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Asymptotic behaviour of solutions of the nonlinear heat equation

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Abstract. The equation arising in biology is studied. Using the method of differential inequalities and Lapunov functions, we describe the behaviour of solutions of (1)–(3) as $t \to \infty$. Usually the solutions tend to one of the stationary solutions 0 or 1. In the end of the work some remarks concerning global in time existence of solutions and a priori estimates are presented.

1. Introduction. The present paper is stimulated by the recent result of Anderson [1] concerning the differential equation

$$u_t = u_{xx} + f(x, u) \tag{1}$$

in the set $D := \{(t, x) \in \mathbf{R}^2;\ t > 0,\ x \in (0, a)\}$, with additional conditions

$$u_x(t, 0) = u_x(t, a) = 0 \quad \text{for } t > 0, \tag{2}$$

$$u(0, x) = \varphi(x) \quad \text{for } x \in (0, a) \tag{3}$$

and nonlinear function f. This equation admits interesting biological interpretation (as in [1]). Suppose, namely, that two competing allospecies X and Y share a breeding range which can be represented as a one-dimensional interval of space $0 \leqslant x \leqslant a$. Suppose that X and Y are presented at location x and time t in proportions u: $(1-u)$, where u is a function of x and t and satisfies $0 \leqslant u(t, x) \leqslant 1$. Then the model of the competitive interaction of species X and Y is given by (1)–(3). The function f represents the dynamics of the local competitive interaction.

The behaviour of solutions of (1)–(3) depends heavily upon the initial function φ, thus it is interesting to determine the regions of stability in dependence on φ and f.

DEFINITIONS. We say that w is a *stationary solution* of problem (1)–(3) if w fulfils the Neumann problem:

$$w_{xx} + f(x, w) = 0 \quad \text{for } x \in (0, a), \tag{4}$$

$$w_x(0) = w_x(a) = 0. \tag{5}$$

We consider the following real Banach spaces: $C^0([0, a])$, $C^{2+\alpha}([0, a])$, $C^{1+\alpha/2, 2+\alpha}(\bar{D})$ (see [6]), $L^2(0, a)$.

We need also the special versions of the Poincaré inequality (see [9]):

$$\underset{c>0}{\exists} \ \underset{v \in C_0^1([0,a])}{\forall} \ \|v\|_{L^2}^2 \leqslant c\,\|v_x\|_{L^2}^2, \tag{6}$$

where C_0^1 denotes the space of all C^1 functions vanishing for $x = 0$ and $x = a$, $L^2 = L^2(0, a)$,

$$\underset{C>0}{\exists} \ \underset{v \in C^1([0,a])}{\forall} \ \|v - \bar{v}\|_{L^2}^2 \leqslant C\,\|v_x\|_{L^2}^2, \tag{7}$$

where $\bar{v} = a^{-1} \int_0^a v(z)\,dz$.

2. Assumptions. Classical solutions of the problem (1)–(3) are investigated. Existence (local in time) of solutions of such problems is shown in [7], [6], see also Section 7 below.

Let f and φ satisfy the conditions:

(A) The function $f\colon [0, a] \times [0, 1] \to R$ belongs to $C^1([0, a] \times [0, 1])$ and, for any fixed $x \in [0, a]$, f has three different roots 0, λ_x, 1 such that $0 < \lambda_x < 1$. The value $f(x, z)$ is strictly negative for $z \in (0, \lambda_x)$, and strictly positive for $z \in (\lambda_x, 1)$.

(B) The function $f\colon [0, 1] \to R$ belongs to $C^{2+\alpha}([0, 1])$, and has three different roots $0 < \lambda < 1$; its value $f(z)$ is strictly negative for $z \in (0, \lambda)$ and strictly positive for $z \in (\lambda, 1)$.

(C) The function $\varphi\colon [0, a] \to [0, 1]$ belongs to $C^{2+\alpha}([0, a])$ and satisfies the compatibility condition

$$\varphi_x(0) = \varphi_x(a) = 0.$$

Such φ is called the *initial condition*.

