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Stability of a transport equation

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Abstract. An application of the lower bound function method for proving the asymptotical stability of solutions of a transport equation is presented. This result contains as a special case the stability theorem for the Chandrasekhar–Münch equation given in [2].

1. Introduction. Let G be an unbounded, Lebesgue measurable subset of R^n. By D we denote a subset of $L^1 = L^1(G)$ which consists of all normalized non-negative functions, i.e.,

$$D = \{f \in L^1 : f \geqslant 0,\ \|f\| = 1\}, \qquad \|f\| = \int_G f(x)\,dx.$$

Any function $f \in D$ is called a *density*. A linear operator $P\colon L^1 \to L^1$ is called a *Markov operator* if $P(D) \subset D$. Let a family $\{P^t\}_{t \geqslant 0}$ be a continuous semigroup of Markov operators [4]. A semigroup $\{P^t\}_{t \geqslant 0}$ is called *asymptotically stable* if there exists a density f_0 such that

$$P^t f_0 = f_0 \quad \text{for } t \geqslant 0$$

and

$$\lim_{t \to \infty} \|P^t f - f_0\| = 0 \quad \text{for every } f \in D.$$

A function $V\colon G \to R$ will be called a *Lapunov function* if it is measurable and satisfies the conditions

$$(1.1) \qquad \lim_{|x| \to \infty} V(x) = \infty \quad \text{and} \quad V(x) \geqslant 0, \quad x \in G.$$

If V is a Lapunov function and f is a density we set

$$(1.2) \qquad E(V|\,f) = \int_G V(x) f(x)\,dx.$$

Given $r > 0$, we denote by D_r the subset of D which consists of all densities satisfying

$$f(x) = 0 \quad \text{for } |x| > r \text{ (a.e.)}.$$

By D_0 we denote an arbitrary dense subset of D. An important role in our further considerations is played by

$$\inf P^t(D_r) = \inf\{P^t f\colon f \in D_r\},$$

where the infimum is understood in the sense of the partial order in $L^1(G)$. The following proposition, proved in [2], played a crucial role in our considerations.

PROPOSITION 1.1. *Let $V\colon G \to R$ be a Lapunov function such that*

$$\limsup_{t\to\infty} E(V|\ P^t f) \leqslant N \quad \textit{for } f \in D_0, \tag{1.3}$$

where the constant N is common for all f. Further, let $r_0 > 0$ be such that

$$\inf_{|x|>r_0} V(x) > N. \tag{1.4}$$

If for some $t_0 \geqslant 0$ the function $\inf P^{t_0}(D_{r_0})$ is not vanishing (a.e.), then the semigroup $\{P^t\}_{t\geqslant 0}$ is asymptotically stable.

2. The transport equation. Let $\Omega = [0, \infty)^n$. We consider the integro-differential equation

$$\frac{\partial u(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i(x)u(t, x)) + u(t, x) = \int_\Omega k(x; y)u(t, y)\,dy, \tag{2.1}$$

$t \geqslant 0$, $x \in \Omega$, with the boundary value conditions

$$\begin{aligned} u(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0, \quad & i = 1, 2, \ldots, n,\ t \geqslant 0, \\ u(0, x) = f(x), \quad & x \in \Omega. \end{aligned} \tag{2.2}$$

The coefficients $a = (a_1, \ldots, a_n)\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ are assumed to have continuous second order derivatives and to satisfy the inequalities

$$0 < a_i(x) \leqslant M \quad \text{for } x \in \Omega,\ i = 1, \ldots, n, \tag{2.3}$$

where M is a constant. In addition we assume that the equation $x'(t) = a(x(t))$ generates a C^2 dynamical system $\pi\colon \boldsymbol{R}\times\boldsymbol{R}^n \to \boldsymbol{R}^n$ such that all coordinates increase to infinity as $t \to \infty$. The kernel k is measurable and stochastic, i.e.,

$$\int_\Omega k(x; y)\,dx = 1 \quad \text{and} \quad k(x; y) \geqslant 0, \quad x, y \in \Omega. \tag{2.4}$$

3. A semigroup corresponding to the transport equation. Define a family $T_0(t)$ ($t \geqslant 0$) of operators on $L^1 = L^1(\Omega)$ by setting

(3.1) $T_0(t)\varphi(x)$

$$= \mathbf{1}_\Omega[\pi(-t, x)]\,\varphi[\pi(-t, x)]\exp\left\{-\int_0^t \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}[\pi(s-t, x)]\,ds\right\}.$$

We will prove the following lemma.

