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On partial stability for delay systems

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Abstract. The paper deals with delay-differential systems of the form (1) and also with perturbed non-linear systems of the form (11). First, a characterization of exponential asymptotic stability with respect to x is given for system (1), in terms of a Liapunov functional. Then, using a comparison technique based on Ważewski's theory of differential inequalities, a theorem on partial stability is proved for the perturbed system (11).

In his famous paper on differential inequalities and their applications [10], T. Ważewski has given a powerful tool for both qualitative and quantitative investigation of differential equations. Many stability results obtained by using differential inequalities in a systematic manner have been included in the monographs [7] and [11].

In our papers [1], [3] we applied the theory of differential inequalities, as built up by T. Ważewski, to the study of partial stability for ordinary differential systems. A recent survey paper by A. S. Oziraner and V. V. Rumyantsev [8] gives an idea of the extent of Liapunov's second method, combined with the theory of differential inequalities (i.e. comparison technique), in investigating partial stability for ordinary differential equations. In [9] the method of Liapunov's vector functions is used in view of obtaining criteria of partial stability.

Using some ideas that already appeared fruitful in [2], [3] and [11], we shall investigate now some problems of partial stability related to differential equations with delay. More precisely, we have in mind the characterization of the asymptotic partial stability of the exponential type for linear systems with delay, in terms of Liapunov's functions, and then to use the differential inequalities in order to find conditions of partial stability for certain perturbed systems.

Let us consider first the linear (homogeneous) system of equations with delay

$$\begin{aligned} \dot{x}(t) &= A(t, x_t) + B(t, y_t), \\ \dot{y}(t) &= C(t, x_t) + D(t, y_t), \end{aligned} \tag{1}$$

where $x \in R^n$, $y \in R^m$, $t \in R_+$ and the subscript t indicates the restriction of the corresponding function to the interval $[t-h, t]$, with $h > 0$ fixed:

$x_t(u) = x(t+u)$, $-h \leqslant u \leqslant 0$. It is further assumed that the standard representation holds for $A(t, \varphi)$ and the other operators that appear in (1):

$$(2) \qquad A(t, \varphi) = \int_{-h}^{0} [d_s a(t, s)]\varphi(s), \qquad t \in R_+,$$

with $a(t, s)$ being an $n \times n$ matrix whose elements are real-valued measurable functions on $R_+ \times [-h, 0]$, such that $\mathrm{Var}_{[-h,0]} a(t, \cdot) \leqslant m(t)$, where $m(t)$ is locally integrable on R_+. Moreover, it is assumed that $a(t, s)$ is continuous from the left on $(-h, 0)$. $B(t, \psi)$ will be represented by means of a matrix $b(t, s)$ of the type $n \times m$, and similarly for $C(t, \varphi)$ and $D(t, \psi)$.

For any $t_0 \geqslant 0$, $\varphi \in C([-h, 0], R^n)$ and $\psi \in C([-h, 0], R^m)$, let us denote by

$$(3) \qquad \begin{aligned} x &= x(t; t_0, \varphi, \psi), \\ y &= y(t; t_0, \varphi, \psi), \end{aligned}$$

the solution of (1), such that

$$(4) \qquad x_{t_0} = \varphi, \quad y_{t_0} = \psi.$$

This solution exists for $t \geqslant t_0$ and $x_t \in C([-h, 0], R^n)$, $y_t \in C([-h, 0], R^m)$ for any such t.

We recall [3] the definition of the asymptotic partial stability (x-stability) of the exponential type. Namely, the trivial solution $x = 0$, $y = 0$ of (1) enjoys the above property if (and only if) there exist two positive numbers M and a, such that

$$(5) \qquad \|x_t(t_0; \varphi, \psi)\| \leqslant M(\|\varphi\| + \|\psi\|)\exp[-a(t-t_0)], \qquad t \geqslant t_0.$$

The following theorem characterizes the concept of asymptotic partial stability of the exponential type, in terms of Liapunov's functionals.

THEOREM 1. *The solution $x = 0$, $y = 0$ of the system* (1) *is asymptotically stable of the exponential type with respect to x (i.e.* (5) *holds), if and only if there exists a continuous Liapunov functional $V(t, \varphi, \psi)$ satisfying the following conditions*:

$$(6) \qquad \|\varphi\| \leqslant V(t, \varphi, \psi) \leqslant M(\|\varphi\| + \|\psi\|),$$

$$(7) \qquad |V(t, \varphi_1, \psi_1) - V(t, \varphi_2, \psi_2)| \leqslant M(\|\varphi_1 - \varphi_2\| + \|\psi_1 - \psi_2\|),$$

$$(8) \qquad V^{\cdot}(t, \varphi, \psi) \leqslant -aV(t, \varphi, \psi).$$

Proof. First, let us note that the derivative $V^{\cdot}(t, \varphi, \psi)$ is meant in the following sense

$$(9) \quad V^{\cdot}(t, \varphi, \psi) = \limsup_{\delta \to 0+} \frac{V\big(t+\delta, x_{t+\delta}(t, \varphi, \psi), y_{t+\delta}(t, \varphi, \psi)\big) - V(t, \varphi, \psi)}{\delta}.$$

