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A differentiable dependence on the right-hand side of solutions of ordinary differential equations

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Abstract. The aim of this paper is to study the differentiability of the solution of certain system of ordinary equations as a function of the right-hand side of the system. The differentiability is understood in the sense expounded in [4]. In that paper the differential $D_f(x_0)\colon K_{X_0}(x_0) \to Y$ of a function $f\colon X_0 \to Y$ at a point $x_0 \in X_0$ is defined by the formula $D_f(x_0)(x) = \lim\limits_{\substack{s\to 0+\\ u\to 0}} (1/s)\big(f(x_0+s(x+u))-f(x_0)\big)$ and $K_{X_0}(x_0)$ is the set of all points $x \in X$ for which there are arbitrarily small $s > 0$ and $u \in X$ so that $x_0+s(x+u) \in X_0$. The result which we present here is similar to those obtained, by means of some supplimentary conditions, in [3] (see Theorem 7.1, Corollary 3) or in [2] (see § 7).

1. In this section we shall describe our problem.

Let R be the real line and let $R_0 = [a, b]$ be a compact interval of R.

Let X be an Euclidean space and let X_0 be an open subset of X.

Let F be the space of all functions $f\colon R_0 \times X_0 \to X$ which have the following Carathéodory's properties:

(i) for every $t \in R_0$, the function $x \in X_0 \to f(t, x) \in X$ is continuous;

(ii) for every $x \in X_0$, the function $t \in R_0 \to f(t, x) \in X$ is measurable;

(iii) for every compact subset P of X_0, there is an integrable function $m\colon R_0 \to R$ such that if $(t, x) \in R_0 \times P$, then $\|f(t, x)\| \leqslant m(t)$.

The properties of F ensure that for every $(x_0, f_0) \in X_0 \times F$ the system

$$\dot{y}_0(t) = f_0\big(t, y_0(t)\big), \qquad y_0(a) = x_0 \tag{1}$$

has at least one solution y_0 defined on an open subinterval of T_0.

In the sequel we shall denote by $D_f(t, x)$ the differential of the function $x \in X_0 \to f(t, x) \in X$ at the point $x \in X_0$, if it exists. It is known from [4] that $D_f(t, x)$, for fixed t and x, is a positively homogeneous, continuous function from $K_{X_0}(x) = X$ to X.

Let Ξ be the linear normed space of all positively homogeneous, continuous functions $\xi\colon X \to X$, with $\|\xi\| = \sup\limits_{\|x\|=1} \|\xi(x)\|$.

Let F_0 be a set of points $f \in F$ which have the following properties:

(iv) for every $t \in R_0$, the function $x \in X_0 \to f(t, x) \in X$ is differentiable;

(v) for every compact subset P of X_0, there is an integrable function m: $R_0 \to R$ such that if $(t, x) \in R_0 \times P$, then $\|D_f(t, x)\| \leqslant m(t)$.

It is easy to prove, using a mean value theorem, that if $f \in F_0$, then for every compact subset P of X_0 and for every number $r > 0$ for which $x \in P$ and $\|\tilde{x}\| \leqslant r$ imply $x + \tilde{x} \in X_0$ (such a number always exists) there is an integrable function m: $R_0 \to R$ for which $(t, x) \in R_0 \times P$ and $\|\tilde{x}\| \leqslant r$ imply $\|f(t, x+\tilde{x}) - f(t, x)\| \leqslant m(t)\|\tilde{x}\|$.

Consequently, the properties of F_0 ensure that for every $(x_0, f_0) \in X_0 \times F_0$ system (1) has at most one, hence unique, local solution y_0.

We are interested in those points $(x_0, f_0) \in X_0 \times F_0$ for which the solution of system (1) is defined on R_0. The set of such pairs we denote by Σ. We shall suppose that Σ is not empty.

Let Y be the linear normed space of all continuous functions y: $R_0 \to X$, with $\|y\| = \sup\limits_{t \in R_0} \|y(t)\|$.

Denote by σ: $\Sigma \to Y$ the function defined by the formula $\sigma(x_0, f_0) = y_0$.

Our purpose is to endow the linear space F with such a compatible topology that Σ and σ have "good properties".

Let Ψ be the space of all functions ψ: $R_0 \times X_0 \to \Xi$ which have property (v); thus property (v) becomes equivalent to the relation $D_f \in \Psi$. We shall use the notation D_{F_0} for the set of all D_f with $f \in F_0$.

The hypothesis concerning the set F_0 will be made in terms of a compatible topology on the linear space Ψ.

2. In what follows we shall expound the announced topological structures.

Let Y_0 be the set of all points $y \in Y$ for which $y(R_0)$ is a subset of X_0.

Let the space F be endowed with the topology generated by the family of seminorms

$$\|f\|_Q = \sup_{y \in Q} \sup_{t_1, t_2 \in R_0} \left\| \int_{t_1}^{t_2} f(\theta, y(\theta))\, d\theta \right\|,$$

where Q's are compact subsets of Y_0.

Before continuing our description, we shall make two comments about the topology on F.