LEMMA 3.1. *The family* $\{T_0(t)\}_{t\geq 0}$ *defined in* (3.1) *is a continuous semigroup of linear operators on* L^1.

In order to apply the semigroup theory of linear operators we are going to consider the problem

$$\frac{du}{dt} = (A - I + K)u, \qquad u(0) = f, \tag{3.2}$$

where

$$I\varphi = \varphi \qquad K\varphi(x) = \int_\Omega k(x; y)\varphi(y)\,dy \qquad \text{for } \varphi \in L^1 \tag{3.3}$$

and A is an infinitesimal generator of the semigroup $\{T_0(t)\}_{t\geq 0}$. For φ sufficiently smooth, we have

$$A\varphi(x) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}(a_i(x)\varphi(x)).$$

The following identity will be used in our posterior considerations:

$$\exp\left\{-\int_0^t \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}(\pi(s, z))\,ds\right\}\left|\frac{\partial \pi_i(t, z)}{\partial x_j}\right|_{i,j=1,\ldots,n} = 1, \tag{3.4}$$

where $|\partial\pi_i(t, z)/\partial x_j|_{i,j=1,\ldots,n}$ denotes the determinant of Jacobi's matrix for the function $x \mapsto \pi(t, x)$.

P r o o f o f i d e n t i t y (3.4). The derivative, with respect to t, of the left-hand side of (3.4) admits the form

$$\exp\left\{-\int_0^t \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}(\pi(s, z))\,ds\right\}\left[\frac{\partial}{\partial t}\left|\frac{\partial \pi_i(t, z)}{\partial x_j}\right|_{i,j=1,\ldots,n} - \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}(\pi(t, z))\left|\frac{\partial \pi_i(t, z)}{\partial x_j}\right|_{i,j=1,\ldots,n}\right].$$

The second factor appearing in the product is equal to zero. This follows from the rule of differentiation of determinants,

$$\frac{d}{dt}\det\begin{bmatrix} g_{11}(t)\ldots g_{1n}(t) \\ \ldots\ldots\ldots\ldots \\ g_{i1}(t)\ldots g_{in}(t) \\ \ldots\ldots\ldots\ldots \\ g_{n1}(t)\ldots g_{nn}(t)\end{bmatrix} = \sum_{i=1}^n \det\begin{bmatrix} g_{11}(t)\ldots g_{1n}(t) \\ \ldots\ldots\ldots\ldots \\ g'_{i1}(t)\ldots g'_{in}(t) \\ \ldots\ldots\ldots\ldots \\ g_{n1}(t)\ldots g_{nn}(t)\end{bmatrix}$$

and the equality

$$\frac{\partial^2 \pi_i(t, z)}{\partial t\, \partial x_j} = \sum_{l=1}^{n} \frac{\partial a_i(\pi(t, z))}{\partial x_l} \frac{\partial \pi_l(t, z)}{\partial x_j}, \qquad i, j = 1, \ldots, n.$$

Thus the left-hand side of identity (3.4) remains constant for all $t \geq 0$. On the other hand, for $t = 0$, equality (3.4) is obvious.

Proof of Lemma 3.1. For every $f \in L^1$ and $t \geq 0$ with the use of (3.4) we verify that

$$\begin{aligned} \|T_0(t) f\| &= \int_{\Omega} \mathbf{1}_{\Omega}[\pi(-t, x)] |f[\pi(-t, x)]| \exp\left\{-\int_0^t \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}[\pi(s-t, x)]\, ds\right\} dx \\ &= \int_{\Omega} |f(y)| \exp\left\{-\int_0^t \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}(\pi(s, y))\, ds\right\} \left|\frac{\partial \pi_i(t, y)}{\partial x_j}\right|_{i,j=1,\ldots,n} dy = \|f\|. \end{aligned}$$