The sufficiency of conditions (6)–(8), in order to assure the validity of (5), can be easily proven. In order to prove the necessity, we shall construct a functional $V(t, \varphi, \psi)$ satisfying (6)–(8). Following [11], let us denote

$$V(t, \varphi, \psi) = \sup_{s\geqslant 0}\{\|x_{t+s}(t, \varphi, \psi)\|\exp(as)\}. \tag{10}$$

For $s = 0$, the quantity in the brackets reduces to $\|\varphi\|$. From (5) it follows that the same quantity is dominated by $M(\|\varphi\| + \|\psi\|)$. Therefore condition (6) holds for $V(t, \varphi, \psi)$ defined by (10). Condition (7) follows easily if we take into account the linearity of $x_t(t_0, \varphi, \psi)$ with respect to φ and ψ. The continuity of $V(t, \varphi, \psi)$ can be established as follows. Since

$$\begin{aligned} |V(t+\delta, \varphi, \psi) - V(t, \varphi_0, \psi_0)| &\leqslant |V(t+\delta, \varphi, \psi) - V(t+\delta, \varphi_0, \psi_0)| + \\ &+ \left|V(t+\delta, \varphi_0, \psi_0) - V\big(t+\delta, x_{t+\delta}(t, \varphi_0, \psi_0), y_{t+\delta}(t, \varphi_0, \psi_0)\big)\right| + \\ &+ \left|V\big(t+\delta, x_{t+\delta}(t, \varphi_0, \psi_0), y_{t+\delta}(t, \varphi_0, \psi_0)\big) - V(t, \varphi_0, \psi_0)\right|, \end{aligned}$$

we see that the first two terms are small when $\|\varphi-\varphi_0\|$, $\|\psi-\psi_0\|$ and δ are small (due to the Lipschitz condition and to the definition of x_t, y_t). The last term on the right-hand side tends to zero as $\delta\to 0+$. It is obviously related to the derivative, and the existence of the latter implies the above assertion. Now let us remark that

$$x_{t+\delta+s}\big(t+\delta, x_{t+\delta}(t, \varphi, \psi), y_{t+\delta}(t, \varphi, \psi)\big) = x_{t+\delta+s}(t, \varphi, \psi).$$

We have

$$\begin{aligned} &\limsup_{\delta\to 0+} \frac{V\big(t+\delta, x_{t+\delta}(t, \varphi, \psi), y_{t+\delta}(t, \varphi, \psi)\big) - V(t, \varphi, \psi)}{\delta} \\ &= \limsup_{\delta\to 0+} \delta^{-1}\Big\{\sup_{s\geqslant 0}\Big[\big\|x_{t+\delta+s}\big(t+\delta, x_{t+\delta}(t, \varphi, \psi), y_{t+\delta}(t, \varphi, \psi)\big)\big\| e^{as}\Big] - \\ &\qquad\qquad - \sup_{s\geqslant 0}[\|x_{t+s}(t, \varphi, \psi)\| e^{as}]\Big\} \\ &= \limsup_{\delta\to 0+} \delta^{-1}\{\sup_{s\geqslant \delta}[\|x_{t+s}(t, \varphi, \psi)\| e^{as}]e^{-a\delta} - \sup_{s\geqslant 0}[\|x_{t+s}(t, \varphi, \psi)\| e^{as}]\} \\ &\leqslant \limsup_{\delta\to 0+} \delta^{-1}\{\sup_{s\geqslant 0}[\|x_{t+s}(t, \varphi, \psi)\| e^{as}](e^{-a\delta} - 1)\} \\ &= -aV(t, \varphi, \psi). \end{aligned}$$

Theorem 1 is thereby proven.

Let us now consider the perturbed (non-linear) system

$$\begin{aligned} \dot{x}(t) &= A(t, x_t) + B(t, y_t) + f(t, x_t, y_t), \\ \dot{y}(t) &= C(t, x_t) + D(t, y_t) + g(t, x_t, y_t), \end{aligned} \tag{11}$$

where $f(t, \varphi, \psi)$ and $g(t, \varphi, \psi)$ are continuous mappings from $R_+ \times \times C([-h, 0], R^n) \times C([-h, 0], R^m)$ into R^n and R^m respectively, satisfying a local Lipschitz condition with respect to φ and ψ, and such that

$$|f(t, \varphi, \psi)| + |g(t, \varphi, \psi)| \leqslant \omega(t, \|\varphi\|), \tag{12}$$

where $\omega(t, u)$ is a continuous scalar function for $t \geqslant 0$, $u \geqslant 0$, which satisfies a local Lipschitz condition and is non-decreasing in the second argument. Moreover, we assume that $\omega(t, 0) \equiv 0$.

