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Difference inequalities of the elliptic type

by Z. KOWALSKI and A. PLIŚ (Kraków)

1. In this paper we shall deal with the difference inequality of the elliptic type:

$$(1.1)\qquad \sum_{i,j=1}^{n} a_{ij}^{M}\cdot u^{Mij}+\sum_{j=1}^{n} b_j^{M}\cdot u^{Mj}+c^{M}\cdot u^{M}\geqslant -\varepsilon,$$

where u^{Mj} and u^{Mij} $(i,j=1,\ldots,n)$ denote difference expressions for the first and second partial derivatives $\partial u/\partial x_j$ and $\partial^2 u/\partial x_i\partial x_j$, respectively, at the nodal point M.

We prove (cf. Theorem 2) that, inequality (1.1) being fulfilled, the maximum value of the function u^M does not exceed some positive quantity $g(h)$, cf. (11.3).

Theorem 3 gives the corresponding result for the difference inequality of the elliptic type of the form (14.1) and the minimum value of the function u^M in the set Q.

These results can be used to obtain the error estimate of the difference method for elliptic differential equations.

Having an approximate solution u^m of a difference equation at the nodal points x^M, we can obtain the effective estimate for $u^M-u(x^M)$ from Theorems 2 and 3, $u(x)$ being the solution of the corresponding differential equation. The estimations (4.1) can easily be computed. The results will be published later.

However, for the theoretical proof of convergence of the difference method for elliptic equations with mixed derivatives the difficulty of proving conditions (4.1) should not be overlooked, cf. A. Pliś [1]. In this paper we assume that conditions (4.1) are fulfilled.

2. We shall denote by Q the set of points of the real n-dimensional space R^n:

$$(2.1)\qquad Q\colon\ 0\leqslant x_j\leqslant\sigma\qquad (j=1,\ldots,n;\ 0<\sigma=\text{const}).$$

We denote by M the sequence of indices

$$(2.2)\qquad M=(m_1,m_2,\ldots,m_n),\qquad 0\leqslant m_j\leqslant N\ (j=1,\ldots,n),$$

[1] A. Pliś, *Loss of uniqueness property in difference approximation of a Dirichlet problem*, Comm. Math. 14 (1970), p. 97–99.

and by x^M the nodal point with the coordinates

$$x^M = (x_1^M, x_2^M, \ldots, x_n^M), \tag{2.3}$$

where $x_j^M = m_j \cdot h$ $(j = 1, \ldots, n)$, and $0 < h = \sigma/N$, N being a natural number.

We shall consider also the nodal points in the set Q characterized by the following sequences of indices:

$$\begin{aligned} j(M) &= (m_1', \ldots, m_n'), \quad m_j' = m_j+1,\ m_i' = m_i \text{ for } i \neq j, \\ -j(M) &= (m_1', \ldots, m_n'), \quad m_j' = m_j-1,\ m_i' = m_i \text{ for } i \neq j, \\ & \qquad (i = 1, \ldots, n;\ j = 1, \ldots, n), \end{aligned} \tag{2.4}$$

and for $i \neq j$:

$$\begin{aligned} ij(M) &= (m_1', \ldots, m_n'), \quad m_i' = m_i+1,\ m_j' = m_j+1, \\ -ij(M) &= (m_1', \ldots, m_n'), \quad m_i' = m_i-1,\ m_j' = m_j+1, \\ -i-j(M) &= (m_1', \ldots, m_n'), \quad m_i' = m_i-1,\ m_j' = m_j-1, \\ i-j(M) &= (m_1', \ldots, m_n'), \quad m_i' = m_i+1,\ m_j' = m_j-1, \end{aligned} \tag{2.5}$$

where $m_s' = m_s$ in formula (2.5) for $s = 1, \ldots, n$; $s \neq i$, $s \neq j$, cf. Fig. 1.

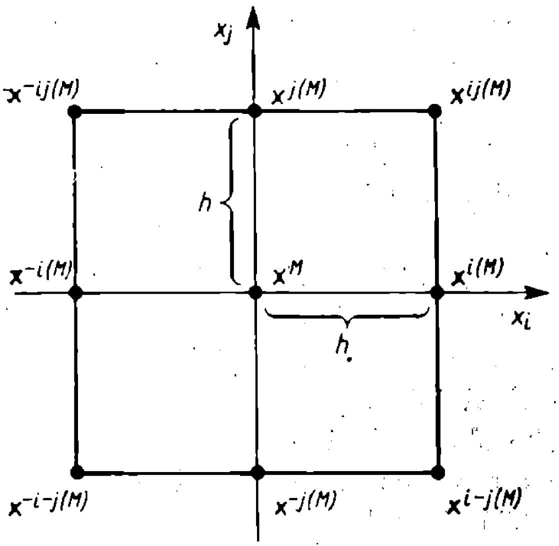

Fig. 1. The nodal points x^M, $x^{i(M)}$, $x^{ij(M)}$, ... For the sake of simplicity we have placed the nodal point M at the origin

