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On the range of analytic functions related to Carathéodory class

by YÛSAKU KOMATU (Tokyo, Japan)

Franciszek Leja in memoriam

Abstract. A one-parameter additive family of operators acting on the class of functions regular in the unit disk and normalized at the origin is observed. In restricting its domain to a subclass related to Carathéodory class, the author determines the range of values of operated functions in a concentric disk. The case of particular family generated by a special measure is also considered.

1. Introduction. Let $\mathscr{F}$ denote the whole class of analytic functions f regular in the unit disk $E = \{|z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. In a previous paper [1] we have introduced a linear operator $\mathscr{L}$ of the form

$$\mathscr{L}f(z) = \int_I \frac{f(zt)}{t} d\sigma(t)$$

defined on $\mathscr{F}$ where σ is a probability measure supported on the interval $I = [0, 1]$. Since $f \in \mathscr{F}$ implies $\mathscr{L}f \in \mathscr{F}$, the iteration $\mathscr{L}^n$ for any positive integer n arises automatically.

It has been shown that there exists the family $\{\mathscr{L}^\lambda\}$ depending on a continuous parameter λ such that it satisfies the additivity $\mathscr{L}^\lambda \mathscr{L}^\mu = \mathscr{L}^{\lambda+\mu}$ together with $\mathscr{L}^0 = \mathrm{id}$ and that under certain restriction on σ every $\mathscr{L}^\lambda$ admits the unique integral representation

$$\mathscr{L}^\lambda f(z) = \int_I \frac{f(zt)}{t} d\sigma_\lambda(t)$$

with a probability measure σ_λ supported on I.

A particular case generated by $\sigma(t) = t$ is distinguished. Then σ_λ is explicitly determined by

$$\sigma_\lambda(t) = \frac{1}{\Gamma(\lambda)} \int_0^t \left(\log \frac{1}{\tau}\right)^{\lambda-1} d\tau$$

and the operator $\mathscr{L}^\lambda$ reduces to the fractional integration of order λ with respect to $\log z$, i.e.,

$$\mathscr{L}^\lambda f(z) = \frac{1}{\Gamma(\lambda)} \int_\infty^{\log z} f(e^\omega)(\log z - \omega)^{\lambda-1}\, d\omega,$$

the path of integration being taken along the half straight line parallel to the real axis which is contained in the half-plane $\{\operatorname{Re}\omega < 0\}$.

2. Range on general case. Let $\mathscr{P}(\alpha)$ with $\alpha < 1$ denote the Carathéodory class of order α which consists of analytic functions p regular in E and satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > \alpha$ in E. It is readily seen that $f(z)/z \in \mathscr{P}(\alpha)$ implies $f_\lambda(z)/z \in \mathscr{P}(\alpha)$; here and also in the following lines we write $f_\lambda = \mathscr{L}^\lambda f$ for the sake of brevity.

Now, we consider the subclass $\mathscr{F}(\alpha)$ of $\mathscr{F}$ which consists of functions f satisfying $f(z)/z \in \mathscr{P}(\alpha)$. As shown by Strohhäcker [3], the class of convex mappings is a subclass of $\mathscr{F}(\frac{1}{2})$. In relation to this fact, we have derived in [2] some results on the range concerning $\mathscr{F}(\frac{1}{2})$. In the present paper we shall show that these results can be generalized to the class $\mathscr{F}(\alpha)$, though the method used below is similar as before.

We begin with a general theorem on the range of $f_\lambda(z)/z$ for $\{|z| \leqslant r\}$.

THEOREM 1. *Any function $f \in \mathscr{F}(\alpha)$ satisfies*

$$\left|\frac{f_\lambda(z)}{z} - \frac{\varphi_\lambda(r;\alpha)}{r}\right| \leqslant \frac{\psi_\lambda(r;\alpha)}{r} - 1$$

for $|z| \leqslant r < 1$, where φ and ψ are elementary functions in $\mathscr{F}$ defined by

$$\frac{\chi(z;\alpha)}{z} = (1-\alpha)\frac{1+z}{1-z} + \alpha, \qquad \frac{\varphi(z;\alpha)}{z} = \frac{\chi(z^2;\alpha)}{z^2} = (1-\alpha)\frac{1+z^2}{1-z^2} + \alpha$$

and

$$\frac{\psi(z;\alpha)}{z} = 1 + (1-2\alpha)z + \varphi(z;\alpha) = 1 + 2(1-\alpha)\frac{z}{1-z^2}.$$

The extremal functions for the estimation are of the form $f(z) = \bar{\varepsilon}\chi(\varepsilon z;\alpha)$ with $|\varepsilon| = 1$, unless σ coincides with the point measure concentrated at 0. Further, the range of $f_\lambda(z)/z$ for $\{|z| \leqslant r\}$ induced from any function f of this form is just the closed circle expressed by the estimation.