The equality

$$\|T_0(t) f\| = \|f\|, \qquad t \geq 0,\ f \in L^1 \tag{3.5}$$

means in particular that $T_0(t)$ is an operator acting in L^1. The identity

$$\mathbf{1}_{\Omega}(\pi(-t-s, x)) = \mathbf{1}_{\Omega}(\pi(-t-s, x))\, \mathbf{1}_{\Omega}(\pi(-t, x))$$

implies that the semigroup condition for $\{T_0(t)\}_{t \geq 0}$ holds true. It remains to show the continuity of $\{T_0(t)\}_{t \geq 0}$, more precisely that for every $f \in L^1$ the function $t \mapsto T_0(t) f$ is continuous at $t = 0$. Let us estimate the difference

$$\begin{aligned} &\|T_0(t) f - f\| \\ &= \int_{\Omega} \left| \mathbf{1}_{\Omega}[\pi(-t, x)] f[\pi(-t, x)] \exp\left\{-\int_0^t \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}[\pi(s-t, x)]\, ds\right\} - f(x) \right| dx \\ &= \int_0^{\pi_n(t,0)} \ldots \int_0^{\pi_1(t,0)} |f(x)|\, dx_1 \ldots dx_n + \\ &\quad + \int_{\pi_n(t,0)}^{\infty} \ldots \int_{\pi_1(t,0)}^{\infty} \left| f[\pi(-t, x)] \exp\left\{-\int_0^t \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}[\pi(s-t, x)]\, ds\right\} - \right. \\ &\qquad\qquad \left. - f(x) \right| dx_1 \ldots dx_n. \end{aligned}$$

We will concentrate on the second integral, denoted by J. For a continuous function f with compact support we have

$$J \leqslant \int_{\Omega} \left| f[\pi(-t, x)] \left(\exp\left\{ -\int_0^t \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}[\pi(s-t, x)]\, ds \right\} - 1 \right) \right| dx + \\ + \int_{\Omega} \left| f(\pi(-t, x)) - f(x) \right| dx.$$

From the mean value theorem we have $|\pi(t, x) - x| \leqslant M\sqrt{n}|t|$, where $|(z_1, \ldots, z_n)|$ denotes the Euclidean norm of the vector $(z_1, \ldots, z_n) \in \boldsymbol{R}^n$. The expression integrated in the first component converges uniformly to zero as $t \to 0$. The second component is small due to uniform continuity of f. Now, approximating an arbitrary function $f \in L^1$ by a continous function g with compact support, we have

$$\begin{aligned} \|T_0(t) f - f\| &\leqslant \|T_0(t) f - T_0(t) g\| + \|T_0(t) g - g\| + \|g - f\| \\ &= 2\|g - f\| + \|T_0(t) g - g\| \end{aligned}$$

what means that $t \mapsto T_0(t) f$ is continuous at zero; this completes the proof.

In the end of this section we are going to define a semigroup $\{T(t)\}_{t \geqslant 0}$ such that every $T(t)$ is a Markov operator and $A - I + K$, (3.3), is an infinitesimal generator of $\{T(t)\}_{t \geqslant 0}$. It is obvious that an operator $A - I$ generates a semigroup $\{e^{-t} T_0(t)\}_{t \geqslant 0}$. Let us note that $\{e^{-t} T_0(t)\}_{t \geqslant 0}$ and the operator K, defined in (3.3), fulfil assumptions of the Phillips Perturbation Theorem [3]. Indeed, by (3.5)

$$\|K e^{-t} T_0(t) f\| \leqslant \|K e^{-t} |T_0(t) f|\| = e^{-t} \|T_0(t) f\| = e^{-t} \|f\|$$

for $f \in L^1$. So, by virtue of the Phillips theorem, $A - I + K$ is the infinitesimal generator for the semigroup

(3.6)
$$T(t) f = e^{-t} \sum_{i=0}^{\infty} T_i(t) f,$$

where

(3.7)
$$T_i(t) f = \int_0^t T_0(t-s) K T_{i-1}(s) f\, ds, \quad f \in L^1.$$

The semigroup $\{T(t)\}_{t \geqslant 0}$ is stochastic, i.e., every $T(t)$ is a Markov operator. In fact, the non-negativity of $T(t) f$ for $f \geqslant 0$ is obvious. Further, using (3.7), (3.5) and the stochasticity of K, it is easy to prove by an induction argument that

(3.8)
$$\|T_i(t) f\| = \frac{t^i}{i!} \|f\|, \quad f \geqslant 0,\ f \in L^1.$$

Finally, from (3.8) and (3.6), it follows immediately that

$$\|T(t)f\| = e^{-t}\sum_{i=0}^{\infty}\|T_i(t)f\| = \|f\|, \quad f \geqslant 0,\ f \in L^1.$$