The nodal point $ij(M)$ can be denoted also by $ji(M)$ since we define

$$\begin{aligned} ij(M) &= ji(M), & -ij(M) &= j-i(M), \\ -i-j(M) &= -j-i(M), & i-j(M) &= -ji(M) \end{aligned} \tag{2.6}$$
$$\text{for } i \neq j \ (i = 1, \ldots, n;\ j = 1, \ldots, n).$$

We shall denote by $\operatorname{int}Q$ the set of nodal points (2.3) which belong to the interior of the set Q (cf. (2.1)) and by $\operatorname{sym}A$ the set of nodal points x^M such that $x^M \in \operatorname{int}Q$ and $x^{M^*} \in \operatorname{int}Q$ simultaneously, x^{M^*} and x^M being symmetric with respect to the nodal point x^A.

3. Let us denote by u^M the value of the function u at the nodal point x^M.

We shall consider the difference quotients

$$u_+^{Mj} = \frac{1}{h}\cdot(u^{j(M)} - u^M), \quad u_-^{Mj} = \frac{1}{h}\cdot(u^M - u^{-j(M)}), \tag{3.1}$$

$$u^{Mj} = \frac{1}{2h}\cdot(u^{j(M)} - u^{-j(M)}) \tag{3.2}$$

or the first partial derivatives and the difference quotients

$$\begin{aligned} u^{Mjj} &= h^{-2}\cdot(u^{j(M)} - 2\cdot u^M + u^{-j(M)}), \\ u^{Mij} &= \tfrac{1}{4}\cdot h^{-2}\cdot(u^{ij(M)} - u^{-ij(M)} - u^{i-j(M)} + u^{-i-j(M)}) \quad (i \neq j) \end{aligned} \tag{3.3}$$

for the second derivatives.

From definitions (3.1), (3.2), (3.3), it follows that

$$u^{Mj} = \tfrac{1}{2}\cdot(u_+^{Mj} + u_-^{Mj}), \quad u^{Mjj} = \frac{1}{h}\cdot(u_+^{Mj} - u_-^{Mj}). \tag{3.4}$$

We shall use also the difference quotients u_{++}^{Mij}, u_{-+}^{Mij}, u_{--}^{Mij}, u_{+-}^{Mij}, cf. Fig. 1:

$$\begin{aligned} u_{++}^{Mij} &= h^{-2}\cdot(u^{ij(M)} - u^{j(M)} - u^{i(M)} + u^M), \\ u_{-+}^{Mij} &= h^{-2}\cdot(u^{j(M)} - u^{-ij(M)} - u^M + u^{-i(M)}), \\ u_{--}^{Mij} &= h^{-2}\cdot(u^M - u^{-i(M)} - u^{-j(M)} + u^{-i-j(M)}), \\ u_{+-}^{Mij} &= h^{-2}\cdot(u^{i(M)} - u^M - u^{i-j(M)} + u^{-j(M)}) \quad \text{for } i \neq j. \end{aligned} \tag{3.5}$$

From (3.5) and (3.3) it follows that

$$u^{Mij} = \tfrac{1}{4}\cdot(u_{++}^{Mij} + u_{-+}^{Mij} + u_{--}^{Mij} + u_{+-}^{Mij}). \tag{3.6}$$

4. Throughout this paper we shall use the following

ASSUMPTIONS H. (1) *Suppose that the function* u^M *is defined at the nodal points* (2.3) *of the set* Q, *cf.* (2.1).

(2) *There exists a positive constant* $L > 0$ (*independent of the mesh size* h) *such that the second order difference quotients satisfy the conditions*

$$\begin{aligned} &|u^{Mjj} - u^{Pjj}| \leqslant h\cdot L, \quad |u_{++}^{Mij} - u_{++}^{Pij}| \leqslant h\cdot L, \quad |u_{-+}^{Mij} - u_{-+}^{Pij}| \leqslant h\cdot L, \\ &|u_{--}^{Mij} - u_{--}^{Pij}| \leqslant h\cdot L, \quad |u_{+-}^{Mij} - u_{+-}^{Pij}| \leqslant h\cdot L \quad (i \neq j) \end{aligned} \tag{4.1}$$

for $P = s(M)$ ($s = \pm 1, \pm 2, \ldots, \pm n$), *the distance between* x^M *and* x^P *being* h *in the direction of the* x_s*-axis.*

(3) *We suppose that*

(4.2) $$|u^{Mij}| \leqslant \Lambda \quad (i = 1, \ldots, n;\ j = 1, \ldots, n),\ x^M \in \operatorname{int} Q,$$

where the constant Λ *is independent of the mesh size* h.

