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Boundary value problems for generalized analytic functions of several complex variables

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Dedicated to the memory of Stefan Bergman

Abstract. Complex methods permit to reduce boundary value problems for non-linear partial differential equations in the plane to the analogous problems for holomorphic functions. On the other hand, it is possible (in the case of several complex variables too) to construct a one-to-one mapping between solutions of the regarded system of differential equations and holomorphic functions. This connection is given by an operator equations depending on a holomorphic function. Boundary value problems for the regarded differential equation one can consequently reduce to boundary value problems for the corresponding holomorphic functions. Therefore for the regarded complex partial differential equations there are solvable all the boundary value problems which are solvable for holomorphic functions. Such a problem it the following one: construct a holomorphic function possessing an arbitrarily prescribed real part on an one-dimensional part of the distinguished surface. In the paper it is considered the same boundary value problem for solutions of certain partial differential equations in several complex variables.

The basic idea in applications of complex methods in the theory of partial differential equations is the transformation of holomorphic functions into solutions of a given partial differential equation. Bergman's integral operators realize this programm. Also I. N. Vekua's *New methods for solving differential equations of elliptic type* [19] and the *Generalized analytic functions* in the sense of I. N. Vekua [20] or in the sense of L. Bers [2], respectively, have the same aim. As it is shown in [14] (a monography is [15]) it is also possible to construct an operator in the case of differential equations for functions of several complex variables, which transforms holomorphic functions of several complex variables into solutions of the given system of partial differential equations. The systems which have been examined in [14], [15], are of type ($i = 1, \ldots, m, j = 0, 1, \ldots, n$)

$$\partial w_i/\partial z_j^* = f_{ij}(z_0, z_1, \ldots, z_n, w_1, \ldots, w_m), \tag{1}$$

where z_j^* denotes the complex conjugate to z_j.

In [14], [15] it is assumed that system (1) is completely integrable and, moreover, that the right-hand sides f_{ij} depend holomorphically on the variables w_σ (the case where the f_{ij} are not holomorphic functions of $w_1, \ldots, w_m$ is discussed by L. Diomeda [4], [5]).

Especially the approach to solution of non-linear systems with the help of complex methods is in accordance with the present tendency of complex analysis (see the note [12] of J. Naas and the author). And thus by a generalization of the method of L. Bers and L. Nirenberg [3] for the quasi-linear case, it was possible to reduce boundary value problems for non-linear systems of partial differential equations to analogous problems for holomorphic functions (see [16], [17] and the monography [18]).

Also in this paper this method is applied to systems of type (1). Naturally, the method is applicable only in the cases in which the corresponding boundary value problems for holomorphic functions are solvable. The following boundary problem considered in this paper is just of that type:

In a polycylindric domain find a solution of (1) whose real part $\operatorname{Re} w = (\operatorname{Re} w_1, \ldots, \operatorname{Re} w_m)$ has given values on a one-dimensional part of the boundary and the imaginary part has a given value at a prescribed point.

1. The formulation of the theorem. Let G_j, $j = 0, 1, \ldots, n$, be a bounded domain in the z_j-plane. It is assumed that the boundary curves ∂G_j possess Hölder-continuous differentiable representations. We will consider the polycylindric domain $G = G_0 \times G_1 \times \ldots \times G_n$ in C^{n+1}.

Let $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_n)$ be a fixed point of G. By D_0 we denote the set of all points $(z_0, \tilde{z}_1, \ldots, \tilde{z}_n) \in G$, where $z_0 \in G_0$. The boundary γ_0 of D_0 is a curve lying on the boundary ∂G.

Let $\mathscr{C}$ be the space of continuous functions, normed by the sup-norm. Analogously, $\mathscr{C}_\alpha$ denotes the space of Hölder-continuous functions with the norm

$$\|w\|_{\mathscr{C}_\alpha} = \max\left(\sup |w|, \sup_{z' \neq z''} \frac{|w(z') - w(z'')|}{|z' - z''|^\alpha}\right).$$

As regards the right-hand sides f_{ij} of system (1), it is assumed that the following conditions are fulfilled:

(a) The functions f_{ij} are defined and continuous for all $z_j \in \bar{G}_j$ and $|w_i| \leqslant R$,

(b) The functions f_{ij} have continuous derivatives relative to $w_1, \ldots, w_m$ and relative to the different $z^*_{j_1}, \ldots, z^*_{j_\lambda}$, whereby all $j_1, \ldots, j_\lambda$ are different from j.

