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On the length of some curves in the unit sphere

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The purpose of this note is to present simple proofs of the following two theorems.

Theorem 1. *If $P_1, \ldots, P_n$ are n points in the unit sphere S of the Euclidean n-dimensional space E^n, $P_i = (p_{i1}, \ldots, p_{in})$, $1 \leqslant i \leqslant n$, and $p_{ii} = 0$ for all $1 \leqslant i \leqslant n$, then*

$$\sum_{i=1}^{n-1} \varrho(P_i, P_{i+1}) \geqslant \pi/2,$$

where $\varrho(x, y)$ is the length of the (shortest) great circle's arc joining the points x and y of S.

Theorem 2. *If $P_1, \ldots, P_n$ are n points in the unit sphere S of E^n, $P_i = (p_{i1}, \ldots, p_{in})$, $1 \leqslant i \leqslant n$, and $p_{ii} = 0$ for all $1 \leqslant i \leqslant n$, then*

$$\sum_{i=1}^{n} \varrho(P_i, P_{i+1}) \geqslant \pi \quad (\textit{where } P_{n+1} = P_1).$$

These theorems were mentioned by Nehari [2], and were proven analitically by Lasota and Olech [1] (Theorem 1) and Schwartz [4] (Theorems 1 and 2). Schwartz observed that Theorem 2 implies Theorem 1, by simply going along the arc twice, in opposite directions, so as to get a closed arc. We have independently obtained the following essentially different and considerably simpler geometric proofs.

Let $T_j \colon E^n \to E^n$ be defined by $T_j(x_1, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, -x_j, \ldots, x_n)$, for all $1 \leqslant j < n$; T_j is the reflection of E^n in the hyperplane H_j, given by the equation $x_j = 0$; clearly $T_j(y) = y$ if and only if $y \in H_j$; it is well known that each one of the T_j is an isometric transformation, hence we will use this without proving it; obviously $T_j(S) = S$ for all $1 \leqslant j \leqslant n$.

We are ready for the

Proof of Theorem 1. Since each T_j is an isometric transformation,

$$\varrho(P_i, P_{i+1}) = \varrho[T_iT_{i-1} \ldots T_1(P_i), T_iT_{i-1} \ldots T_1(P_{i+1})] \tag{1}$$

for all $1 \leqslant i \leqslant n-1$,

and therefore

$$(2)\qquad \sum_{i=1}^{n-1} \varrho(P_i, P_{i+1}) = \sum_{i=1}^{n-1} \varrho[T_iT_{i-1}\dots T_1(P_i), T_iT_{i-1}\dots T_1(P_{i+1})].$$

Observe that $T_1(P_1) = P_1$, and that for all $1 \leqslant i \leqslant n-1$, $T_{i+1}[T_iT_{i-1} \dots T_1(P_{i+1})] = T_iT_{i-1}\dots T_1(P_{i+1})$, since this last point is in H_{i+1}. In addition, $T_{n-1}T_{n-2}\dots T_1(P_n) = -P_n$ since each one of the first $n-1$ coordinates of P_n has been multiplied exactly once by -1, and $p_{nn} = 0$.

It therefore follows that the right-hand side of (2) is the length l of an arc in S, joining P_1 to $-P_n$; l is equal, by (2), to the length of the given arc $P_1P_2\dots P_n$ in S, joining P_1 to P_n. Therefore $2l$ is the length of an arc in S, joining P_n to $-P_n$, and it is well known that an arc in S that connects two antipodal points has length $\geqslant \pi$, therefore $l \geqslant \pi/2$, and the proof is complete.

Proof of Theorem 2. As in the previous proof,

$$(1')\qquad \varrho(P_i, P_{i+1}) = \varrho[T_iT_{i-1}\dots T_2(P_i), T_iT_{i-1}\dots T_2(P_{i+1})] \quad \text{for all } 2 \leqslant i \leqslant n,$$

is true and implies

$$(2')\qquad \sum_{i=1}^{n} \varrho(P_i, P_{i+1}) = \sum_{i=1}^{n} \varrho[T_iT_{i-1}\dots T_2(P_i), T_iT_{i-1}\dots T_2(P_{i+1})].$$

Here the first term in the right-hand side of (2′) is $\varrho(P_1, P_2)$, and $T_i[T_{i-1}\dots T_2(P_i)] = T_{i-1}T_{i-2}\dots T_2(P_i)$, for all $2 \leqslant i \leqslant n$; in addition, $T_nT_{n-1}\dots T_2(P_{n+1}) = T_nT_{n-1}\dots T_2(P_1) = -P_1$, since $P_{n+1} = P_1$ and it has $p_{11} = 0$.

Therefore the left-hand side of (2′) is equal, by (2′), to the length of an arc in S joining P_1 to $-P_1$, which is (again) $\geqslant \pi$; this completes the proof.

Remark. The idea of the proofs is to keep the first i parts of the arc while replacing the rest of the arc by the T_i-image of that rest, doing it successively for $i = 1$, $i = 2, \dots, i = n-1$ (with a slight variation for the other proof).

Observe that n applications of Theorem 1 yield a result, weaker than that of Theorem 2; namely with π being replaced by $\frac{n}{2(n-1)}\pi$.

Our proofs can easily be applied to the following:

COROLLARY 1. *If a path in S contains at least one point on every H_j, for all $1 \leqslant j \leqslant n$, then it is of length $\geqslant \pi/2$; if, in addition, it is closed, then it is of length $\geqslant \pi$.*

COROLLARY 2. *If a path in the boundary C of the unit cube in E^d contains at least one point on every H_j, for all $1 \leqslant j \leqslant n$, then it is of length $\geqslant 2$; if, in addition, the path is closed, then it is of length $\geqslant 4$.*

This is a particular case of

COROLLARY 3. *If P is a (centrally symmetric) polytope in E^d, such that $T_j(P) = P$ for all $1 \leqslant j \leqslant n$ and the minimal length of a path in the boundary $Bd(P)$ of P, connecting a point x of $Bd(P)$ to $-x$, is p; then the length of a path in $Bd(P)$ that contains at least one point of every H_j, for all $1 \leqslant j \leqslant n$, is $\geqslant p/2$; if, in addition, the path is closed, then it is of length $\geqslant p$.*

Furthermore, it need not be assumed in Corollary 3 that P is convex, nor polyhedral; the collection of the metric transformations can be replaced by an appropriate finite set of isometric transformations; we omit the rest for obvious reasons.

Using reflections in a similar way, Shaer and Wetzel proved ([3], Lemma 1) that if a is a closed path that meets each and every $(d-1)$-dimensional face of a hyperbox in E^d of diagonal p, than the length of a is at least $2p$.

References

[1] A. Lasota and C. Olech, *An optimal solution of Nicoletti's boundary value problem*, Ann. Polon. Math. 18 (1966), p. 131-139.

[2] Z. Nehari, *Oscillation theorems for systems of linear differential equations*, Trans. Amer. Math. Soc. 139 (1969), p. 339-347.

[3] J. Schaer and J. E. Wetzel, *Boxes for curves of constant length* (to appear).

[4] B. Schwartz, *Curves on the unit sphere and disconjugacy of differential systems*, Technion's preprint series No. MT-71, Haifa, Israel; see also J. Math. Anal. and Appl. 39 (1972), p. 75-86.

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