(4) *Let us consider the difference inequality for the function* u^M:

(4.3) $$\sum_{i,j=1}^{n} a_{ij}^M \cdot u^{Mij} + \sum_{j=1}^{n} b_j^M \cdot u^{Mj} + c^M \cdot u^M \geqslant -\varepsilon,$$

for $x^M \in \operatorname{int} Q$, $0 < \varepsilon = \text{const}$. *We suppose that* 1° *inequality* (4.3) *is of elliptic type, which means that* $\sum_{i,j=1}^{n} a_{ij}^M \lambda_i \lambda_j$ $(x^M \in Q)$, *is a positively defined quadratic form, and* 2° *the coefficients* a_{ij}^M, b_j^M, c^M *satisfy the conditions*

(4.4) $$|a_{ij}^M| \leqslant \gamma, \quad |b_j^M| \leqslant \beta, \quad c^M \leqslant \eta < 0,$$

the constants γ, β *and* η *being independent of the mesh size* h.

(5) *We suppose that* $u^M = 0$ *for* $x^M \in \partial Q$, *where* ∂Q *denotes the boundary of the set* Q.

(6) *The characteristic roots* s_k, $s_k > 0$ $(k = 1, \ldots, n)$ *of the form* $\sum_{i,j=1}^{n} a_{ij}^M \lambda_i \lambda_j$ *are bounded*:

(4.5) $$0 < \delta_1 \leqslant s_k \leqslant \delta_2 \quad (k = 1, \ldots, n),$$

δ_1 *and* δ_2 *being independent of the mesh size* h.

5. Remark 1. Let us denote by F^M the linear form and by S^M the quadratic form

(5.1) $$F^M = \sum_{j=1}^{n} u^{Aj} \cdot (m_j - a_j) h; \qquad S^M = \sum_{i,j=1}^{n} \cdot (m_i - a_i)(m_j - a_j) h^2,$$

for a fixed nodal point $x^A \in \operatorname{int} Q$ and for an arbitrary nodal point $x^M \in Q$.

Let z^M $(x^M \in Q)$ be the function

(5.2) $$z^M = u^M - u^A - F^M - S^M \quad (x^M \in Q).$$

Under these assumptions the function z^M vanishes together with the first and the second order difference quotients at the fixed nodal point x^A:

(5.3) $$z^A = 0, \quad z^{Ap} = 0, \quad z^{Apq} = 0 \quad (p = 1, 2, \ldots, n;\ q = 1, 2, \ldots, n).$$

The simple proof will be omitted.

6. We shall now give Lemma 1 on the extended mean value theorem for the function u^M.

Lemma 1. *Let us suppose that the function* u^M *satisfies Assumptions* H *and denote by* x^A *an arbitrary fixed nodal point*, $x^A \in \operatorname{int} Q$.

Under these assumptions

$$(6.1)\quad u^M = u^A + \sum_{j=1}^{n} u^{Aj}\cdot(m_j - a_j)\cdot h + \tfrac{1}{2}\cdot \sum_{i,j=1}^{n} u^{Aij}\cdot(m_i - a_i)(m_j - a_j)\cdot h^2 + R,$$

for $x^M \in Q$. *In formula* (6.1) *the rest term* R *is of the form*

$$(6.2)\qquad R = 2\theta\cdot h^3\cdot L\cdot|m-a|^3,$$

where $M = (m_1, \ldots, m_n)$, $A = (a_1, \ldots, a_n)$, θ *is a number* $|\theta| \leqslant 1$, *and* $|m-a|$ *denotes by definition*

$$(6.3)\qquad |m-a| = \sum_{j=1}^{n} |m_j - a_j|.$$

The proof of Lemma 1 will be omitted.

7. LEMMA 2. *Let us suppose that the function* u^M *assumes the maximum value at the nodal point* x^A *inside the set* Q, *cf.* (2.1):

$$(7.1)\qquad u^A \geqslant u^M \quad \text{for } x^M \in Q,\ x^A \in \operatorname{int} Q.$$

Under these assumptions the difference quotients of the first order u^{Aj}, u_+^{Aj}, u_-^{Aj} *satisfy the conditions*

$$(7.2)\qquad |u^{Aj}| \leqslant h\cdot|u^{Ajj}|, \quad |u_+^{Aj}| \leqslant h\cdot|u^{Ajj}|, \quad |u_-^{Aj}| \leqslant h\cdot|u^{Ajj}|,$$

for $j = 1, \ldots, n$, *at the nodal point* $x^A \in \operatorname{int} Q$.

The simple proof of Lemma 2 will be omitted.