(c) Let g be one of the functions f_{ij} or of the derivatives specified in (b). Then it is assumed that for all $z_j \in \bar{G}_j$ and $|w_i| \leqslant R$ the following inequalities are fulfilled:

$$|g| \leqslant K_R, \tag{2}$$

$$|g(z, w) - g(z, \tilde{w})| \leqslant L_R \sum_{i=1}^{m} |w_i - \tilde{w}_i|. \tag{3}$$

(d) System (1) is completely integrable and the right-hand sides f_{ij} depend holomorphically on w_σ.

(e) All functions g specified in (b), regarded as functions of z_0, fulfil the Hölder condition with the exponent $\alpha, 0 < \alpha < 1$.

(f) The Hölder–Lipschitz-condition is fulfilled:

$$\|g(\cdot, w)-g(\cdot, \tilde{w})\|_{\mathscr{C}_\alpha(\bar{D}_0)} \leqslant l_R \|w-\tilde{w}\|_{\mathscr{C}_\alpha(\bar{D}_0)}. \tag{4}$$

Remark 1. Assumption (d) is fulfilled, iff the right-hand sides f_{ij} fulfil the relations (see [14], [15]):

$$\frac{\partial f_{ij}}{\partial z_k^*} + \sum_{\sigma=1}^{m} \frac{\partial f_{ij}}{\partial w_\sigma} f_{\sigma k} = \frac{\partial f_{ik}}{\partial z_j^*} + \sum_{\sigma=1}^{m} \frac{\partial f_{ik}}{\partial w_\sigma} f_{\sigma j}.$$

Remark 2. For $w \in \mathscr{C}(\bar{G}) \cap \mathscr{C}_\alpha(\bar{D}_0)$ assumption (e) together with (3) implies that $g(\cdot, w)$ belongs to $\mathscr{C}_\alpha(\bar{D}_0)$, because then

$$\begin{aligned} &\left|g_j(z_0', \ldots, w(z_0', \ldots)) - g_j(z_0'', \ldots, w(z_0'', \ldots))\right| \\ &\qquad \leqslant \left|g_j(z_0', \ldots, w(z_0', \ldots)) - g_j(z_0'', \ldots, w(z_0', \ldots))\right| + \\ &\qquad\qquad + \left|g_j(z_0'', \ldots, w(z_0', \ldots)) - g_j(z_0'', \ldots, w(z_0'', \ldots))\right|. \end{aligned}$$

Let g be a real-valued vector $g = (g_1, \ldots, g_m)$, which belongs to $\mathscr{C}_\alpha(\partial D_0)$. Moreover, let $c = (c_1, \ldots, c_m)$ denote a given constant vector (c_j are real numbers). Then for functions of class $\mathscr{C}(\bar{G})$ we will consider the following boundary condition:

$$\operatorname{Re} w = g \quad \text{on } \gamma_0, \tag{5}$$

$$\operatorname{Im} w[\tilde{z}] = c. \tag{6}$$

Let ψ be a holomorphic function in D_0 which satisfies conditions (5), (6). Defining

$$\psi(z_0, z_1, \ldots, z_n) = \psi(z_0)$$

one gets a holomorphic function, which belongs to $\mathscr{C}(\bar{G})$, and for which conditions (5), (6) are fulfilled. It is assumed that this function $\psi = (\psi_1, \ldots, \psi_m)$ satisfies the inequality

$$|\psi_j| < R.$$

Then the following is true:

THEOREM. *If the polycylindric domain G has a sufficiently small diameter, then there exists a solution w of system* (1), *which satisfies the boundary conditions* (5), (6).

Remark. The solution of the boundary value problem (5), (6) is not uniquely determined.

