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On ultra-weak convergence in L^P

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Abstract. Let $\{\varphi_n\}$ be a sequence in L^P on the unit circle such that

$$\operatorname*{Lim}_{n\to\infty}\int_T f(e^{i\theta})\varphi_n(\theta)\,d\theta = l(f) \quad \text{exists for all } f\in H^q,\ 1/p+1/q=1,\ 1\leqslant p<\infty.$$

Then there exists $\varphi\in L^P$ such that

$$l(f) = \int_T f(e^{i\theta})\varphi(\theta)\,d\theta$$

for all $f\in H^q$. The result is known for $p=1$, $q=\infty$, the purpose of this paper is to supply the proofs for the remaining cases.

1. Introduction. Let Δ denote the unit disk and T its boundary. L^P denotes the usual Lebesgue space considered on T and H^q the Hardy space on Δ. If f is in H^q, then $f(e^{i\theta})$, the boundary function of f, is considered as an element of L^P.

Piranian, Shields and Wells [5] proved the following; which was conjectured by Taylor [6]

THEOREM 1. *Let the sequence $\{a_0, a_1, \ldots\}$ of complex numbers have the property that for each function $\sum b_n z^n$ in H^∞ the limit*

$$\operatorname*{Lim}_{r\to 1}\sum a_n b_n r^n$$

exists and is finite. Then there exists a function $\varphi\in L^1(0,2\pi)$ such that

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}\varphi(t)e^{int}\,dt = \hat{\varphi}(n) \qquad (n\geqslant 0).$$

The converse is true. At the end of [5], they conjecture Theorem 2 which if true would imply Theorem 1. Kahane [2] has shown that Theorem 2 is true if H^∞ is replaced by A. A denotes the subspace, of H^∞, of functions having continuous boundary values. Mooney [4] then completed the proof of Theorem 2 utilizing Kahane's result.

THEOREM 2. *Let $\{\varphi_n\}\subset L^1$ such that*

$$\operatorname*{Lim}_{n\to\infty}\int_0^{2\pi} f(e^{i\theta})\varphi_n(\theta)\,d\theta = l(f)$$

for all $f \in H^{\infty}$. *Then there exists* $\varphi \in L^1$ *such that* $l(f) = \int_0^{2\pi} f(e^{i\theta})\varphi(\theta)d\theta$ *for all* $f \in H^{\infty}$.

In this paper we will extend Theorem 2 by replacing L^1 by L^P and H^{∞} by H^q, where $1/p+1/q = 1$, $1 \leqslant p \leqslant \infty$. Although the method of the proof is similar it is necessary to separate the cases $1 < p$, $q < \infty$ and $p = \infty$, $q = 1$ since L^1 is not reflexive.

2. Comments on the proof of Theorem 2 and the more general result. Since $H^{\infty} \subset L^1$, $1 < p < \infty$, the hypotheses of Theorem 2 are strengthened if L^1 is replaced by L^P and H^{∞} by H^q. Therefore, Theorem 2 still asserts the existence of $\varphi \in L^1$ such that

$$l(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi(\theta)d\theta$$

for $f \in H^{\infty}$ if the stronger hypotheses are satisfied. To obtain the stronger conclusion by using Theorem 1 would require two seemingly difficult steps, (1) to show that $\varphi \in L^P$, rather than $\varphi \in L^1$, (2) to show that the representation is valid for all $f \in H^q$, instead of just H^{∞}. Although Mooney [3] did complete the proof of Theorem 2 by extending the validity of the representation from a subspace to all of H^{∞}, this does not seem viable when comparing H^{∞} with H^q. In fact, it is much simpler to proceed directly. However, the attempt to proceed from Theorem 2 makes the general result seem plausible. The case $p = \infty$ and $q = 1$ does not seem to be suggested by Theorem 2.

3. The case $1 < p, q < \infty$.

THEOREM 3. *Let* $\{\varphi_n\} \subset L^P$, $1 < p < \infty$, *such that*

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi_n(\theta)d\theta = l(f)$$

exists for all $f \in H^q$, $1/p+1/q = 1$.

Then there exists $\varphi \in L^P$ *such that*

$$l(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi_n(e^{i\theta})\varphi(\theta)d\theta$$

for all $f \in H^q$.