8. We shall now give a modification of the (well-known) lemma on quadratic forms.

LEMMA 3. *We shall assume that the quadratic forms* f_1 *and* f_2 *satisfy the conditions*

$$(8.1)\qquad f_1 = \sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j \geqslant 0, \quad f_2 = \sum_{i,j=1}^{n} b_{ij}\mu_i\mu_j \leqslant d(\mu)$$

for $\mu \in R^n$, *where* $d(\mu) > 0$ *for* $0 \neq \mu = (\mu_1, \ldots, \mu_n) \in R^n$, $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$ $(i, j = 1, \ldots, n)$.

Let us denote by a_{kl} $(k, l = 1, \ldots, n)$ *the orthogonal matrix which transforms* f_1 *into the canonical form, and let* s_k $(k = 1, \ldots, n)$ *be the characteristic roots* $(s_k \geqslant 0;\ k = 1, \ldots, n)$ *of the form* f_1.

Under these assumptions

$$(8.2)\qquad \sum_{i,j=1}^{n} a_{ij}\cdot b_{ij} \leqslant \sum_{k=1}^{n} d(s_k^{1/2}\cdot a_k),$$

where $a_k = (a_{k1}, \ldots, a_{kn})$ $(k = 1, \ldots, n)$.

The simple proof of Lemma 3 will be omitted.

9. LEMMA 4. *Let us suppose that the function u^M satisfies Assumptions* **H**, *and assumes the maximum value at the nodal point $x^A \in \operatorname{int} Q$.*

Under these assumptions the quadratic form S^M

$$S^M = \tfrac{1}{2}\cdot\sum_{i,j=1}^{n} u^{Aij}\cdot(m_i-a_i)(m_j-a_j)\cdot h^2, \tag{9.1}$$

cf. (5.1), *satisfies the inequality:*

$$S^M \leqslant 2\cdot|\theta|\cdot h^3\cdot L\cdot|m-a|^3 \quad \textit{for } x^M \in \operatorname{sym} A. \tag{9.2}$$

Proof. The function u^M ($x^M \in Q$) can be expressed with the aid of formula (6.1) (cf. Lemma 1); hence

$$u^M - u^A = \sum_{j=1}^{n} u^{Aj}\cdot(m_j-a_j)h + \tfrac{1}{2}\cdot\sum_{i,j=1}^{n} u^{Aij}\cdot(m_i-a_i)(m_j-a_j)h^2 + \\ +2\cdot\theta\cdot h^3\cdot L\cdot|m-a|^3 \quad \text{for } x^M \in Q. \tag{9.3}$$

The left-hand member of (9.3) is non-positive by assumption: $u^M - u^A \leqslant 0$ for $x^M \in Q$. On the right-hand side of formula (9.3) the linear form changes the sign and the quadratic form does not change the sign when the nodal point x^M is replaced by the nodal point x^{M^*} ($x^{M^*} \in Q$), x^M and x^{M^*} being symmetric relative to x^A.

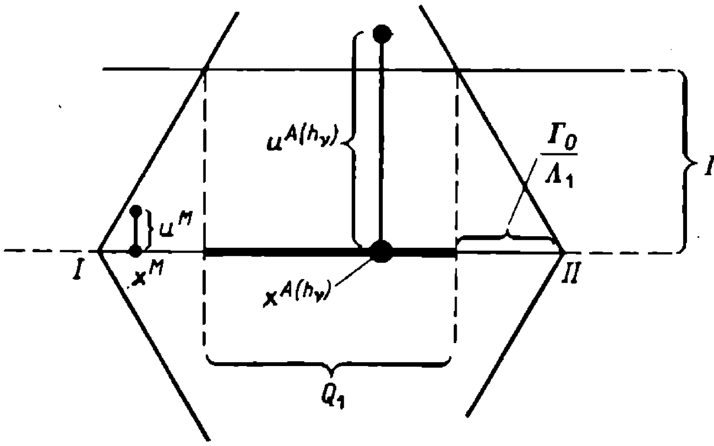

Fig. 2. The sets $Q = I\text{–}II$ and Q_1 in the two-dimensional case ($n = 2$) as seen from the edge. The nodal points $x^{A(h_\nu)}$ ($\nu = 1, 2 \ldots$) belong to the set Q_1 ($Q_1 \subset Q$)

We shall estimate the quadratic form S^M at that nodal point (x^M or x^{M^*}), where the linear form (with coefficients u^{Aj}) is non-negative. Then that linear form can be omitted and we obtain from (9.3) the inequality

$$S^M \leqslant 2\cdot|\theta|\cdot h^3\cdot L\cdot|m-a|^3 \quad \text{for } x^M \in \operatorname{sym} A. \tag{9.4}$$

This ends the proof of relation (9.2).