3. Preliminaries. We need a special version of Theorem 64.3 of [11]:

PROPOSITION 1. *Let the functions u and w satisfy the differential inequalities*

$$\begin{aligned} u_t &\leqslant u_{xx} + f(x, u) \\ w_t &\geqslant w_{xx} + f(x, w) \end{aligned} \quad \text{in } D,$$

together with the conditions

$$u(0, x) = \varphi(x), \quad w(0, x) = \psi(x) \quad \text{for } x \in [0, a],$$

$$u_x(t, 0) = u_x(t, a) = 0 = w_x(t, 0) = w_x(t, a) \quad \text{for } t > 0.$$

Let the function f be continuous in (x, u) and Lipschitz with respect to the second argument:

$$\underset{c>0}{\exists}\ \underset{h,l\in[0,1]}{\forall}\ \underset{x\in[0,a]}{\forall}\ |f(x,h)-f(x,l)| \leqslant c|h-l|.$$

If $\varphi(x) \leqslant \psi(x)$ for $x \in [0, a]$, then the classical solutions satisfy $u(t, x) \leqslant w(t, x)$ in D.

Proposition 1 implies the following corollaries:

COROLLARY 1. *Let the function f satisfy*

$$\underset{\lambda>0}{\exists}\ \underset{x\in[0,a]}{\forall}\ \underset{z\leqslant\lambda}{\forall}\ f(x,z) \leqslant 0,$$

and let, for the initial condition φ, $0 \leqslant \varphi(x) \leqslant m \leqslant \lambda$ holds. Then the solution u of (1)–(3) is bounded from above by m.

COROLLARY 2. *Uniqueness of the solution of problem (1)–(3) under assumption (A) is an easy consequence of Proposition 1.*

COROLLARY 3. *If the solution u corresponding to the initial condition φ tends to zero as t tends to infinity, then any solution corresponding to an initial condition ψ such that $\psi(x) \leqslant \varphi(x)$, $x \in [0, a]$, also tends to zero.*

The next lemma describes the behaviour of the linear heat equation at infinity, the special case of the fundamental theorem of [10].

LEMMA 1. *All the solutions of the heat equation*

$$u_t = u_{xx}, \quad u_x(t, 0) = u_x(t, a) = 0, \quad u(0, x) = \varphi(x)$$

converge to $\bar{\varphi} = a^{-1}\int_0^a \varphi(x)\,dx$ uniformly in $[0, a]$.

Proof. Integrating the equation over $[0, a]$, we verify that the average $\bar{u}(t) = a^{-1}\int_0^a u(t, x)\,dx$ is constant in time and equal to $\bar{\varphi}$. Also the function $U(t, x) := u(t, x) - \bar{\varphi}$ is given by the Fourier series

$$U(t, x) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{n^2\pi^2}{a^2}t\right)$$

with

$$c_n = 2a^{-1}\int_0^a (\varphi(x)-\bar{\varphi})\cos\left(\frac{n\pi}{a}x\right)dx, \quad n = 1, 2, \ldots, \quad c_0 = 0.$$

Hence $(u(t, \cdot)-\bar{\varphi})$ tends to 0 as $t \to \infty$.

4. The method of differential inequalities. We shall consider stability of stationary solutions 0 and 1 of the problem

$$u_t = u_{xx} + f(u) \quad \text{in } D, \tag{8}$$

$$u_x(t, 0) = u_x(t, a) = 0, \quad t > 0, \tag{9}$$

$$u(0, x) = \varphi(x), \quad x \in [0, a]. \tag{10}$$

Since the properties of $f(u)$ in the interval $[0, \lambda]$ are analogous to the properties of $-f(1-u)$ in the interval $[\lambda, 1]$, we can restrict our studies to the first interval and the convergence of the solutions to zero.