2. The construction of an operator, corresponding to the boundary value problem. Let T_{G_j}, $j = 0, 1, \ldots, n$, denote the operator defined by

$$T_{G_j} h(z_0, z_1, \ldots, z_n) = -\frac{1}{\pi} \iint_{G_j} \frac{h(z_0, \ldots, \zeta_j, \ldots, z_n)}{\zeta_j - z_j} \, d\xi_j \, d\eta_j,$$

where $\zeta_j = \xi_j + i\eta_j$. With the help of this operator one can explicitly express a solution of the non-homogeneous Cauchy–Riemann-equation

$$\frac{\partial w}{\partial z_j^*} = h_j, \quad j = 0, 1, \ldots, n, \tag{7}$$

in the following way (see [13], [14], [15]):

$$w = \sum_{\lambda=0}^{n} (-1)^\lambda \sum_{j_0, \ldots, j_\lambda}^{*} T_{j_0} \ldots T_{j_\lambda} \frac{\partial h_{j_0}^\lambda}{\partial z_{j_1}^* \ldots \partial z_{j_\lambda}^*} \tag{8}$$

(as in [14], [15] the symbol $\sum^*$ denotes summation over all distinct $(\lambda+1)$-tuples of numbers $j_0, \ldots, j_\lambda$ between 0 und n). Of course, in order that representation (8) of a solution of system (7) might be written, must be assumed that the integrability conditions are fulfilled:

$$\frac{\partial h_i}{\partial z_j^*} = \frac{\partial h_j}{\partial z_i^*}.$$

Now we assume that w is a given solution of system (1). Then the functions

$$\Phi_i = w_i - \sum_{\lambda=0}^{m} (-1)^\lambda \sum_{j_0, \ldots, j_\lambda}^{*} T_{j_0} \ldots T_{j_\lambda} \frac{\partial}{\partial z_{j_\lambda}^*} \ldots \frac{\partial}{\partial z_{j_1}^*} f_{ij_0} \tag{9}$$

are holomorphic. Here

$$\frac{\partial}{\partial z_k^*} f_{ij} \doteq \frac{\partial f_{ij}}{\partial z_k^*} + \sum_{\sigma=1}^{m} \frac{\partial f_{ij}}{\partial w_\sigma} \frac{\partial w_\sigma}{\partial z_k^*} \tag{10}$$

and so on, where the w_σ are replaced by the given functions $w_\sigma = w_\sigma(z)$. Relations analogous to (10) can be written also for the higher derivatives in (9). Since the function w has been assumed to be a solution of the given system (1), the derivatives $\partial w_\sigma / \partial z_k^*$ in (10) can be replaced by the $f_{\sigma k}$. Therefore, in solutions of (1), the right-hand side of (10) can be written as a polynomial in the $f_{\sigma\tau}$. Analogously it is seen that, for solutions of (1), the term

$$\frac{\partial}{\partial z_{j_\lambda}^*} \ldots \frac{\partial}{\partial z_{j_1}^*} f_{ij_0}$$

is a polynomial $A_{j_0 \ldots j_\lambda}^i$ in the $f_{\sigma\tau}$.

Just like the functions $f_{\sigma\tau}$, these polynomials $A_{j_0 \ldots j_\lambda}^i$ depend on z and w. Equation (9) can be rewritten as

(11) $$w = \Phi + \sum_{\lambda=0}^{n} (-1)^{\lambda} \sum_{j_0 \cdots j_\lambda}^{*} T_{j_0} \cdots T_{j_\lambda} A_{j_0 \cdots j_\lambda},$$

where $A_{j_0 \cdots j_\lambda} = (A^1_{j_0 \cdots j_\lambda}, \ldots, A^m_{j_0 \cdots j_\lambda})$.

For each solution $w = (w_1, \ldots, w_m)$ of the given system (1) there exists a holomorphic function $\Phi = (\Phi_1, \ldots, \Phi_m)$ such that w fulfils system (11) of integral equations.