10. Lemma 5. *Let us denote by $S(x)$, $x = (x_1, \ldots, x_n) \in R^n$, the quadratic form*

$$S(x) = \tfrac{1}{2}\cdot\sum_{i,j=1}^{n} u^{Aij}\cdot(x_i - x_i^A)(x_j - x_j^A) \qquad (x \in R^n), \tag{10.1}$$

where $A = (a_1, \ldots, a_n)$ and x^A is the nodal point in the set Q_1, $Q_1 \neq 0$, $Q_1 \subset \operatorname{int} Q$ (cf. Fig. 3).

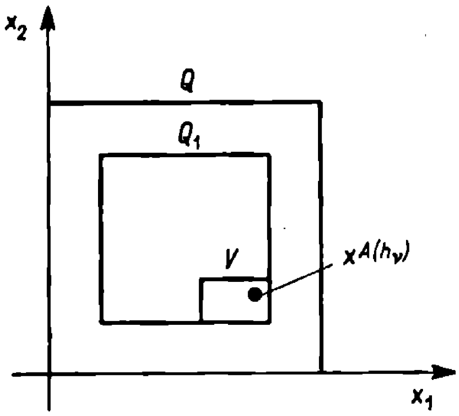

Fig. 3. The sets Q, Q_1 and V in the two-dimensional case ($n = 2$)

Let us denote by S^M the discrete values of $S(x)$ at the nodal points x^M in the set Q (cf. (5.1)):

$$S^M = \tfrac{1}{2}\cdot\sum_{i,j=1}^{n} u^{Aij}\cdot(m_i - a_i)(m_j - a_j)h^2. \tag{10.2}$$

We shall suppose that u^M has a maximum in the set Q at the nodal point x^A, $x^A \in Q_1$:

$$u^M \leqslant u^A \quad \textit{for } x^M \in Q. \tag{10.3}$$

We shall denote by V, $V \subset Q$, the set

$$V = \{x\colon |x_j - x_j^M| \leqslant h \ (j = 1, \ldots, n);\ h\cdot|m - a| \leqslant h^a\}, \tag{10.4}$$

where a = const, $0 < a < 1$ (cf. Fig. 3).

Under these assumptions we have the following estimate for the quadratic form $S(x)$ in the set V:

$$S(x) \leqslant C(h) \quad \textit{for } x \in V, \tag{10.5}$$

where

$$C(h) = 2\cdot|\theta|\cdot L\cdot h^{3a} + n\cdot 2\Lambda\cdot h^{1+a} + n^2\cdot\Lambda\cdot h^2. \tag{10.6}$$

Proof. The expansion of the function $S(x)$ in the neighbourhood of the nodal point x^M has the form

$$(10.7)\quad S(x) = S^M + \sum_{j=1}^{n} (S_{x_j})_M \cdot (x_j - x_j^M) + \tfrac{1}{2} \cdot \sum_{i,j=1}^{n} (S_{x_i x_j})_M \cdot (x_i - x_i^M)(x_j - x_j^M).$$

For the derivatives S_{x_j} and $S_{x_i x_j}$ we obtain from (10.1)

$$(10.8)\qquad S_{x_i} = 2 \cdot \sum_{j=1}^{n} u^{Aij} \cdot (x_j - x_j^A); \qquad S_{x_i x_j} = 2 \cdot u^{Aij};$$

hence

$$(10.9)\qquad |(S_{x_i})_M| \leqslant 2 \cdot \sum_{j=1}^{n} |u^{Aij}| \cdot |x_j^M - x_j^A| \leqslant 2 \cdot \Lambda \sum_{j=1}^{n} |m_j - a_j| h = 2 \cdot \Lambda \cdot h |m - a|; \qquad (S_{x_i x_j})_M \leqslant 2\Lambda.$$

From (10.9) it follows that the second and last term in (10.7) has the estimate

$$(10.10)\qquad \sum_{j=1}^{n} (S_{x_j})_M \cdot (x_j - x_j^M) + \tfrac{1}{2} \cdot \sum_{i,j=1}^{n} (S_{x_i x_j})_M (x_i - x_i^M)(x_j - x_j^M) \leqslant n \cdot 2\Lambda h^2 |m - a| + n^2 \cdot \Lambda h^2 \quad \text{for } |x_j - x_j^M| \leqslant h \quad (j = 1, \ldots, n);$$

therefore from (10.10), (10.7) and Lemma 4 we obtain

$$(10.11)\qquad S(x) \leqslant 2|\theta| L \cdot h^3 |m - a|^3 + n \cdot 2\Lambda h^2 |m - a| + n^2 \cdot \Lambda h^2, \quad \text{for } x^M \in \operatorname{sym} A \text{ and } |x_j - x_j^M| \leqslant h \ (j = 1, \ldots, n).$$

But in the set V (cf. (10.4)) we have $h|m-a| \leqslant h^a$; hence (10.11) and (10.4) imply that

$$(10.12)\qquad S(x) \leqslant 2 \cdot |\theta| L \cdot h^{3a} + n \cdot 2\Lambda \cdot h^{1+a} + n^2 \cdot \Lambda h^2 \quad \text{for } x \in V.$$

This ends the proof of Lemma 5.