4. Stability of the semigroup $\{T(t)\}_{t\geqslant 0}$. We are going to show, under some additional assumptions concerning the kernel k, that the semigroup $\{T(t)\}_{t\geqslant 0}$ is asymptotically stable. We will use Proposition 1.1. We shall restrict ourselves to some special class of Lapunov functions. A Lapunov function $V\colon \Omega \to R$ will be called *admissible* if the following conditions are satisfied

$$\sup_{|s|\leqslant 1} V(s) < \infty, \quad V(x+y) \leqslant V(x)+V(y), \quad x, y \in \Omega, \tag{4.1}$$

and

$$\text{if } x_j \leqslant y_j \text{ for every } j = 1, \ldots, n, \text{ then } V(x) \leqslant V(y). \tag{4.2}$$

Our first result concerning the asymptotical behavior of $\{T(t)\}_{t\geqslant 0}$ is given by the following

PROPOSITION 4.1. *Let V be an admissible Lapunov function and let*

$$\int_{\Omega} V(y)k(y; z)\,dy \leqslant \alpha V(z)+\beta, \quad z \in \Omega \tag{4.3}$$

with some real constants α, β, $0 < \alpha < 1$. Then there exists a constant N such that

$$\limsup_{t\to\infty} E(V|\ T(t)f) \leqslant N \tag{4.4}$$

for every $f \in D$ with compact support.

Proof. Using (3.6), we may rewrite the expression $E(t) = E(V|\ T(t)f)$ in the form

$$E(t) = e^{-t}\sum_{i=0}^{\infty} e_i(t), \quad e_i(t) = \int_{\Omega} V(x)\,T_i(t)f(x)\,dx. \tag{4.5}$$

We are going to show that e_i satisfies a system of integral inequalities, which in turn imply (4.4). Setting

$$f_{is} = T_{i-1}(s)f \quad \text{and} \quad q_{is}(t) = \int_{\Omega} V(x)\,T_0(t-s)\,f_{is}(x)\,dx$$

and using (3.1), (3.4), we obtain

$$q_{is}(t) = \int_{\Omega} V(x)\,Kf_{is}[\pi(-t+s, x)]\,1_{\Omega}[\pi(-t+s, x)]\times$$

$$\times\exp\left\{-\int_0^{t-s}\sum_{j=1}^{n}\frac{\partial a_j}{\partial x_j}[\pi(\tau-t+s, x)]\,d\tau\right\}dx$$

$$= \int_{\Omega} V[\pi(t-s, y)]\,Kf_{is}(y)\,dy.$$

From the mean value theorem it follows that

$$\pi_j(t-s,\, y) \leqslant y_j + M(t-s), \qquad j = 1, \ldots, n.$$

Now, using (4.1), (4.2), we have

$$\begin{aligned} q_{is}(t) &\leqslant \int_\Omega V\big(y_1 + M(t-s), \ldots, y_n + M(t-s)\big)\, Kf_{is}(y)\, dy \\ &\leqslant \int_\Omega V(y)\, Kf_{is}(y)\, dy + M \int_\Omega V(t-s, \ldots, t-s)\, Kf_{is}(y)\, dy \\ &= \int_\Omega V(y) \Big\{\int_\Omega k(y;\, z) f_{is}(z)\, dz\Big\}\, dy + MV(t-s, \ldots, t-s)\, \|f_{is}\|. \end{aligned}$$

From this, changing the order of integration and using (4.3) and (3.8), we obtain

$$q_{is}(t) \leqslant \alpha \int_\Omega V(z) f_{is}(z)\, dz + \big(\beta + MV(t-s, \ldots, t-s)\big) \frac{s^{i-1}}{(i-1)!}.$$

The above and the definitions of e_i and f_{is} give

$$q_{is}(t) \leqslant \alpha e_{i-1}(s) + \big(\beta + MV(t-s, \ldots, t-s)\big) \frac{s^{i-1}}{(i-1)!},$$

or finally

$$\text{(4.6)} \qquad e_i(t) \leqslant \alpha \int_0^t e_{i-1}(s)\, ds + \beta \frac{t^i}{i!} + M \int_0^t V(t-s, \ldots, t-s) \frac{s^{i-1}}{(i-1)!}\, ds, \qquad i = 1, 2, \ldots$$