Proof. Since $f \in \mathscr{F}(\alpha)$ implies $(f - \alpha z)/(1-\alpha) \in \mathscr{F}(0)$, we get in view of Herglotz representation on $\mathscr{P}(0)$ the expression

$$\frac{f(z)}{z} = (1-\alpha)\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\tau(\theta) + \alpha = \int_{-\pi}^{\pi} \frac{\chi(e^{-i\theta}z;\alpha)}{e^{-i\theta}z}\, d\tau(\theta),$$

where τ is a probability measure supported on the interval $(-\pi, \pi]$. Now, the range of $\chi(z; 0)/z \in \mathscr{P}(0)$ for $\{|z| \leqslant r\}$ is contained in the closed circle with the segment $[\chi(-r; 0)/(-r), \chi(r; 0)/r]$ as a diameter. Hence that of $\chi(z; \alpha)/z$ is contained in the closed circle with the segment $[\chi(-r; \alpha)/(-r), \chi(r; \alpha)/r]$ as a diameter, of which the center and the radius are given by

$$\tfrac{1}{2}\left(\frac{\chi(r; \alpha)}{r} + \frac{\chi(-r; \alpha)}{-r}\right) = (1-\alpha)\frac{1+r^2}{1-r^2} + \alpha = \frac{\varphi(r; \alpha)}{r}$$

and

$$\tfrac{1}{2}\left(\frac{\chi(r; \alpha)}{r} - \frac{\chi(-r; \alpha)}{-r}\right) = 2(1-\alpha)\frac{r}{1-r^2} = \frac{\psi(r; \alpha)}{r} - 1,$$

respectively. Consequently, we have

$$\left|\frac{\chi(z; \alpha)}{z} - \frac{\varphi(r; \alpha)}{r}\right| \leqslant \frac{\psi(r; \alpha)}{r} - 1$$

for $|z| \leqslant r$. On the other hand, by taking the definition of $\mathscr{L}^\lambda$ into account, we obtain

$$\frac{f_\lambda(z)}{z} - \frac{\varphi_\lambda(r; \alpha)}{r} = \int_I \left(\int_{-\pi}^{\pi} \frac{\chi(e^{-i\theta} zt; \alpha)}{e^{-i\theta} zt} d\tau(\theta) - \frac{\varphi(rt; \alpha)}{rt} \right) d\sigma_\lambda(t)$$

$$= \int_I \left(\int_{-\pi}^{\pi} \left(\frac{\chi(e^{-i\theta} zt; \alpha)}{e^{-i\theta} zt} - \frac{\varphi(rt; \alpha)}{rt} \right) d\tau(\theta) \right) d\sigma_\lambda(t).$$

Thus, by remembering the above inequality, we have

$$\left|\frac{f_\lambda(z)}{z} - \frac{\varphi_\lambda(r; \alpha)}{r}\right| \leqslant \int_I \left(\int_{-\pi}^{\pi} \left(\frac{\psi(rt; \alpha)}{rt} - 1 \right) d\tau(\theta) \right) d\sigma_\lambda(t)$$

$$= \int_I \left(\frac{\psi(rt; \alpha)}{rt} - 1 \right) d\sigma_\lambda(t) = \frac{\psi_\lambda(r; \alpha)}{r} - 1.$$

Concerning the extremal functions it is readily seen that the equality sign at a point on $\{|z| \leqslant r\}$ and necessarily on $\{|z| = r\}$ in the estimation holds if and only if τ is the point measure concentrated at a single point θ, and hence f reduces to $f(z) = \bar{\varepsilon}\chi(\varepsilon z; \alpha)$ with $\varepsilon = e^{-i\theta}$.

3. A distinguished case. We now observe the dinstinguished case generated by $\sigma(t) = t$. Then, by taking into account the familiar formula

$$\int_0^1 t^{\nu-1}\left(\log\frac{1}{t}\right)^{\lambda-1} dt = \frac{\Gamma(\lambda)}{\nu^\lambda} \quad (\nu = 1, 2, \ldots),$$

we obtain the series form of f_λ:

$$f_\lambda(z) = \sum_{\nu=1}^{\infty} \frac{f^{(\nu)}(0)}{\nu!}\frac{z^\nu}{\nu^\lambda}$$

for any $f \in \mathscr{F}$. By making use of this relation, we state a particularization of Theorem 1.