Remark 2. The estimate of the quadratic form $S(x)$ for $x \in R^n$ can easily be found. For this purpose let us write

$$(10.13)\qquad G(x - x^A, h) = \begin{cases} C(h) & \text{for } x \in V, \\ l^2 \cdot C(h) & \text{for } x \in R^n \setminus V, \end{cases}$$

where $x - x^A = l \cdot (x^b - x^A)$ $(l \geqslant 1)$, x^b being the intersection point of the boundary ∂V with a segment joining the points x^A and x.

From the definition of the quadratic form $S(x)$ and from (10.5) it follows that

$$(10.14)\qquad S(x) \leqslant G(x - x^A, h) \quad \text{for } x \in R^n.$$

Remark 3. From the estimate (10.14) of the quadratic form $S(x)$ and from Lemma 3 on quadratic forms it can easily be deduced that

(10.15) $$\sum_{i,j=1}^{n} a_{ij}^{A}\cdot u^{Aij} \leqslant E(h),$$

where

(10.16) $$E(h) = \sum_{k=1}^{n} G(s_k^{1/2}\cdot \alpha_k, h).$$

Here s_k $(k = 1, \ldots, n)$ denote the positive characteristics roots of the form $f_1 = \sum_{i,j=1}^{n} a_{ij}^{A}\lambda_i\lambda_j$, α_{kl} $(k, l = 1, \ldots, n)$ is the orthogonal matrix transforming f_1 into the canonical form, and $\alpha_k = (\alpha_{k1}, \ldots, \alpha_{kn})$ $(k = 1, \ldots, n)$.

In fact, f_1 is the positive defined form because of Assumptions H. On the other hand, the form

(10.17) $$S(x) = \sum_{i,j=1}^{n} u^{Aij}\cdot(x_i - x_i^A)(x_j - x_j^A) = \sum_{i,j=1}^{n} u^{Aij}\mu_i\mu_j \quad \text{for } \mu_i = x_i - x_i^A \ (i = 1, \ldots, n),$$

posesses the estimate (10.14); hence (10.17), (10.14) and Lemma 3 yield the estimate (10.15).

11. THEOREM 1. *Let us suppose that the function u^M satisfies Assumptions* H. *Suppose in addition that the difference inequality of the elliptic type*

(11.1) $$\sum_{i,j=1}^{n} a_{ij}^{M}\cdot u^{Mij} + \sum_{j=1}^{n} b_j^{M}\cdot u^{Mj} + c^M\cdot u^M \geqslant -\varepsilon,$$

holds if $u^M > 0$ for $x^M \in \operatorname{int} Q$.

Under these assumptions we have

(11.2) $$\max_{M\in Q} u^M \leqslant g(h),$$

for

(11.3) $$0 < g(h) = -\eta^{-1}\cdot[E(h) + D(h) + \varepsilon],$$

where $0 < D(h) = n\beta h\Lambda$, and $E(h)$ is defined by (10.16) (cf. Remark 3).

Proof. Assuming the contrary, suppose that

(11.4) $$\max_{x^M\in Q} u^M > g(h).$$

Let us denote the left-hand member in formula (11.4) by u^A $(x^A \in \operatorname{int} Q)$. Thus we have

(11.5) $$u^A > g(h).$$

With (11.1) in mind we shall verify the following inequalities:

$$\sum_{i,j=1}^{n} a_{ij}^{A}\cdot u^{Aij} \leqslant E(h), \tag{11.6}$$

$$\sum_{j=1}^{n} b_{j}^{A}\cdot u^{Aj} \leqslant D(h), \tag{11.7}$$

$$c^{A}\cdot u^{A} \leqslant \eta\cdot u^{A}. \tag{11.8}$$

In fact, (11.6) follows from Remark 3.

From Lemma 2, cf. (7.2), (4.4) and (4.2), we obtain

$$\sum_{j=1}^{n} b_{j}^{A}\cdot u^{Aj} \leqslant n\beta h\Lambda = D(h), \tag{11.9}$$

which completes the proof of (11.7).

Inequality (11.8) follows immediately from assumption (4.4): $c^{A} \leqslant \eta < 0$, since $u^{A} > 0$ because of (11.5) and (11.3).