THEOREM 1. *Let the functions f and φ satisfy conditions* (B), (C); *additionally, let $f'(0^+) < 0$ and*

$$0 \leqslant \varphi(x) \leqslant \lambda, \quad \bar{\varphi} < \lambda. \tag{11}$$

Then the solution u of (8)–(10) *tends to* 0 *as $t \to \infty$ uniformly in* $[0, a]$.

Proof. The proof consists of two parts. First we shall prove that for t sufficiently large $u(t, x) < \lambda$. Consider the comparison system

$$w_t = w_{xx}, \quad w_x(t, 0) = w_x(t, a) = 0, \quad w(0, x) = \varphi(x).$$

Since $0 \leqslant \varphi(x) \leqslant \lambda$, by Proposition 1 we have $0 \leqslant w(t, x) \leqslant \lambda$; hence from condition (B) $f(w(t, x)) \leqslant 0$ and

$$w_t \geqslant w_{xx} + f(w).$$

Applying Proposition 1 to u and w, we get

$$u(t, x) \leqslant w(t, x) \quad \text{in } D.$$

Since by Lemma 1 w tends to $\bar{\varphi}$, then for $T_0 > 0$

$$\underset{\varepsilon > 0}{\exists} \ \underset{t \geqslant T_0 > 0}{\forall} \ w(t, x) \leqslant \bar{\varphi} + \varepsilon \leqslant \lambda - \varepsilon$$

and, moreover, $u(t, x) \leqslant \lambda - \varepsilon$ for $t \geqslant T_0$.

Next we show that u converges to zero. The new comparison system in the cylinder $D_{T_0} := D \cap \{t > T_0\}$ is considered. Initial condition is given for $t = T_0$:

$$u_t = u_{xx} + f(u) \quad \text{in } D_{T_0}, \tag{12}$$

$$u_x(t, 0) = u_x(t, a) = 0 \quad \text{for } t > T_0, \tag{13}$$

$$u(T_0, x) = \varphi_1(x) \quad \text{for } x \in [0, a], \tag{14}$$

where φ_1 is the value of the solution of (8)–(10) for $t = T_0$. We have $0 \leqslant \varphi_1(x) \leqslant \lambda - \varepsilon$, and by Corollary 1, for $t \geqslant T_0$

$$0 \leqslant u(t, x) \leqslant \max_{[0,a]} \varphi_1(x) \leqslant \lambda - \varepsilon.$$

Condition $f'(0^+) < 0$ and strict negativity of f on the interval $(0, \lambda)$ imply that

$$\underset{\varrho>0}{\exists} \underset{z\in[0,\lambda-\varepsilon]}{\forall} f(z) \leqslant -\varrho z,$$

and, in particular, since $0 \leqslant u(t, x) \leqslant \lambda - \varepsilon$; $t \geqslant T_0$, we have

$$f(u(t, x)) \leqslant -\varrho u(t, x) \quad \text{for } t \geqslant T_0. \tag{15}$$

Consider together with (12)–(14) the comparison problem

$$v_t = v_{xx} - \varrho v \quad \text{in } D_{T_0} \tag{16}$$

with conditions (13)–(14) for v. Proposition 1 gives $0 \leqslant u(t, x) \leqslant v(t, x)$, and since the transformed function $\eta(t, x) := v(t, x)\exp(\varrho(t-T_0))$ satisfies the linear heat equation (and hence is bounded), we verify that

$$0 \leqslant v(t, x) = \eta(t, x)\exp(-\varrho(t-T_0)) \to 0, \qquad t \to \infty$$

uniformly in $[0, a]$.

Remark 1. Theorem 1 remains true under assumptions (A), (C) with f depending on x if the value ϱ in (15) will be chosen common for $x \in [0, a]$.

