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Addenda to "Periodic solutions of $x''+f(x)x'^{2n}+g(x) = 0$ with arbitrarily large period"

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For the differential equation

$$x''+f(x)x'^{2n}+g(x) = 0 \tag{1}$$

the author has recently discussed ([1]) the existence od periodic solutions with arbitrarily large period. The proof of the theorem, however, is incomplete(1). Following is the corrected form of the theorem.

THEOREM. *Suppose that* $n \geqslant 1$ *and that*

(a) f, g *are continuous for* $-\infty < x < \infty$,

(b) *initial value problems for* (1) *are unique,*

(c) $xg(x) > 0$ *for* $x \neq 0$, $f(x) > 0$ *for all* x, *and*

(d) *there exists a function* $h \in C'(-\infty, 0]$ *such that*

(i) $h(x) > 0$,

(ii) $h'(x) > -2f(x)\big(h(x)\big)^n - 2g(x)$ *for* $x \leqslant 0$, *and*

(iii) $\displaystyle\int_{-\infty}^{0} \frac{ds}{\big(h(s)\big)^{1/2}} = \infty$.

Then equation (1) *has periodic solutions with arbitrarily large* (*smallest*) *periods.*

Correction of the proof. The error in the proof given in [1] is on page 346, line 9. It must be shown that the solution

$$x(t) = x(t, 0, 0, y_1), \quad \dot{x}(t) = y(t, 0, 0, y_1)$$

exists for all $t \leqslant 0$.

To see this note that

$$\big(\dot{x}(t)\big)^2 = z\big(x(t)\big) \leqslant h\big(x(t)\big)$$

(1) The author is indebted to Professor Milos Rab for calling this oversight to his attention.

as long as $x(t)$ exists. Thus $-\dot{x}(t)/\big(h(x(t))\big)^{1/2} \leqslant 1$ and integrating from t to 0, where $t < 0$, yields

$$\int_{x(t)}^{x(0)} \frac{ds}{\big(h(s)\big)^{1/2}} \leqslant -t.$$

By (d), (iii) $t \to -\infty$ if $x(t) \to -\infty$ which establishes the existence of $x(t)$ for all $t \leqslant 0$.

Remark. For $n = 1$ a better result can be attained, as discussed in [1]. A corresponding correction must be made to Theorem 2 in [1]. The details are omitted here.

Sedziwy ([2]) and Villari ([3]) have also given criteria for periodic solutions of (1) with arbitrarily large period. The purpose here is to show that Sedziwy's and Villari's criteria follow as corollaries to the above theorem.

COROLLARY 1 (Villari [3], Theorem 1). *Suppose that the hypotheses of the above theorem hold except that* (d) *is replaced by*

(e) $$\int_0^{-\infty} g(u)\,du < \infty.$$

Then equation (1) *has periodic solutions with arbitrarily large periods.*

Proof. Let $h(x) = C - 2\int_0^x g(u)\,du$ for $x \leqslant 0$, where $C = \int_0^{-\infty} g(u)\,du$. It is easily verified that h satisfies (d) and the theorem can be applied.

COROLLARY 2 (Villari [3], Theorem 2). *Suppose that the hypotheses of the above theorem hold except that* (d) *is replaced by*

(f) $$f(x)x^{2n} \geqslant -x - g(x) \quad \textit{for } x \leqslant -a,\ a > 0.$$

Then equation (1) *has periodic solutions with arbitrarily large periods.*

Proof. Let $h(x) = x^2 + k$, where $k > 0$. It is easily established that

$$h' > -2fh^n - 2g \quad \text{for } x \leqslant -a.$$

It is clear that we can find a $k > 0$ so that

$$h' > -2fh^n - 2g \quad \text{for } -a \leqslant x \leqslant 0.$$

Thus (d) is satisfied and the theorem applies.

COROLLARY 3 (Sedziwy [2], Theorem 1). *Suppose that the hypotheses of the theorem are satisfied except that* (d) *is replaced by*

(g) $$\limsup_{x \to -\infty} \frac{g(x)}{x} = a < \infty; \quad f(x) \geqslant b > 0 \quad \textit{for } x \leqslant -\beta,\ \beta > 0.$$

Then equation (1) *has periodic solutions with arbitrarily large periods.*

Proof. Clearly there is an $a > 0$ such that (f) holds. Thus Corollary 2 applies.

For completeness we will state one further condition which implies the existence of a function h satisfying (d). This condition was established in [1].

COROLLARY 4. *Suppose that the hypotheses of the theorem are satisfied except that* (d) *is replaced by*

(h) *there exist positive constants* C_1 *and* C_2 *such that*

$$-g(x) < f(x)\Big[C_1\Big(\int_0^x g(s)\,ds\Big)^n + C_2\Big] \text{ for } x \leqslant 0 \quad \text{and} \quad \limsup_{x\to-\infty} \frac{g(x)}{x} < \infty.$$

Then equation (1) *has periodic solutions with arbitrarily large periods.*

References

[1] J. W. Heidel, *Periodic solutions of* $x'' + f(x)\,x'^{2n} + g(x) = 0$ *with arbitrarily large period*, Ann. Polon. Math. 24 (1971), p. 343–348.

[2] S. Sedziwy, *Periodic solutions of* $x'' + f(x)\,x'^{2n} + g(x) = \mu p(t)$, ibidem 21 (1969), p. 16–22.

[3] Gaetano Villari, *Soluzioni periodiche di una classe di equazioni del secondo ordine non lineari*, Matematiche 24 (1969), p. 368–374.

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