Analogously, we may derive the inequality

$$\text{(4.7)} \qquad e_0(t) \leqslant m(f) + MV(t-s, \ldots, t-s),$$

where

$$m(f) = \int_\Omega V(x) f(x)\, dx.$$

As a result of (4.6) and (4.7) it follows that the functions

$$E_r(t) = e^{-t} \sum_{i=0}^{r} e_i(t), \qquad r = 1, 2, \ldots,$$

satisfy

$$\text{(4.8)} \qquad E_r(t) \leqslant \alpha \int_0^t e^{-(t-s)} E_r(s)\, ds + m(f) e^{-t} + c, \qquad t \geqslant 0$$

with

$$c = \beta + M \sup_t e^{-t} V(t, \ldots, t) + M \int_0^\infty e^{-s} V(s, \ldots, s)\, ds.$$

Conditions (4.1) imply that $V(x)$ is bounded by the function $y = a|x|+b$. Consequently the values c and $m(f)$ are finite for every f with compact support. Now, from standard results of the theory of integral inequalities, it follows that $E_r(t)$ is bounded by the solution of the integral equation corresponding to (4.8). Thus an elementary calculation shows that

$$E_r(t) \leqslant \frac{c}{1-\alpha} + m(f)e^{-(1-\alpha)t}$$

which immediately implies (4.4) with $N = c/(1-\alpha)$ and completes the proof.

Now, in order to establish the stability of the semigroup $\{T(t)\}_{t \geqslant 0}$ given by formulas (3.6), (3.1), (3.7), it sufficies to verify that for some $t_0 > 0$ the function $\inf P^{t_0}(D_{r_0})$ is not equal to zero. The first step in this direction will be done by studing the term $T_1(t)f$ appearing in series (3.6). For every $t_0 > 0$, we have

$$T(t_0)f(x) \geqslant e^{-t_0} T_1(t_0)f(x) = c_0 \int_0^{t_0} T_0(t_0-s)KT_0(s)f(x)\,ds$$

and using (3.3), (3.1)

$$T(t_0)f(x) \geqslant c_0 \int_0^{t_0}\int_\Omega k[\pi(-t_0+s, x); y]\,\mathbf{1}_\Omega[\pi(-s, y)]\,f[\pi(-s, y)] \times$$
$$\times \exp\left\{-\int_0^s \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}[\pi(\tau-s, y)]\,d\tau\right\} \mathbf{1}_\Omega[\pi(-t_0+s, x)] \times$$
$$\times \exp\left\{-\int_0^{t_0-s} \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}[\pi(\tau-t_0+s, x)]\,d\tau\right\} dy\,ds.$$

Changing the variables, we obtain

$$T(t_0)f(x) \geqslant c_0 \int_0^{t_0}\int_\Omega k[\pi(-t_0+s, x); \pi(s, z)]\,f(z)\,\mathbf{1}_\Omega[\pi(-t_0+s, x)] \times$$
$$\times \exp\left\{-\int_0^{t_0-s} \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}[\pi(\tau-t_0+s, x)]\,d\tau\right\} dz\,ds.$$

Let $e_0(x) > 0$ denote minimum of the function

$$s \mapsto \exp\left\{-\int_0^{t_0-s} \sum_{i=1}^n \frac{\partial a_i}{\partial x_i}[\pi(\tau-t_0+s, x)]\,d\tau\right\}$$

attained on the interval $[0, t_0]$. We can also define

$$\text{(4.9)} \qquad h_1(x) = \inf\left\{\int_0^{t_0} \mathbf{1}_\Omega[\pi(-t_0+s, x)]\,k[\pi(-t_0+s, x); \pi(s, z)]\,ds:\right.$$
$$\left. 0 \leqslant z_i \leqslant r_0,\ i = 1, \ldots, n\right\}.$$

where $r_0 > 0$ is a constant such that (1.4) is satisfied. Then, whenever $f \in D_{r_0}$, we verify that

$$T(t_0)f(x) \geqslant c_0 e_0(x) h_1(x) \int_{\Omega} f(z)\,dz = c_0 e_0(x) h_1(x).$$

From Proposition 1.1 we obtain the following

THEOREM 4.1. *If there exists an admissible Lapunov function V satisfying* (4.3) *and if there is a $t_0 > 0$ such that h_1 does not vanish (a.e.) in Ω, then the semigroup $\{T(t)\}_{t \geqslant 0}$ corresponding to the transport equation* (2.1) *is asymptotically stable.*

EXAMPLE 1. As a first example we shall consider the equation

$$\frac{\partial u(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(a_i(x) u(t,x)\big) + u(t,x) = \varphi(x), \tag{5.1}$$

$t \geqslant 0$, $x \in \Omega$ with condition (2.2). Let the coefficients a_i $(i = 1, \ldots, n)$ satisfy all assumptions formulated in Section 2.