In order to construct a solution of the given system (1) satisfying the prescribed boundary conditions, the function Φ on the right-hand side of (11) will be chosen as

$$\Phi = \Psi + \Phi_{(w)},$$

where the function Ψ is the holomorphic solution of the boundary value problem (5), (6) and $\Phi_{(w)}$ is the holomorphic solution of the boundary value problem

(12) $$\operatorname{Re} \Phi_{(w)} = -\operatorname{Re} \sum_{\lambda=0}^{n} (-1)^{\lambda} \sum_{j_0 \cdots j_\lambda}^{*} T_{j_0} \cdots T_{j_\lambda} A_{j_0 \cdots j_\lambda} \quad \text{on } \gamma_0,$$

(13) $$\operatorname{Im} \Phi_{(w)}[\tilde{z}] = -\operatorname{Im} \sum_{\lambda=0}^{n} (-1)^{\lambda} \sum_{j_0 \cdots j_\lambda}^{*} T_{j_0} \cdots T_{j_\lambda} A_{j_0 \cdots j_\lambda}[\tilde{z}].$$

If it is required that $\Psi, \Phi_{(w)}$ depend only on z_0, then the functions $\Psi, \Phi_{(w)}$ are uniquely determined.

By this choice of the function Φ the right-hand side of (11) defines an operator $\tilde{T}$, which maps a given w into $W = \tilde{T}w$, where

$$W = \Psi + \Phi_{(w)} + \sum_{\lambda=0}^{n} (-1)^{\lambda} \sum_{j_0 \cdots j_\lambda}^{*} T_{j_0} \cdots T_{j_\lambda} A_{j_0 \cdots j_\lambda}.$$

From the definition of this operator it follows immediately that a fixed point of $\tilde{T}$ realizes the given boundary conditions (5), (6). In order to prove the theorem, it will be shown that the operator $\tilde{T}$ possesses a fixed point and, moreover, that this fixed point is solution of the given system (1).

Remark 1. It is well known that a holomorphic function in a polycylindric domain is uniquely determined by its values on the surface of determination. But it is not possible to construct a holomorphic function with arbitrarily prescribed values of the real part on the surface of determination (example: consider the polycylindric domain defined by $|z_1| < 1$, $|z_2| < 1$. Then it is easy to see that $u(z_1, z_2) = x_1 x_2 + y_1 y_2$, if $f = u + iv$ and u has on the surface of determination the values $\cos(\vartheta_1 - \vartheta_2)$, $\vartheta_j = \arg z_j$. On the other hand, this function does not possess a function v harmonic conjugate to u, because in this case one would have $\partial v/\partial x_1 = -\partial u/\partial y_1 = -y_2$, $\partial v/\partial y_2 = \partial u/\partial x_2 = y_1$. This is impossible because $\partial^2 v/\partial x_1 \partial y_2 \neq \partial^2 v/\partial y_2 \partial x_1$).

Remark 2. The sum on the right-hand side of equation (11) is a modification of a corresponding sum in [14], [15]. This modification is due to H. Meden [11]. In contrast to [14], [15], the representation (11), obtained by H. Meden, does not contain derivatives of the function w.

Remark 3. Extensions of given generalized analytic functions, which are given in a neighbourhood of the boundary, to the whole domain, are discussed by Le-hung-Son [9], [10].

3. The adequate function space. Let $\mathscr{R}$ be the function-space containing all $w = (w_1, \ldots, w_m)$ with the following properties:

(a) $w \in \mathscr{C}(\bar{G})$,

(b) w possesses in G derivatives in Sobolev's sense relative to z_j^*, $j = 0, 1, \ldots, n$; the derivatives $\partial w/\partial z_j^*$ belong to $\mathscr{C}(\bar{G})$.

(c) If w is regarded as a function of z_0, one has $w \in \mathscr{C}_\alpha(\bar{D}_0)$.

The space $\mathscr{R}$ is normed by

$$\|w\|_{\mathscr{R},\varkappa} = \max_{i,j} \left(\|w_i\|_{\mathscr{C}(\bar{G})}, \|w_i\|_{\mathscr{C}_\alpha(\bar{D}_0)}, \left\| \frac{\partial w_i}{\partial z_j^*} \right\|_{\mathscr{C}(\bar{G})} \right),$$

where $\varkappa$ is a positive number which will be fixed later. The completeness of $\mathscr{R}$ can be proved in the usual way (see [14], [15]).