THEOREM 2. *In the distinguished case generated by $\sigma(t) = t$ the estimation given in Theorem 1 for $f \in \mathscr{F}(\alpha)$ may be brought into the series form*

$$\left|\frac{f_\lambda(z)}{z} - \left(1 + 2(1-\alpha)\sum_{n=1}^{\infty} \frac{r^{2n}}{(2n+1)^\lambda}\right)\right| \leqslant 2(1-\alpha)\sum_{n=1}^{\infty} \frac{r^{2n-1}}{(2n)^\lambda}.$$

Proof. In view of the remark mentioned just above, the expansions of $\varphi(r;\alpha)$ and $\psi(r;\alpha)$ in the power series with respect to r yield readily those of $\varphi_\lambda(r;\alpha)$ and $\psi_\lambda(r;\alpha)$, respectively, and hence the desired result.

Finally, we supplement a theorem on the range of values of $f_\lambda(z)/z$ in the whole disk E.

THEOREM 3. *In the distinguished case the range of values of $f_\lambda(z)/z$ with $\lambda > 1$ in E for $f \in \mathscr{F}(\alpha)$ is contained in the circular disk with the segment*

$$\left(2(1-\alpha)\left(1-\frac{1}{2^{\lambda-1}}\right)\zeta(\lambda)-(1-2\alpha),\ 2(1-\alpha)\zeta(\lambda)-(1-2\alpha)\right)$$

as a diameter, i.e., the estimation

$$\left|\frac{f_\lambda(z)}{z} - \left(1 + 2(1-\alpha)\left(\left(1-\frac{1}{2^\lambda}\right)\zeta(\lambda)-1\right)\right)\right| < 2(1-\alpha)\frac{1}{2^\lambda}\zeta(\lambda)$$

holds for $z \in E$, where ζ denotes the Riemann zeta function.

Proof. The range-circle for $f_\lambda(z)/z$ ($|z| \leqslant r$) swells as a point set together with $r \in [0, 1)$. If $\lambda > 1$, the center and the radius of the limit circle as $r \to 1-0$ are given by

$$1 + 2(1-\alpha)\sum_{n=1}^{\infty}\frac{1}{(2n+1)^\lambda} = 1 + 2(1-\alpha)\left(\left(1-\frac{1}{2^\lambda}\right)\zeta(\lambda)-1\right)$$

and

$$2(1-\alpha)\sum_{n=1}^{\infty}\frac{1}{(2n)^\lambda} = 2(1-\alpha)\frac{1}{2^\lambda}\zeta(\lambda),$$

respectively. Hence the result follows.

When $0 < \lambda \leqslant 1$, the center and the radius of the limit circle as $r \to 1-0$ diverge both to positive infinity. However, the left endpoint of the diameter on the real axis of the range-circle remains always finite; in fact, it lies on the left of α. Moreover, it tends to the point

$$\lim_{r\to 1-0}\left(1+2(1-\alpha)\left(\sum_{n=1}^{\infty}\frac{r^{2n}}{(2n+1)^{\lambda}}-\sum_{n=1}^{\infty}\frac{r^{2n-1}}{(2n)^{\lambda}}\right)\right)$$
$$=1+2(1-\alpha)\sum_{\nu=2}^{\infty}\frac{(-1)^{\nu-1}}{\nu^{\lambda}}=2(1-\alpha)\left(1-\frac{1}{2^{\lambda-1}}\right)\zeta(\lambda)-(1-2\alpha),$$

where $\zeta(\lambda)$ is understood to be analytically prolonged; here the Abel's continuity theorem is taken into account. In particular, when $\lambda = 1$, the limit point lies at $2(1-\alpha)\log 2-(1-2\alpha)$. When $\lambda = 0$, while the limit point lies necessarily at α, the derivative of the limit point as a function of λ is equal to $(1-\alpha)\log(2\pi)-(1-2\alpha)$.

In conclusion, it is noted that the way used here may be regarded as a model showing how to deal with similar problems concerning linear functionals of f in a subclass of $\mathscr{F}$.

References

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CHUO UNIVERSITY
TOKYO, JAPAN

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