First, this topology is the trace of the topology of compact convergence on the space F, regarded by means of the mapping $f \to \hat{f}$, where $\hat{f}(y)(t) = \int_a^t f(\theta, y(\theta))\, d\theta$, as a subspace of the space of all continuous functions from Y_0 to Y, because:

$$\sup_{y \in Q} \|\hat{f}(y)\| \leqslant \|f\|_Q \leqslant 2 \sup_{y \in Q} \|\hat{f}(y)\|.$$

Second, this topology is the trace of the topology used in [1] on the space F, regarded by means of the mapping $f\to\hat{f}$, where $\hat{f}$ assigns to every point of $R_0\times Y_0$ a point of X by the formula $\hat{f}(t,y)=f(t,y(t))$, as a subspace of all functions from $R_0\times Y_0$ to X having Carathéodory's properties, because:

$$\sup_{y\in Q}\sup_{t_1,t_2\in R_0}\left\|\int_{t_1}^{t_2}\hat{f}(\theta,y)\,d\theta\right\|=\|f\|_Q.$$

This means that the theory of quasi-convexity of [1] (see Lemma 4.1) remains true in F.

Now, let us continue our exposition.

For a function $\mu\colon R_0\to R$ which is not necessarily measurable but bounded from above by an integrable function, we put

$$\int_{R_0}\mu(\theta)\,\delta\theta=\inf\Big\{\int_{R_0}m(\theta)\,d\theta;\ m\colon R_0\to R\text{ integrable and }\mu(t)\leqslant m(t)\text{ for every }t\in R_0\Big\}.$$

Let the space Ψ be endowed with the topology generated by the family of seminorms

$$\|\psi\|_P=\int_{R_0}\sup_{x\in P}\|\psi(\theta,x)\|\,\delta\theta,$$

where P's are compact subsets of X_0.

3. In order to state and prove our result, we need the system in variation ($f_0\in F_0$ and y_0 is the solution of (1))

$$\dot{y}(t)=D_{f_0}\big(t,y_0(t)\big)\big(y(t)\big)+f\big(t,y_0(t)\big),\qquad y(a)=x. \tag{2}$$

It is easy to prove by using a theorem of [4], a Lebesgue theorem and a mean value theorem, that the following propositions hold:

for every $t\in R_0$, the function $x\in X\to D_{f_0}\big(t,y_0(t)\big)(x)\in X$ is continuous;

for every $x\in X$, the function $t\in R_0\to D_{f_0}\big(t,y_0(t)\big)(x)\in X$ is measurable;

there is an integrable function $m\colon R_0\to R$ such that if $t\in R_0$, $x_1\in X$ and $x_2\in X$, then

$$\big\|D_{f_0}\big(t,y_0(t)\big)(x_1)-D_{f_0}\big(t,y_0(t)\big)(x_2)\big\|\leqslant m(t)\|x_1-x_2\|.$$

Consequently, for every $(x,f)\in X\times F$ system (2) has a unique solution $y\in Y$.

The promised result is the following

THEOREM. *Let the set D_{F_0} be bounded. Then the function σ is differentiable. Moreover, $K_\Sigma(x_0,f_0)=X\times K_{F_0}(f_0)$ for every $(x_0,f_0)\in\Sigma$ and $D_\sigma(x_0,f_0)(x,f)=y$ for every $(x,f)\in K_\Sigma(x_0,f_0)$, where y is the solution of* (2).

Proof of the theorem. Let $(x_0, f_0) \in \Sigma$. Then $K_\Sigma(x_0, f_0) \subseteq K_{X_0 \times F_0}(x_0, f_0) = X \times K_{F_0}(f_0)$. Now, let $(x, f) \in X \times K_{F_0}(f_0)$.

For $s > 0$, $(\tilde{x}, \tilde{f}) \in X \times F$ and $(x_0, f_0) + s\big((x, f) + (\tilde{x}, \tilde{f})\big) \in X_0 \times F_0$ (it is known that there are arbitrarily small such s and $(\tilde{x}, \tilde{f})$), let us consider the system

$$(3) \qquad \begin{aligned} \big(y_0 + s(y + \tilde{y})\big)'(t) &= \big(f_0 + s(f + \tilde{f})\big)\Big(t, \big(y_0 + s(y + \tilde{y})\big)(t)\Big), \\ \big(y_0 + s(y + \tilde{y})\big)(a) &= x_0 + s(x + \tilde{x}), \end{aligned}$$

which has a unique local solution $\tilde{y}$.

We shall show that if s and $(\tilde{x}, \tilde{f})$ are sufficiently small, then $\tilde{y} \in Y$ and $\tilde{y}$ is small.

This will mean that $(x_0, f_0) + s\big((x, f) + (\tilde{x}, \tilde{f})\big) \in \Sigma$; hence $(x, f) \in K_\Sigma(x_0, f_0)$, and $\sigma\Big((x_0, f_0) + s\big((x, f) + (\tilde{x}, \tilde{f})\big)\Big) = y_0 + s(y + \tilde{y})$; hence σ is differentiable at (x_0, f_0) and $D_\sigma(x_0, f_0)(x, y) = y$.