5. The combined method. Theorem 1 gives information about the behaviour of solutions only when $0 \leqslant \varphi(x) \leqslant \lambda$ or $\lambda \leqslant \varphi(x) \leqslant 1$. The initial function φ could not intersect the value λ. We want to study now possible limit states of solutions. This will be done with the use of the concept of Lapunov functions. As was shown by Chafee [2], all possible limit states of trajectories are contained in the set of stationary solutions of (8)–(10). We want to study some special cases of this convergence.

As a consequence of (B), (C) the solution u of (8)–(10) is smooth; its derivatives $D_x^4 u$, $D_t D_x^2 u$ are Hölder continuous (exponent α) in any cylinder $[\gamma, T] \times [0, a]$ with $\gamma > 0$. Moreover, we have (see Section 7) global in time estimates of the derivatives of u in $C^0([0, a])$:

$$\|u_x(t, \cdot)\|_{C^0} \leqslant \text{const}, \qquad \|u_{xx}(t, \cdot)\|_{C^0} \leqslant \text{const}, \qquad \|u_t(t, \cdot)\|_{C^0} \leqslant \text{const}. \tag{17}$$

Hence, as a consequence of the Ascoli–Arzela theorem, the trajectorie $\{u(t, \cdot)\}_{t\geqslant 0}$ is compact in $C^1([0, a])$ phase space; for any sequence $\{t_n\}_{n\in N}$, $t_n \to \infty$, we can extract a subsequence such that for some $v \in C^1([0, a])$

$$u(t_{n_k}, \cdot) \to v \quad \text{in } C^1([0, a]).$$

We can study now stability of the constant stationary solutions $0, \lambda, 1$.

THEOREM 2. *Let conditions* (B), (C) *hold and let u be a solution of* (8)–(10) *such that*

$$\underset{t\geqslant 0}{\forall} \int_0^a f(u(t, x))u(t, x)\,dx \leqslant 0. \tag{18}$$

Then any sequence $\{u(t_n, \cdot)\}$, $t_n \to \infty$, *convergent in* $C^1([0, a])$ *tends to one of the limits* 0, λ *or* 1.

Proof. Multiplying (8) by u and integrating over $[0, a]$, we get ($L^2 = L^2(0, a)$)

$$\frac{d}{dt}\|u(t, \cdot)\|_{L^2}^2 = -2\int_0^a (u_x)^2\,dx + 2\int_0^a f(u)\,u\,dx. \tag{19}$$

Now the quantity $L\Phi := \|\Phi\|_{L^2}^2$ will play a role of the Lapunov function (see [8]) for our problem. More precisely, L is a Lapunov function on the closed subset of $C^1([0, a])$

$$C := \{\Phi \in C^1([0, a]):\ \int_0^a f(\Phi(x))\,\Phi(x)\,dx \leqslant 0,\ \Phi_x(0) = \Phi_x(a) = 0\}.$$

The derivative of L along the solution u satisfying (18) is non-positive; hence as a result of the general theory [8] the set of possible limit states consists of such functions v belonging to C, for which the right-hand side of (19) vanishes. In particular,

$$\int_0^a (v_x)^2\,dx = 0$$

which is possible only for $v(x) \equiv \text{const} = c$. Now, since also the second component must vanish, we have

$$\int_0^a f(c)\,c\,dx = acf(c) = 0,$$

which is possible only for c equal to 0, λ or 1.

Next theorem shows realizations of (generally implicit) assumption (18) of Theorem 2.

THEOREM 3. *Let f be as in Theorem* 1. *Then for every positive $\varepsilon \leqslant \lambda$ there exists a $\delta(\varepsilon) > 0$, such that if*

$$a^{-1}\int_0^a \varphi(x)\,dx = \bar{\varphi} < \lambda - \varepsilon \quad \textit{and} \quad 0 \leqslant \varphi(x) < \lambda + \delta(\varepsilon),$$

then corresponding to the initial condition φ solution u of the problem (8)–(10) *tends uniformly to zero.*

Proof. If $0 \leqslant \varphi(x) \leqslant \lambda$, then by Corollary 1 for all $t \geqslant 0$ we conclude that $0 \leqslant u(t, x) \leqslant \lambda$, and hence, as a consequence of Theorem 1, u tends to zero.

