\varphi, \psi) - (\varphi, \psi^{\cdot})_n| &\leqslant |(\varphi, \psi) - (\varphi, \psi^{\cdot})| + |(\varphi, \psi^{\cdot}) - (\varphi, \psi^{\cdot})_n| \\ &\leqslant \max|df/ds| e n^{-4} + 17\varrho_\varphi n^{-4}\log n\,;\end{aligned} \tag{26}$$

so by (25), (26), Lemma 3 and (24)

$$\|\psi - \varphi\|^2 = (\|\psi\|^2 - \|\psi^{\cdot}\|_n^2) - 2\big((\varphi, \psi) - (\varphi, \psi^{\cdot})_n\big) + (\|\varphi\|^2 - \|\varphi\|_n^2) + \|\psi^{\cdot} - \varphi\|_n^2 = O_n.$$

On the other hand (§ 4),

$$|\psi' - \varphi'| \leqslant n^4/e + \varrho_\varphi \leqslant n^4 \quad (n \geqslant N).$$

Now, we proved in [9] the theorems which enable us to estimate $[\sigma]$ (20) in terms of $\|\sigma\|$ and of a bound for density. Namely, [9] (32) gives us for $\sigma = \psi - \varphi$

$$\frac{c_1 \dfrac{\log n}{n}}{[\sigma]^2} \geqslant \frac{\|\sigma\|^2}{[\sigma]^2} \geqslant \frac{c_2}{\log \dfrac{c_3 n^4}{M[\sigma]}},$$

where c_1, c_2, c_3 and M are positive constants. This implies (18) by an elementary calculation. Now (19) follows from the general inequality

$$\Big|\int_C h(\zeta)\,d\sigma(\zeta)\Big| \leqslant [\sigma]\,\mathrm{Var}_{\zeta\in C}\, h(\zeta) \tag{27}$$

valid for any measure σ on C and any h of bounded variation. The proof is given in [9] (37) (formulated for the special case we need: $h = \log|z-\zeta|^{-1}$). This gives the proof of our theorem.

11. The derivatives also converge, and we have e.g.

$$\begin{aligned}\Big|\frac{\partial}{\partial x} U^{\psi_n^{\cdot}}(x+iy) - \frac{\partial}{\partial x} U^{\varphi}(x+iy)\Big| &= \Big|\frac{\partial}{\partial x}\int \log|z-\zeta|^{-1} d(\psi_n^{\cdot} - \varphi)\Big| \\ &= \Big|\int -\frac{\cos\arg(z-\zeta)}{|z-\zeta|}\, d(\psi_n^{\cdot} - \varphi)\Big| \leqslant [\psi_n^{\cdot} - \varphi]\, V_x(z),\end{aligned}$$

$$V_x(z) = \mathrm{Var}_{\zeta\in C} \frac{\cos\arg(z-\zeta)}{|z-\zeta|}, \quad z = x+iy \in D.$$

The last inequality is proved by (27).

12. $-U^{\varphi}(z) = u(z; f, q_0)$ with $q_0 = \varphi(C)$. If we are interested in $z \in D$ only, the value of q_0 is immaterial. If we have to obtain, in D_∞, $u(z; f, q_1)$, it is sufficient to add $\{q_1 - \varphi(C)\}(U^\eta - \|\eta\|^2)$ (see (4)). We then approximate $\varphi(C)$ by $\psi_n^{\cdot}(C)$ and η by Leja's method ([15]) or our variant of it ([10]); the convergence of both approximations is as in (19) [11], [10].

The possibility of obtaining $u(z; f, q_0)$ in D_∞ by the (original) extreme points method was pointed out by Siciak ([19]).

13. Computing hints. The calculator has nothing to do with ϑ, θ and C_{in}. Simply z_{in} are to be choosen on C at approximately equal distances (along C)—e.g. by sketching C and inscribing an equilateral polygon with an arbitrarily chosen side a. By counting the vertices we then obtain n. The coordinates (or directly mutual distances) of z_{in}'s will be approximate of course, but an error of c/n^2 is admissible, since it increases the error in $I(\psi_n^{\cdot})$ by $O(1/n)$ only. By the same reasoning m_{in} can be assumed to be a number of the form k/n^2 with integer k.

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JAGELLONIAN UNIVERSITY
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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