Let w be a solution of (1), fulfilling the estimate $|w_i| \leqslant R$, $i = 1, \ldots, m$. From (2) one gets

$$\left\| \frac{\partial w_i}{\partial z_j^*} \right\|_{\mathscr{C}(\bar{G})} = \|f_{ij}(\cdot, w)\|_{\mathscr{C}(\bar{G})} \leqslant K_R. \tag{14}$$

We remark that, by any choice of $\varkappa > 0$, the ball

$$\mathscr{M} = \{w \in \mathscr{R}: \|w\|_{\mathscr{R},\varkappa} \leqslant R\}$$

is closed. From the definition of the norm one gets

$$\varkappa \|\partial w_i/\partial z_j^*\|_{\mathscr{C}(\bar{G})} \leqslant R, \quad \text{if } \varkappa \leqslant R/K_R. \tag{15}$$

4. Estimates of the operator $\tilde{T}$. In order to estimate the operator $\tilde{T}$ we will make use of the following preliminary theorems:

(a) For the holomorphic solution ψ of the boundary value problem (5), (6) one has

$$\|\psi\|_{\mathscr{C}_\alpha(\bar{D}_0)} \leqslant K_1(\alpha) \|g\|_{\mathscr{C}_\alpha(\gamma_0)} + \|c\| \tag{16}$$

(theorem of Privalov, see for instance [2], [18]).

(b) The operator T_{G_j} is bounded and maps $\mathscr{C}(\bar{G}_j)$ into $\mathscr{C}_\alpha(\bar{G}_j)$. Moreover, its norm is arbitrary small, if $\operatorname{diam}(G_j)$, the diameter of G_j, is sufficiently small (see for instance [18], [20]).

(c) For $i \neq j$ the operator T_{G_j} is bounded and maps $\mathscr{C}_\alpha(\bar{G}_i)$ into itself. Its norm is arbitrarily small, if the measure mG_j of G_j is small enough (this follows immediately from the estimation of an integral by mG_j).

Let R_1 be a positive number satisfying the inequality $R_1 < R$. Then we regard only such boundary values g, c for which the holomorphic solution ψ of the boundary value problem (5), (6) (see section 1) fulfils the condition

$$\|\psi\|_{\mathscr{C}_\alpha(\bar{D}_0)} \leqslant R_1. \tag{17}$$

Remark. If $\|\psi\|_{\mathscr{C}(\bar{D}_0)} < R$, then it is possible to admit also such boundary values g, c, for which one gets $\|\psi\|_{\mathscr{C}_\alpha(\bar{D}_0)} \geqslant R$. In this case in the definition of $\|w\|_{*,\varkappa}$ the number $\|w_i\|_{\mathscr{C}_\alpha(\bar{D}_0)}$ must be replaced by $\varkappa\|w_i\|_{\mathscr{C}_\alpha(\bar{D}_0)}$. If it will be chosen sufficiently small then also in this case one can prove that a solution of the boundary value problem (5), (6) exists in the ball $\mathscr{M}$. Let w be an arbitrary element of $\mathscr{M}$. Since the functions $A^i_{j_0\dots j_\lambda}$ are continuous and bounded in $\mathscr{M}$, one gets the estimate

$$\Big\|\sum_{\lambda=0}^{n}(-1)^\lambda \sum_{j_0\dots j_\lambda}^{*} T_{j_0}\dots T_{j_\lambda} A^i_{j_0\dots j_\lambda}\Big\|_{\mathscr{C}(\bar{G})} \leqslant C_1, \tag{18}$$

where $C_1 \to 0$ if $\operatorname{diam}(G) \to 0$.

On the one hand, the functions $A^i_{j_0\dots j_\lambda}$ are continuous, so that

$$\sum_{\lambda=0}^{n}(-1)^\lambda \sum_{j_0\dots j_\lambda}^{*} T_{j_0}\dots T_{j_\lambda} A^i_{j_0\dots j_\lambda}$$

are also continuous. On the other hand, these functions depend holomorphically on w_i, because $A^i_{j_0\dots j_\lambda}$ have the same property.