Consider the general case. Let the initial condition φ satisfy our assumptions (with $\delta(\varepsilon)$ which will be determined later) and let $\psi(x) := \min\{\varphi(x); \lambda\}$. Writing $\gamma := \max_{[0,a]}(\varphi(x)-\psi(x))$, we have

$$\varphi(x) \leqslant \lambda+\gamma, \qquad 0 \leqslant \varphi(x)-\psi(x) \leqslant \gamma.$$

However, the function ψ is only continuous and satisfies (which is easy to check) $\psi'(0^+) = \psi'(a^-) = 0$; it will be taken as the initial function, moreover, Proposition 1 works also for such initial functions. Consider the comparison system for the problem (8)–(10):

$$v_t = v_{xx}, \qquad v_x(t, 0) = v_x(t, a) = 0, \qquad v(0, x) = \psi(x).$$

Proposition 1 and Lemma 1 used for v and the solution w of the problem (8)–(9) with the initial function ψ give:

$$\underset{T_0>0}{\exists}\ \underset{t \geqslant T_0}{\forall}\ w(t, x) \leqslant v(t, x) < \lambda-\tfrac{1}{2}\varepsilon.$$

Moreover, T_0 depends on ε only, because it will be find from the estimates

$$\begin{aligned} v(T_0, x) &= a^{-1}\int_0^a \psi(x)\,dx+\\ &\quad+\sum_{n=1}^{\infty} 2a^{-1}\int_0^a \psi(y)\cos\left(\frac{n\pi}{a}y\right)dy\cos\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{n^2\pi^2}{a^2}T_0\right)\\ &\leqslant \lambda-\varepsilon+2a^{-1}\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2}{a^2}T_0\right)\int_0^a |\psi(y)|\,dy < \lambda-\tfrac{1}{2}\varepsilon. \end{aligned}$$

If the difference $|u(T_0, x)-w(T_0, x)|$ is less than $\frac{1}{2}\varepsilon$, then $u(T_0, x) < \lambda$ (because $w(T_0, x) < \lambda-\frac{1}{2}\varepsilon$), and the preliminary case in our proof shows that u converges to zero. Hence it remains to estimate the difference $|u(T_0, x)-w(T_0, x)|$ for $x \in [0, a]$.

The function $z := u-w$ is non-negative for $t \geqslant 0$ (because u, w satisfy (8)–(9) and $\varphi(x) \geqslant \psi(x)$), and satisfies

$$z_t = z_{xx}+f(u)-f(w) \leqslant z_{xx}+c\,|u-w| = z_{xx}+cz,$$

where $c := \max_{[0,1]}|f'(s)|$. Now, z may be estimated as a solution of the linear problem by

$$\begin{aligned} &|z(t, x)|\\ &\quad\leqslant \exp(ct)\left\{a^{-1}\int_0^a(\varphi(x)-\psi(x))\,dx+\sum_{n=1}^{\infty} c_n\cos\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{n^2\pi^2}{a^2}t\right)\right\}, \end{aligned}$$

with

$$c_n = 2a^{-1}\int_0^a (\varphi(x)-\psi(x))\cos\left(\frac{n\pi}{a}x\right)dx,$$

or, farther with the use of our assumptions by

$$|z(t,x)| \leqslant \exp(ct)\left\{\gamma+2\gamma\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2}{a^2}t\right)\right\}.$$

Since (by definitions) $\gamma \leqslant \delta(\varepsilon)$, we also have

$$|z(T_0,x)| \leqslant \exp(cT_0)\,\delta(\varepsilon)\left\{1+2\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2}{a^2}T_0\right)\right\} \equiv \mathrm{const}\,\delta(\varepsilon)$$

and the value of $\delta(\varepsilon)$ will now be choosen equal to $\varepsilon/(2\,\mathrm{const})$. The proof is finished.