The function $\varphi\colon R^n \to R$ is assumed to be measurable and to satisfy

$$\int_{\Omega} \varphi(x)\,dx = 1, \qquad \varphi(x) \geqslant 0 \quad \text{for } x \in \Omega \tag{5.2}$$

and

$$\int_{\Omega} V(x)\varphi(x)\,dx < \infty \tag{5.3}$$

together with some admissible Lapunov function V. Semigroup $\{T(t)\}_{t \geqslant 0}$ generated by (5.1) is asymptotically stable. In fact, the kernel $k(x; y) = \varphi(x)$ is stochastic for (5.2) and satisfies (4.3) because of (5.3). Formula (4.9) admits the form

$$h_1(x) = \int_0^{t_0} 1_{\Omega}[\pi(-t_0+s, x)]\,\varphi[\pi(-t_0+s, x)]\,ds. \tag{5.4}$$

Because of (5.2) there exists an open set $U \subset \Omega$ with positive Lebesgue measure such that $\varphi(x) > 0$ for $x \in U$. If s is sufficiently close to t_0, then $\pi(-t_0+s, x) \in U$ which in turn implies that $h_1(x) > 0$.

EXAMPLE 2. We are going to apply Theorem 4.1, in the case where $n = 1$, to the kernel

$$k(x; y) = \begin{cases} \lambda b(\lambda x - y), & 0 \leqslant y \leqslant \lambda x, \\ 0, & 0 \leqslant \lambda x < y. \end{cases} \tag{5.5}$$

Here $\lambda > 1$ is a constant and $b\colon R \to R$ is a measurable function satisfying

$$b(z) \geqslant 0, \qquad \int_0^{\infty} b(z)\,dz = 1, \tag{5.6}$$

(5.7) $$\int_0^\infty z b(z)\,dz < \infty .$$

Thanks to (5.6) the kernel k is stochastic. In this case equation (2.1) has the form

(5.8) $$\frac{\partial u(t, x)}{\partial t} + \frac{\partial u(t, x)}{\partial x} + u(t, x) = \lambda \int_0^{\lambda x} b(\lambda x - y)\, u(t, y)\, dy$$

and generates an asymptotically stable semigroup. In fact, condition (4.3) is satisfied for the Lapunov function $V(x) = x$ with constants $\alpha = 1/\lambda$ and $\beta = (1/\lambda) \int_0^\infty z b(z)\,dz$. Let $\delta > 0$ be such that $\int_0^\delta b(z)\,dz > 0$. To define the function $h_1(x)$ it is sufficient to fix $t_0 > (\delta + r_0)/(\lambda - 1)$ (see (2.4) and Proposition 2.1). Now we are going to check that for x satisfying $(\delta + r_0 + t_0)/\lambda < x < t_0$ the function $h_1(x)$ is positive. To show this, it is sufficient to consider the integral appearing in (4.9), which now can be calculated as follows:

$$\int_0^{t_0} \mathbf{1}_{[0,\infty)}(x - t_0 + s)\, k(x - t_0 + s; z + s)\, ds = \int_{t_0 - x}^{t_0} k(x - t_0 + s; z + s)\, ds$$

$$= \lambda \int_{[z + \lambda(t_0 - x)]/(\lambda - 1)}^{t_0} b[\lambda(x - t_0 + s) - (z + s)]\, ds.$$

Changing the variables, we obtain

$$\int_0^{t_0} \mathbf{1}_{[0,\infty)}(x - t_0 + s)\, k(x - t_0 + s; z + s)\, ds = \frac{\lambda}{\lambda - 1} \int_0^{\lambda x - z - t_0} b(y)\, dy$$

$$\geqslant \frac{\lambda}{\lambda - 1} \int_0^{\lambda x - r_0 - t_0} b(y)\, dy \geqslant \frac{\lambda}{\lambda - 1} \int_0^{\delta} b(y)\, dy > 0.$$

So h_1 does not vanish a.e. and we have the asymptotical stability of the semigroup $\{T(t)\}_{t \geqslant 0}$ corresponding to (5.8).