The preliminary theorems (b) and (c) show that

$$\Big\|\sum_{\lambda=0}^{n}(-1)^\lambda \sum_{j_0\dots j_\lambda}^{*} T_{j_0}\dots T_{j_\lambda} A^i_{j_0\dots j_\lambda}\Big\|_{\mathscr{C}_\alpha(\bar{D}_0)} \leqslant C_2 \tag{19}$$

and $C_2 \to 0$, if $\operatorname{diam}(G) \to 0$. Applying (16) in the case of $\Phi_{(w)}$ one gets

$$\|\Phi_{(w)}\|_{\mathscr{C}_\alpha(\bar{D}_0)} \leqslant K(\alpha)C_2 + C_1. \tag{20}$$

Since ψ and $\Phi_{(w)}$ belong to $\mathscr{C}_\alpha(\bar{D}_0)$ and, consequently, also to $\mathscr{C}(\bar{G})$, $\tilde{T}w$ belongs to $\mathscr{C}(\bar{G}) \cap \mathscr{C}_\alpha(\bar{D}_0)$. In order to show that $\tilde{T}w$ belongs to $\mathscr{R}$ it has still to be proved that the $\partial W_i/\partial z_k^*$ belong to $\mathscr{C}(\bar{G})$.

From the definition of W we get immediately (see [15], formula (4.9.10))

$$\frac{\partial W_i}{\partial z_k^*} \tag{21}$$

$$= \sum_{\lambda=0}^{n-1}(-1)^\lambda \sum_{j_0\dots j_\lambda}^{*k} T_{j_0}\dots T_{j_\lambda}\frac{\partial}{\partial z_k^*}A^i_{j_0\dots j_\lambda} + f_{ik} + \sum_{\lambda=1}^{n}(-1)^\lambda \sum_{j_0\dots j_{\lambda-1}}^{*k} A^i_{j_0\dots k\dots j_{\lambda-1}}.$$

Here $\sum^{*k}$ means that all $j_0, \dots, j_\lambda$ are different from k. Replacing $\lambda-1$ by λ in the second sum in formula (21) we obtain

$$(22)\qquad \frac{\partial W_i}{\partial z_k^*}-f_{ik} = \sum_{\lambda=0}^{n-1}(-1)^\lambda \sum_{j_0\dots j_\lambda}^{**} T_{j_0}\dots T_{j_\lambda}\left(\frac{\partial}{\partial z_k^*}A^i_{j_0\dots j_\lambda}-A^i_{j_0\dots k\dots j_\lambda}\right).$$

On the other hand, the following lemma is valid:

LEMMA. *$(\partial/\partial z_k^*)A^i_{j_0\dots j_\lambda}-A^i_{j_0\dots k\dots j_\lambda} = B^{ik}_{j_0\dots j_\lambda}$, where $B^{ik}_{j_0\dots j_\lambda}$ denotes a polynomial in $\partial w_\sigma/\partial z_\tau^*-f_{\sigma\tau}$.*

Proof. By $F_{\sigma\tau}$ we denote the difference $\partial w_\sigma/\partial z_\tau^*-f_{\sigma\tau}$, by $\Lambda^{(r)}_{j_0\dots j_\lambda}$ a homogeneous polynomial in the $F_{\sigma j_\nu}$ and their derivatives relative to different $z_{j_0}^*,\dots,z_{j_{\nu-1}}^*, z_{j_{\nu+1}}^*,\dots,z_{j_\lambda}^*$ up to order r.

It has been proved ([15], 4.8, Theorem 2) that

$$(23)\qquad \frac{\partial}{\partial z_{j_\lambda}^*}\dots\frac{\partial}{\partial z_{j_1}^*}f_{ij_0}-\frac{\partial}{\partial z_{i_\lambda}^*}\dots\frac{\partial}{\partial z_{i_1}^*}f_{ij_0} = \Lambda^{(\lambda-1)}_{j_0\dots j_\lambda},$$

if $(i_0,\dots,i_\lambda)$ is a permutation of $(j_0,\dots,j_\lambda)$. On the other hand, from the definition of $A^i_{j_0\dots j_\lambda}$ it follows immediately that

$$(24)\qquad A^i_{j_0\dots j_\lambda}-\frac{\partial}{\partial z_j^*}\dots\frac{\partial}{\partial z_{j_1}^*}f_{ij_0} = \Lambda^{(\lambda-1)}_{j_0\dots j_\lambda},$$

and consequently also

$$(25)\qquad \frac{\partial}{\partial z_k^*}A^i_{j_0\dots j_\lambda}-\frac{\partial}{\partial z_k^*}\frac{\partial}{\partial z_{j_\lambda}^*}\dots\frac{\partial}{\partial z_{j_1}^*}f_{ij_0} = \Lambda^{(\lambda)}_{j_0\dots j_\lambda k}.$$