Consider next the case where the nonlinear term f satisfies

$$\max_{[0,1]} f'(s) < \frac{1}{2a^2}. \tag{20}$$

THEOREM 4. *Let the nonlinear term f satisfy* (20), (B) *and the conditions* $f'(0^+) < 0$, $f'(1^-) < 0$. *Then the solution u of* (8)–(10) *tends to* 0, λ *or* 1 *as t tends to infinity.*

Proof. Differentiating (8) with respect to x and using the symbols $v := u_x$, $g(u) := f'(u)$, we get:

$$v_t = v_{xx}+g(u)v, \quad v(t,0) = v(t,a) = 0, \quad v(0,x) = \varphi'(x).$$

Applying the transformation $V(t,x) := v(t,x)\exp(-kt)$ with $k := \max_{[0,1]} g(s)$, we get for the function V:

$$V_t = V_{xx}+[g(u)-k]\,V$$

with analogous conditions as for v. Now

$$h^u(t,x) := g(u)-k \leqslant 0,$$

so, applying Theorem 1 of [5] to any u separately, we get for V the following estimate independent of u:

$$|V(t,x)| \leqslant 2\max_{[0,a]}|\varphi'(x)|\exp(-t/2a^2).$$

Hence, for the function v

$$|u_x(t,x)| = |v(t,x)| = |V(t,x)\exp(kt)| \leqslant 2\max_{[0,a]}|\varphi'(x)|\exp\left(kt-\frac{1}{2a^2}t\right), \tag{21}$$

which together with the estimate (20) ensures the convergence of u_x to zero. Therefore, for sufficiently large $T > 0$ one of the following possibilities holds:

$u(T, x) > \lambda$ (then as a consequence of Theorem 1 $u(t, x) \to 1$ as $t \to \infty$),

$$u(T, x) < \lambda \quad (\text{then } u(t, x) \to 0 \text{ as } t \to \infty),$$

for all $t > 0$ $u(t, x)$ intersects λ (then $u(t, x) \to \lambda$ as $t \to \infty$).

6. Finally, consider the set of the solutions which do not converge to 0 or 1.

LEMMA 2. *For symmetric f, weakly nonlinear;*

$$f(z) = -f(1-z), \quad f'(z) \leqslant \beta < C^{-1} \quad \textit{for } z \in [0, 1], \tag{22}$$

satisfying (B) *and the initial condition satisfying*

$$0 \leqslant \varphi(x) = 1 - \varphi(a-x) \leqslant 1, \quad x \in [0, a]$$

the solution u of (8)–(10) *corresponding to φ tends uniformly to $\lambda = \frac{1}{2}$.*

Proof. It is easy to see that $(1 - u(t, a-x))$ solves the same equation as $u(t, x)$, and moreover, with the same initial condition; hence by uniqueness

$$0 \leqslant u(t, x) = 1 - u(t, a-x) \leqslant 1, \quad t \geqslant 0. \tag{23}$$

Consider now the equation for u_x

$$u_{tx} = u_{xxx} + f'(u) u_x.$$

Multiplying the above by u_x and integrating over $[0, a]$, we get

$$\frac{d}{dt} \int_0^a u_x^2 \, dx = -2 \int_0^a u_{xx}^2 \, dx + 2 \int_0^a f'(u) u_x^2 \, dx.$$