With (11.6), (11.7), (11.8) at hand we obtain first

$$\sum_{i,j=1} a_{ij}^{A}\cdot u^{Aij} + \sum_{j=1}^{n} b_{j}^{A}\cdot u^{Aj} + c^{A}\cdot u^{A} \leqslant E(h)+D(h)+\eta\cdot u^{A}. \tag{11.10}$$

But from assumption (11.5) and the definition of $g(h)$ (cf. (11.3)) it follows that

$$E(h)+D(h)+\eta\cdot u^{A} < -\varepsilon; \tag{11.11}$$

hence, from (11.10) and (11.11) we obtain the inequality

$$\sum_{i,j=1}^{n} a_{ij}^{A}\cdot u^{Aij} + \sum_{j=1}^{n} b_{j}^{A}\cdot u^{Aj} + c^{A}\cdot u^{A} < -\varepsilon. \tag{11.12}$$

Since inequalities (11.12) and (11.1) are contradictory, we conclude that the maximum value u^{A} satisfies (11.2).

This ends the proof of Theorem 1.

12. LEMMA 6. *Let us suppose that the function u^{M} satisfies Assumptions* H. *Assume in addition that there exist a constant* $0 < \Gamma_0 = \text{const}$ *and a sequence* h_ν $(\nu = 1, 2, \ldots)$ *such that*

$$0 < h_\nu \to 0, \quad \textit{as } \nu \to +\infty, \tag{12.1}$$

$$u^{A(h_\nu)} \geqslant \Gamma_0 \quad (\nu = 1, 2, \ldots), \tag{12.2}$$

x^{A}, $A = A(h_\nu)$, *being the nodal point, where the maximum value is attained*: $u^{M} \leqslant u^{A}$, *for* $x^{M} \in Q$.

Under these assumptions the nodal points x^{A}, $A = A(h_\nu)$ $(\nu = 1, 2, \ldots)$ *are in the set* Q_1, $Q_1 \neq 0$, $Q_1 \subset Q$:

$$Q_1 = \{x \in Q\colon r(x, \partial Q) \geqslant \Gamma_0/\Lambda_1\}, \tag{12.3}$$

where $r(x, \partial Q)$ *denotes the distance between the point* x *and the boundary* ∂Q *of the set* Q, $\Lambda_1 > 0$ *is a constant* (cf. Fig. 2) *and the set* Q_1 *does not depend on the value of* h_ν $(\nu = 1, 2, \ldots)$.

Proof. The difference quotients of the second order u^{Mij} are bounded by the constant Λ for all $h > 0$ because of Assumptions H. This and the boundedness of the set Q imply that the difference quotients of the first order u^{Mj} are bounded by some constant $\Lambda_1 > 0$, Λ_1, obviously dependent on Λ.

But $u^M = 0$ for $x^M \in \partial Q$; therefore we have

$$|u^M| \leqslant \Lambda_1 \cdot r(x^M, \partial Q) \quad \text{for } x^M \in Q, \tag{12.4}$$

where $r(x^M, \partial Q)$ denotes the distance between the nodal point x^M and the boundary ∂Q (cf. Fig. 2). From inequality (12.2) it follows that the nodal points x^A, $A = A(h_\nu)$ $(\nu = 1, 2, \ldots)$ cannot be in neighbourhood of the boundary ∂Q, since in the neighbourhood of the boundary ∂Q the values of u^M are less than Γ_0 (cf. (12.4) and (12.3)):

$$|u^M| \leqslant \Lambda_1 \cdot r(x^M, \partial Q) < \Gamma_0 \tag{12.5}$$

for the nodal points x^M satisfying $r(x^M, \partial Q) < \Gamma_0 / \Lambda_1$.

This proves that the nodal points x^A, $A = A(h_\nu)$ $(\nu = 1, 2, \ldots)$ are in the set Q_1.

This ends the proof of Lemma 6.

13. Theorem 2. *Let us suppose that the function* u^M *satisfies Assumptions* H, *and the quantity* ε *in* (4.3) *depends on* h:

$$0 < \varepsilon = \varepsilon(h) \to 0, \quad \textit{as } h \to 0. \tag{13.1}$$

Let us write

$$u^A = \max_{x^M \in Q} u^M, \quad A = A(h). \tag{13.2}$$

Under these assumptions

$$u^{A(h)} \to 0, \quad \textit{as } h \to 0. \tag{13.3}$$

Proof. We shall proceed by assuming the contrary, and we suppose that there exists a sequence h_ν $(\nu = 1, 2, \ldots)$, $0 < h_\nu \to 0$, as $\nu \to +\infty$, such that

$$u^{A(h_\nu)} \geqslant \Gamma_0 \quad (\nu = 1, 2, \ldots), \tag{13.4}$$

where $0 < \Gamma_0 = \text{const}$.

From Lemma 6 it follows that $x^A \in Q_1$, $A = A(h_\nu)$ $(\nu = 1, 2, \ldots)$ (cf. (12.3) and Fig. 2).