Proof. Set

$$l_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi_n(\theta)d\theta;$$

then $l_n \in (H^q)^*$ which by the Hahn–Banach Theorem has an extension $\hat{l}_n \in (L^q)^*$. Moreover, by the Uniform Boundedness Principle the l_n's are uniformly bounded and hence the $\hat{l}_n$'s. Since $(L^q)^*$ may be identified with L^P the l_n's may be identified with a bounded subset of L^P. Since L^P is reflexive bounded subsets are weakly compact, there exists $\varphi \in L^P$ and a subsequence $\{\varphi_{n_k}\} \subset L^P$ which converges weakly to φ, i.e.

$$\operatorname*{Lim}_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\varphi_{n_k}(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\varphi(\theta)\,d\theta$$

for all $g \in L^q$. If $f \in H^q$, then

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi_{n_k}(\theta)\,d\theta = \hat{l}_{n_k}(f) = l_{n_k}(f)$$

but $\operatorname*{Lim}_{n\to\infty} l_n(f) = l(f)$ so that

$$l(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi(\theta)\,d\theta.$$

The proof of Theorem 3 is considerably shorter than that of Theorem 2 for several reasons, although it is basically similar. In Kahane's construction it is necessary to restrict l_n to A in order to obtain the integral representation for $l(f)$. Unfortunately the representation is given by a measure so it is then necessary to show that it is absolutely continuous and hence given by an L^1 function. Because of the restriction to A, Mooney's construction is necessary to show that the representation is valid for H^∞.

4. The case $p = \infty$, $q = 1$.

THEOREM 4. *Let $\{\varphi_n\} \subset L^\infty$ such that*

$$\operatorname*{Lim}_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi_n(\theta)\,d\theta = l(f)$$

exists for all $f \in H^1$. Then there exists $\varphi \in L^\infty$ such that

$$l(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi(\theta)\,d\theta$$

exists for all $f \in H^1$.

Proof. Proceeding as in the proof of Theorem 3 set

$$l_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi_n(\theta)\,d\theta.$$

Then $l_n \in (H^1)^*$ and extends to $\hat{l}_n \in (L^1)^*$. Since $(L^1)^*$ may be identified with L^∞, the l_n's may be identified with a bounded subset of L^∞ (use the U.B. Principle again). By Alaoglu's Theorem, the unit ball of L^∞ is weak* compact. Without loss of generality we may assume $\|l_n\| \leqslant 1$ so that there is a $\varphi \in L^\infty$ and a subsequence $\{\hat{\varphi}_{n_k}\}$ which converges weak* to φ, i.e.,

$$\lim_{k\to\infty} \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\hat{\varphi}_{n_k}(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\varphi(\theta)d\theta$$

for all $g \in L^1$, where $\hat{\varphi}_{n_k}$ is identified with $\hat{l}_{n_k}$ which in the extension of l_n. For $f \in H^1$

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\hat{\varphi}_{n_k}(\theta)d\theta = \hat{l}_{n_k}(f) = l_{n_k}(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi_{n_k}(\theta)d\theta$$

and by hypothesis

$$\lim_{n\to\infty} l_n(f) = l(f)$$

so

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi_n(\theta)d\theta = l(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\varphi(\theta)d\theta.$$

Combining Theorems 2, 3, 4 gives the desired complete general result.

4. Results. E. A. Heard [1] has announced a new proof of the weak sequential completeness of L^1 using Kahane's results. A similar approach to the weak sequential completeness of L^p, $1 < p < \infty$, is of no consequence, however, since the reflexive property was used in the proof of Theorem 3.

Unlike Kahane's and Mooney's results, Theorems 3 and 4 are independent of dimension, that is both generalize to Δ^N and T^N without any change in the proofs as follows.

THEOREM 5. *Let Δ^N and T^N denote the N dimensional polydisc in C^N and its distinguished boundary, $N \geqslant 1$. Let $1 < p \leqslant \infty$, $1/p + 1/q = 1$. If $\{\varphi_n\} \subset L^P(T^N)$ such that*

$$\lim_{n\to\infty} l_n(f) = \lim_{n\to\infty}\int_{T^N} f^*\varphi_n = l(f)$$

exists for all $f \in H^q(\Delta^N)$ (f^* is the boundary function of f), then there exists $\varphi \in L^p(T^N)$ such that

$$l(f) = \int_{T^N} f^*\varphi$$

for all $f \in H^q(\Delta^N)$.

As observed by Kahane the hypothesis of Theorem 2 is the existence of

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{n,k}b_k, \text{ where } \varphi_n(\theta) = \sum_{k=-\infty}^{\infty} a_{n,k}e^{-ik\theta}, \text{ for all } \sum_{k=0}^{\infty} b_k e^{ik} \epsilon H^{\infty}(\Delta).$$

The conclusion of the theorem is $\lim_{n\to\infty} a_{n,k} = \int \varphi(\theta)e^{ik\theta}d\theta$ for some $\varphi \epsilon L^1(T)$. Theorem 1 is a special case of this re-statement. In like manner Theorems 3 and 4 can be re-stated to give results analogous to Theorem 1.

An example. In the proofs of Theorems 2, 3, 4, 5 a crucial step is the extraction of a convergent subsequence. This convergent subsequence is obtained by the weak* sequential compactness of the unit ball, rather than just weak* compactness; and separability is a sufficient condition. The following example found in [3], p. 311, shows that in general separability can not be omitted.

By the natural imbedding l^1 is a subspace of $(l^{\infty})^*$ then $\{e_k\} \subset l^1$ is a bounded sequence in $(l^{\infty})^*$ but no subsequence is weakly convergent in $(l^{\infty})^*$.

References

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