Now with the use of (22) and as a consequence of the Poincaré inequality (6) we have

$$\frac{d}{dt} \int_0^a u_x^2 \, dx \leqslant 2(-C^{-1} + B) \int_0^a u_x^2 \, dx, \tag{24}$$

and hence u_x tends to zero in $L^2(0, a)$ as t tends to infinity. Using the Sobolev inequality [9]

$$\underset{B>0}{\exists} \; \underset{w \in C^1([0,a])}{\forall} \; \|w\|^2_{C^0([0,a])} \leqslant B(\|w\|^2_{L^2} + \|w_x\|^2_{L^2})$$

for the function $w = v - \bar{v}$ ($\bar{v}$ denotes the average of v) and (7), we obtain the estimate

$$\underset{k>0}{\exists} \; \underset{v \in C^1([0,a])}{\forall} \; \|v - \bar{v}\|^2_{C^0([0,a])} \leqslant k \|v_x\|^2_{L^2},$$

which together with (24) gives

$$\|u(t,\cdot)-\bar{u}(t)\|^2_{C^0([0,a])} \leqslant k\|u_x(t,\cdot)\|^2_{L^2} \to 0, \qquad t \to \infty.$$

Since, (23), $\bar{u}(t) = \frac{1}{2}$, this gives the uniform convergence of u to $\frac{1}{2}$.

In general case, using Ważewski's method, we may show that for arbitrary fixed function f satisfying $f'(0^+) < 0$, $f'(1^-) < 0$ there exists continuum initial conditions φ such that the solution of (8)–(10) does not converge to 0 or 1.

DEFINITION. By a *non-decreasing homotopy family* (parameter μ) of functions we mean every set of functions γ: $[0, 1]\times[0, a] \to [0, 1]$ of the arguments (μ, x) which satisfy:

if $\mu_1 \geqslant \mu_2$, then

$$\gamma(\mu_1, x) \geqslant \gamma(\mu_2, x) \quad \text{for } x\in[0, a],$$

$$\gamma(0, x) = 0, \quad \gamma(1, x) = 1, \quad x\in[0, a],$$

$$\gamma_x(\mu, 0) = \gamma_x(\mu, 1) = 0, \quad \mu\in[0, 1],$$

and the function $\mu \to \gamma(\mu, \cdot)$ is continuous in $C^{2+\alpha}([0, a])$ for $\mu\in[0, 1]$.

Let us consider an arbitrary fixed family. For small μ the solution of (8)–(9) with initial condition $\gamma(\mu, x)$ tends to 0, for μ near 1 the solution tends to 1. By Corollary 3 the set of μ for which the solutions tend to 0 coincides with an interval $[0, a_1)$ or $[0, a_1]$; analogously, the set of μ for which the solutions tend to 1 is an interval $(a_2, 1]$ or $[a_2, 1]$, where $a_1 \leqslant a_2$. But both these intervals must be open from one side. If the first of them is closed, then arguing as in the second part of the proof of Theorem 3 (the trajectorie $\{u(t,\cdot)\}_{t\geqslant 0}$, $u(0, x) = \gamma(a_1, x)$ is compact in $C^1([0, a])$ and $u(t, x)$ tends to 0 pointwise as t tends to infinity; hence for a sequence $\{t_n\}_{n\in N}$, $t_n \to \infty$, $u(t_n, \cdot)$ tends to 0 uniformly; the rest is a consequence of the continuous dependence of solutions on the initial condition as in Theorem 3) we see that for some $\mu > a_1$ the corresponding solution tends to zero. This contradicts the maximality of a_1. The solutions corresponding to the parameters μ from the remaining set $[a_1, a_2]$ must tend neither to 0 nor to 1.

7. Some remarks concerning global in time existence of the solutions. As a consequence of our assumptions concerning f, when the values of φ belong to [0, 1], the same holds true for u. As a standard consequence of Theorem 7.4, p. 560 of [7], in any bounded in time cylinder there exists a $C^{1+\alpha/2, 2+\alpha}$ solution of (1)–(3). Additional smoothness assumed for f in (B) ensures existence of the derivatives $D_x^4 u$, $D_t D_x^2 u$ for the solution of (8)–(10). There are several ways to show global in time a priori estimates for the derivatives u_x, u_{xx} and u_t. This is done, for example, in [4], Appendix B. Another approach is given in [3], Theorems 11 and 12, and in [2].

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