Remark. The strict inequality $\alpha < 1$ in (4.3) is necessary for the asymptotical stability of the semigroup $\{T(t)\}_{t \geqslant 0}$ given by (3.6), (3.1) and (3.7). To note this consider the following example. Let

(5.9) $$\frac{\partial u(t, x)}{\partial t} + \frac{\partial u(t, x)}{\partial x} + u(t, x) = e^{-x} \int_0^x e^y u(t, y)\, dy.$$

Here the kernel k, given by

(5.10) $$k(x, y) = \begin{cases} e^{-x+y}, & 0 \leqslant y \leqslant x, \\ 0, & x < y, \end{cases}$$

is stochastic and for every admissible Lapunov function V

$$\int_0^\infty V(x)k(x,y)\,dx = \int_y^\infty V(x)e^{-x+y}\,dx = \int_0^\infty V(z+y)e^{-z}\,dz$$

$$\leqslant \int_0^\infty V(z)e^{-z}\,dz + V(y)\int_0^\infty e^{-z}\,dz = V(y)+\beta.$$

It is easy to see that the function h_1 defined by (4.9) does not vanish. But now the semigroup corresponding to (5.9) is not asymptotically stable. Suppose the contrary. Then there exists a stationary density f_0 of this semigroup. This density is a fixed point of the operator $A-I+K$. It means that f_0 is an absolutely continuous solution of the equation

$$f'(x) = -f(x)+e^{-x}\int_0^x e^y f(y)\,dy. \tag{5.11}$$

But any absolutely continuous solution of (5.11) must belong to C^∞. So, differentiating both sides of (5.11), we obtain the identity $f''(x) = -2f'(x)$, from which it follows that

$$f(x) = \tfrac{1}{2}f(0)e^{-2x}+\tfrac{1}{2}f(0)$$

is the only solution of (5.11). Now it is evident that there is no density in $D(A)$ which could be a fixed point of $A-I+K$.

Example 3. We shall show the stability of semigroup corresponding to the equation

$$\frac{\partial u(t,x)}{\partial t}+\sum_{i=1}^n \frac{\partial u(t,x)}{\partial x_i}+u(t,x) = \int_\Omega \frac{1}{y_1}e^{-x_1/y_1}\ldots\frac{1}{y_n}e^{-x_n/y_n}u(t,y)\,dy. \tag{5.12}$$

It is easy to check that $V(x) = \sum_{i=1}^n \sqrt{x_i}$ is an admissible Lapunov function satisfying condition (4.3). For given $x=(x_1,\ldots,x_n)$, let $\bar{x} = \max_i x_i$. We are going to estimate the integral appearing in (4.9). For an x such that $x_i > t_0$, $i = 1,\ldots,n$, we have

$$\int_0^{t_0} 1_\Omega(x_1-t_0+s,\ldots,x_n-t_0+s)k(x_1-t_0+s,\ldots,x_n-t_0+s;$$
$$z_1+s,\ldots,z_n+s)\,ds$$

$$= \int_0^{t_0}\frac{1}{z_1+s}\exp\left(-\frac{x_1-t_0+s}{z_1+s}\right)\ldots\frac{1}{z_n+s}\exp\left(-\frac{x_n-t_0+s}{z_n+s}\right)ds$$

$$\geqslant \int_0^{t_0}\frac{1}{z_1+s}\exp\left(-\frac{\bar{x}-t_0+s}{z_1+s}\right)\ldots\frac{1}{z_n+s}\exp\left(-\frac{\bar{x}-t_0+s}{z_n+s}\right)ds = J.$$

The function

$$z_i \mapsto \frac{1}{z_i + s} \exp\left(-\frac{\bar{x} - t_0 + s}{z_i + s} \right)$$

appearing in our estimate is increasing for $0 \leqslant z_i \leqslant \bar{x} - t_0$. So we have

$$J \geqslant \int_0^{t_0} \frac{1}{s^n} \exp^n\left(-\frac{\bar{x} - t_0 + s}{s} \right) ds = \frac{1}{(\bar{x} - t_0)^{n-1}} \int_{\bar{x}/t_0}^{x - t_0} (y-1)^{n-2} e^{-ny}\, dy.$$

Since the last integral is positive for $t_0 > 1$ and $\bar{x}$ sufficiently large, we can use Theorem 4.1 to verify that equation (5.12) with conditions (2.2) generates asymptotically stable semigroup.

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