Applying formulas (23), (24) with $j_0,\dots,j_\lambda,k$ instead of $j_0,\dots,j_\lambda$ one gets, in view of (25),

$$(26)\qquad \frac{\partial}{\partial z_k^*}A^i_{j_0\dots j_\lambda}-A^i_{j_0\dots k\dots j_\lambda} = \Lambda^{(\lambda+1)}_{j_0\dots j_\lambda k}.$$

Now the left-hand side of (26) contains only derivatives relative to z_k^*; this proves the lemma.

Using the lemma we get from (22)

$$(27)\qquad \frac{\partial W_i}{\partial z_k^*} = f_{ik}+\sum_{\lambda=0}^{n-1}(-1)^\lambda\sum_{j_0\dots j_\lambda}^{**}T_{j_0}\dots T_{j_\lambda}B^{ik}_{j_0\dots j_\lambda}.$$

This means that the derivatives $\partial W_i/\partial z_k^*$ belong to $\mathscr{C}(\bar{G})$. Moreover, we have the estimate

$$\varkappa\left\|\frac{\partial W_i}{\partial z_k^*}\right\|_{\mathscr{C}(\bar{G})} \leqslant \varkappa K_R+\varkappa C_3,$$

where $C_3\to 0$ if $\operatorname{diam}(G)\to 0$. If it is assumed

$$(28)\qquad \varkappa \leqslant \tfrac{1}{2}R/K_R$$

instead of the earlier assumption $\varkappa \leqslant R/K_R$, one get consequently

(29) $$\varkappa \, \|\partial W_i/\partial z_k^*\|_{\mathscr{C}(\bar{G})} \leqslant \tfrac{1}{2} R + \varkappa C_3.$$

Estimates (17)–(20), (29) result in the following estimate for $W = \tilde{T}w$:

(30) $$\|\tilde{T}w\|_{\mathscr{B}} \leqslant \max\left(R_1 + K(\alpha) C_2 + 2C_1,\, R_1 + K(\alpha)(C_1 + C_2) + C_1,\, \tfrac{1}{2} R + \varkappa C_3\right)$$

(provided that (27) holds).

Now we ask, under what conditions the operator $\tilde{T}$ is contractive. For two elements w, $\tilde{w} \in \mathscr{M}$ and $W = \tilde{T}w$, $\tilde{W} = \tilde{T}\tilde{w}$ we examine the difference

$$W - \tilde{W} = (\Phi_{(w)} - \Phi_{(\tilde{w})}) + \sum_{\lambda=0}^{n} (-1)^\lambda \sum_{j_0 \dots j_\lambda}^{*} T_{j_0} \dots T_{j_\lambda} (A_{j_0 \dots j_\lambda} - \tilde{A}_{j_0 \dots j_\lambda}),$$

where $\tilde{A}_{j_0 \dots j_\lambda}$ indicates that the function w is replaced by $\tilde{w}$. Because

$$A^i_{j_0 \dots j_\lambda} = P_1 \dots P_l,$$

where the P_ν denotes functions f_{ij} or their derivatives, we have

$$\begin{aligned} A^i_{j_0 \dots j_\lambda} - \tilde{A}^i_{j_0 \dots j_\lambda} &= (P_1 \dots P_l - \tilde{P}_1 \dots \tilde{P}_l) \\ &= \sum(\dots + (P_1 \dots P_\nu \tilde{P}_{\nu+1} \dots \tilde{P}_l - P_1 \dots \tilde{P}_\nu \tilde{P}_{\nu+1} \dots \tilde{P}_l) + \dots); \end{aligned}$$

consequently

(31) $$\begin{aligned} |A^i_{j_0 \dots j_\lambda} - \tilde{A}^i_{j_0 \dots j_\lambda}| &\leqslant \text{const} \cdot L_R \cdot \sum_{\sigma=1}^{m} |w_\sigma - \tilde{w}_\sigma| \\ &\leqslant \text{const} \cdot L_R \cdot m \, \|w - \tilde{w}\|_{\mathscr{C}(\bar{G})} \end{aligned}$$