We shall prove that we have

$$u^{A(h_\nu)} < \Gamma_0 \quad \text{for } h_\nu \text{ sufficiently small}. \tag{13.5}$$

In fact, from Theorem 1 we have estimate (11.2), (11.3). In formula (11.3) we have $D(h_\nu) = n\beta h_\nu \Lambda \to 0$, as $h_\nu \to 0$, and $\eta = \text{const}$.

Let us consider the quantity $E(h_\nu)$ defined by formula (10.16). We shall verify that

$$E(h_\nu) \to 0, \quad \text{as } h_\nu \to 0. \tag{13.6}$$

For this purpose let us observe first that the points $a_k = (a_{k1}, \ldots, a_{kn}) \in R^n$ $(k = 1, 2, \ldots, n)$ (cf. (10.16)) are on the unit sphere, since a_{kl} $(k, l = 1, \ldots, n)$ is the orthogonal matrix. The characteristic roots s_k, $s_k > 0$ $(k = 1, \ldots, n)$ of the positive defined quadratic form $\sum_{i,j=1} a_{ij}^A \lambda_i \lambda_j$ (cf. Assumptions H) are bounded:

$$0 < \delta_1 \leqslant s_k \leqslant \delta_2 \quad (k = 1, \ldots, n), \tag{13.7}$$

δ_1 and δ_2 being independent of the mesh size h. This implies that the points $s_k^{1/2} \cdot a_k \in R^n$ $(k = 1, \ldots, n)$ are in some bounded set Q_2 for all $h > 0$:

$$s_k^{1/2} \cdot a_k \in Q_2 \quad \text{for } h > 0. \tag{13.8}$$

On the other hand, $G(x - x^A, h) = C(h)$ for $x \in V$, cf. (10.13) and (10.6).

The value $C(h)$ of the function G at the point $x^b \in \partial V$ divided by $|x^b - x^A|^2$ approaches zero as $h \to 0$. Indeed, we have (cf. (10.4) and (12.3)) $|x^b - x^A|^2 \geqslant h^{2a}$. Hence (cf. (10.6))

$$\frac{C(h)}{|x^b - x^A|^2} \leqslant \frac{C(h)}{h^{2a}} \to 0, \quad \text{as } h \to 0, \tag{13.9}$$

since

$$\begin{aligned} &\frac{h^{3a}}{h^{2a}} = h^a \to 0, \quad \frac{h^{1+a}}{h^{2a}} = h^{1-a} \to 0, \\ &\frac{h^2}{h^{2a}} = h^{2(1-a)} \to 0, \quad \text{as } h \to 0. \end{aligned} \tag{13.10}$$

In the set $R^n \setminus V$ we have (cf. (10.13)):

$$G(x - x^A, h) = \frac{|x - x^A|^2}{|x^b - x^A|^2} \cdot C(h).$$

This and (13.9) imply that

$$G(x - x^A, h) \to 0, \quad \text{as } h \to 0, \tag{13.11}$$

the convergence being uniform with respect to x in every closed and bounded set, e.g. in the set Q_2.

In particular, (13.11) and (13.8) yield

$$G(s_k^{1/2} \cdot a_k, h) \to 0, \quad \text{as } h \to 0. \tag{13.12}$$

From (13.12) and definition (10.16) of $E(h)$ follows (13.6).

This implies that there exists an $h_\nu > 0$ sufficiently small for $g(h_\nu)$ in formula (11.2), (11.3) to satisfy the inequality

$$g(h_\nu) < \Gamma_0 \quad \text{for } h_\nu \text{ sufficiently small.} \tag{13.13}$$

Hence, from (13.13) and (11.2) it follows that

$$u^{A(h_\nu)} < \Gamma_0 \quad \text{for } h_\nu \text{ sufficiently small,} \tag{13.14}$$

which contradicts assumption (13.4).

This ends the proof of Theorem 2.

14. THEOREM 3. *Let us suppose that the function u^M satisfies Assumptions* H, *inequality* (4.3) *being replaced by the inequality*

$$\sum_{i,j=1}^{n} a_{ij}^M \cdot u^{Mij} + \sum_{j=1}^{n} b_j^M \cdot u^{Mj} + c^M \cdot u^M \leqslant +\varepsilon, \tag{14.1}$$

where $0 < \varepsilon = \varepsilon(h) \to 0$, *as* $h \to 0$.

Let us write

$$u^B = \min_{x^M \in Q} u^M, \qquad B = B(h). \tag{14.2}$$

Under these assumptions we have

$$u^{B(h)} \to 0, \qquad as\ h \to 0. \tag{14.3}$$

Theorem 3 can be proved in the same manner as Theorem 2. It is sufficient to repeat the argument with the minimum value (14.2), the sense of the corresponding inequalities being reversed.

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