Let $\alpha > 0$ be such that if $\bar{s} \in [0, \alpha]$ and $\|\bar{y}\| \leqslant \alpha$, then $y_0 + \bar{s}(y + \bar{y}) \in Y_0$ (such a number always exists). Let us set

$$P = \big\{y_0(t) + \bar{s}\big(y(t) + \bar{x}\big);\ t \in R_0,\ \bar{s} \in [0, \alpha],\ \|\bar{x}\| \leqslant \alpha\big\}$$

(P is a compact subset of X_0);

$$Q = \{y_0 + \bar{s}y;\ \bar{s} \in [0, \alpha]\} \qquad (Q \text{ is a compact subset of } Y_0);$$

$$\beta(\bar{s}) = \int_{R_0} \Big\| (1/\bar{s})\Big(f_0\big(\theta, (y_0 + \bar{s}y)(\theta)\big) - f_0\big(\theta, y_0(\theta)\big)\Big) - D_{f_0}\big(\theta, y_0(\theta)\big)\big(y(\theta)\big) + $$

$$+ f\big(\theta, (y_0 + \bar{s}y)(\theta)\big) - f\big(\theta, y_0(\theta)\big)\Big\| d\theta \qquad (\lim_{\bar{s} \to 0} \beta(\bar{s}) = 0);$$

$$\gamma = \sup_{\tilde{f} \in F_0} \|D_{\tilde{f}}\|_P.$$

We shall complete the proof of the theorem by proving the following

Lemma. *Let $s < \alpha$ and $\big(\|\tilde{x}\| + \|\tilde{f}\|_Q + \beta(s)\big)\exp\gamma < \alpha$. Then $\tilde{y} \in Y$ and $\|\tilde{y}\| \leqslant \big(\|\tilde{x}\| + \|\tilde{f}\|_Q + \beta(s)\big)\exp\gamma$.*

Proof of the lemma. It follows from systems (1), (2) and (3) that the function $\tilde{y}$ verifies the following integral equality:

$$\tilde{y}(t) = \tilde{x} + \int_a^t \tilde{f}\big(\theta, (y_0 + sy)(\theta)\big) d\theta +$$

$$+ \int_a^t (1/s)\Big(f_0\big(\theta, (y_0 + sy)(\theta)\big) - f_0\big(\theta, y_0(\theta)\big)\Big) - D_{f_0}\big(\theta, y_0(\theta)\big)\big(y(\theta)\big) +$$

$$+ f\big(\theta, (y_0 + sy)(\theta)\big) - f\big(\theta, y_0(\theta)\big) d\theta +$$

$$+ \int_a^t (1/s)\Big(\big(f_0 + s(f + \tilde{f})\big)\Big(\theta, \big(y_0 + s(y + \tilde{y})\big)(\theta)\Big) - \big(f_0 + s(f + \tilde{f})\big)\big(\theta, (y_0 + sy)(\theta)\big)\Big) d\theta.$$

Let $m\colon R_0 \to R$ be any integrable function which satisfies, for every $t \in R_0$, the following inequality:

$$\sup_{\bar{x} \in P} \|D_{f_0+s(f+\tilde{f})}(t, \bar{x})\| \leqslant m(t).$$

For those $t > a$ for which $\tilde{y}$ is defined and $\theta \in [a, t]$ implies $\|\tilde{y}(\theta)\| \leqslant \alpha$, it follows that

$$\|\tilde{y}(t)\| \leqslant \|\tilde{x}\| + \|\tilde{f}\|_Q + \beta(s) + \int_a^t m(\theta)\|\tilde{y}(\theta)\|d\theta.$$

From Gronwall's lemma it follows that

$$\|\tilde{y}(t)\| \leqslant \big(\|\tilde{x}\| + \|\tilde{f}\|_Q + \beta(s)\big)\exp\Big(\int_a^t m(\theta)\,d\theta\Big).$$

We may replace, successively, in the above inequality, $\int_a^t m(\theta)\,d\theta$ by $\int_{R_0} m(\theta)\,d\theta$, $\|D_{f_0+s(f+\tilde{f})}\|_P$ and γ. Consequently

$$\sup_t \|\tilde{y}(t)\| < \alpha.$$

A standard argument shows that $\tilde{y}$ can be continued on R_0. Thus the proof of the lemma and of the theorem is complete.

References

[1] R. V. Gamkrelidze, *On some extremal problems in the theory of differential equations with applications to the theory of optimal control*, SIAM J. Control 3 (1965), p. 106–128.

[2] — and G. L. Haratišvili, *Ekstremal'nye zadači v lineĭnyh topologičeskih prostranstvah*, Izv. Akad. Nauk SSSR Ser. Mat. 33 (1969), p. 781–839.

[3] L. W. Neustadt, *On the solutions of certain integral-like operator equations. Existence, uniqueness and dependence theorems*, Arch. Rational Mech. Anal. 38 (1970), p. 131–160.

[4] C. Ursescu, *Sur la notion de differentiabilité*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (to appear).

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