If we denote by $V'(t, \varphi, \psi)$ the derivative of the function $V(t, \varphi, \psi)$ with respect to the system (11), then we easily get

$$V'(t, \varphi, \psi) \leqslant V^{\cdot}(t, \varphi, \psi) + M(|f| + |g|),$$

and considering (6), (8) and (12) we can write the following differential inequality for V:

$$V'(t, \varphi, \psi) \leqslant -aV(t, \varphi, \psi) + \omega\big(t, V(t, \varphi, \psi)\big). \tag{13}$$

In inequality (13), we can replace φ by x_t and ψ by y_t. Then

$$V'(t, x_t, y_t) = D^+ V(t, x_t, y_t),$$

where D^+ denotes the upper right derivative number. One obtains from (13), for $t \geqslant t_0$ and $V(t_0, \varphi_0, \psi_0) \leqslant u_0$,

$$V\big(t, x_t(t_0, \varphi_0, \psi_0), y_t(t_0, \varphi_0, \psi_0)\big) \leqslant u(t; t_0, u_0), \tag{14}$$

where $u(t; t_0, u_0)$ denotes the solution of the comparison equation

$$\dot{u} = -au + M\omega(t, u), \qquad u(t_0) = u_0. \tag{15}$$

The following theorem on partial stability for (11) can be stated.

THEOREM 2. *Assume that the solution $x = 0$, $y = 0$ of the system* (1) *is asymptotically stable of the exponential type with respect to x. Let f and g be as specified above. Then the zero solution of* (11) *enjoys the same kind of stability with respect to x as that of the solution $u = 0$ of equation* (15).

Proof. The basic tool is inequality (14), with $V(t, \varphi, \psi)$ as constructed above. Indeed, (6) implies

$$\|x_t(t_0, \varphi_0, \psi_0)\| \leqslant u(t, t_0, u_0), \qquad t \geqslant t_0 \tag{16}$$

as soon as $V(t_0, \varphi_0, \psi_0) \leqslant u_0$. But the last inequality takes place, according to (8), if $M(\|\varphi_0\| + \|\psi_0\|) \leqslant u_0$. The assertion of Theorem 2 follows immediately from (16).

In particular, the solution $x = 0$, $y = 0$ of the system (11) is asymptotically stable with respect to x if (12) holds with

$$\omega(t, u) = m(t)u, \tag{17}$$

where $m(t) \geqslant 0$ is a continuous function on R_+, such that

$$(18) \qquad -at+M\int_0^t m(s)\,ds \to -\infty \quad \text{as } t\to+\infty.$$

Indeed, the comparison equation is now $\dot{u} = -[a-Mm(t)]u$, and condition (18) assures asymptotic stability.

Remark. It is obvious from the above considerations that the comparison equation (15) can be used in order to obtain various estimates for the solutions of the perturbed system (11). As shown in [3], for instance, boundedness results can be derived under rather general assumptions on the perturbing terms.

In conclusion, let us call the reader's attention to some problems related to partial stability for delay systems.

First, using the representation of solution of (1) in the form

$$x_t(t_0, \varphi, \psi) = X_1(t, t_0)\varphi + X_2(t, t_0)\psi,$$
$$y_t(t_0, \varphi, \psi) = Y_1(t, t_0)\varphi + Y_2(t, t_0)\psi,$$

where $X_i(t, t_0)$, $Y_i(t, t_0)$, $i = 1, 2$, are some families of operators acting on $C([-h, 0], R^n)$ and $C([-h, 0], R^m)$, find necessary and sufficient conditions of partial stability (i.e. adequate estimates for the norms of these operators). The usual case has been treated in [6] for equations with delay. In [3], the author obtained such estimates in the case of ordinary differential equations.

Secondly, alternate results on partial stability for delay equations could be obtained by using several Liapunov's functions (see [9]) or even families of such functions (see [4]). To the best of our knowledge, only ordinary differential equations have been investigated in this respect.

Thirdly, starting from the remark that in investigating partial stability only the x-components of the unknown functions are under consideration with respect to their behaviour, it would be interesting to search simultaneously for information on both x-components and y-components of the solution. The use of two Liapunov functionals seems to be the right approach.

Finally, several other problems (x-periodicity, small parameter problems), already investigated in the usual case (see e.g. [5]), make sense when we keep under consideration only some of the variables.

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