(this follows from (3)). This estimate immediately implies:

(32) $$\Big\| \sum_{\lambda=0}^{n} (-1)^\lambda \sum_{j_0 \dots j_\lambda}^{*} T_{j_0} \dots T_{j_\lambda} (A_{j_0 \dots j_\lambda} - \tilde{A}_{j_0 \dots j_\lambda}) \Big\|_{\mathscr{C}(\bar{G})} \leqslant C_4 \, \|w - \tilde{w}\|_{\mathscr{C}(\bar{G})},$$

where $C_4 \to 0$ if $\operatorname{diam} G \to 0$.

Applying the preliminary theorems (b) and (c) and assumption (4), we obtain analogously to (32):

(33) $$\Big\| \sum_{\lambda=0}^{n} (-1)^\lambda \sum_{j_0 \dots j_\lambda}^{*} T_{j_0} \dots T_{j_\lambda} (A_{j_0 \dots j_\lambda} - \tilde{A}_{j_0 \dots j_\lambda}) \Big\|_{\mathscr{C}_\alpha(\bar{D}_0)} \leqslant C_5 \cdot l_{R'} \cdot \|w - \tilde{w}\|_{\mathscr{C}_\alpha(\bar{G})},$$

where $C_5 \to 0$ if $mG \to 0$. Using Privalov's theorem (preliminary theorem (a)), we get from (33):

(34) $$\|\Phi_{(w)} - \Phi_{(\tilde{w})}\|_{\mathscr{C}_\alpha(\bar{G})} \leqslant K(\alpha) \, C_5 \, l_R \, \|w - \tilde{w}\|_{\mathscr{C}_\alpha(\bar{G})}.$$

From (27) one gets

$$(35)\quad \frac{\partial W_i}{\partial z_k^*}-\frac{\partial \tilde{W}_i}{\partial z_k^*}$$

$$= f_{ik}(\ldots, w)-f_{ik}(\ldots, \tilde{w})+\sum_{\lambda=0}^{n-1} (-1)^{\lambda} \sum_{j_0\ldots j_\lambda}^{*k} T_{j_0} \ldots T_{j_\lambda}(B^{ik}_{j_0\ldots j_\lambda}-\tilde{B}^{ik}_{j_0\ldots j_\lambda}).$$

By definition the $B^{ik}_{j_0\ldots j_\lambda}$ contain also the derivatives $\partial w_\sigma/\partial z_\tau^*$.

Similarly to (31) one gets

$$|B^{ik}_{j_0\ldots j_\lambda}-\tilde{B}^{ik}_{j_0\ldots j_\lambda}| \leqslant \text{const}\cdot L_R \sum_{\sigma=1}^{m} |w_\sigma-\tilde{w}_\sigma|+\text{const}\cdot \max_{\sigma,\tau} \left|\frac{\partial w_\sigma}{\partial z_\tau}-\frac{\partial \tilde{w}_\sigma}{\partial z_\tau^*}\right|.$$

Applying this estimate to (35), we get immediately:

$$(36)\quad \varkappa \left\|\frac{\partial W_i}{\partial z_k^*}-\frac{\partial \tilde{W}_i}{\partial z_k^*}\right\|_{\mathscr{C}(\bar{G})}$$

$$\leqslant mL_R\, \|w-\tilde{w}\|_{\mathscr{C}(\bar{G})}+\varkappa C_6 L_R\, \|w-\tilde{w}\|_{\mathscr{C}(\bar{G})}+\varkappa C_7 \max \left\|\frac{\partial w_\sigma}{\partial z_\tau^*}-\frac{\partial \tilde{w}_\sigma}{\partial z_\tau^*}\right\|_{\mathscr{C}(\bar{G})}$$

(where $C_6 \to 0$, $C_7 \to 0$ if $\operatorname{diam} G \to 0$). Applying the definition of $\|\cdot\|_{\mathscr{M}}$, and formulas (32), (33), (34) and (36), we obtain

$$(37)\quad \|W-\tilde{W}\|_{\mathscr{M}} = \|\tilde{T}w-\tilde{T}\tilde{w}\|_{\mathscr{M}}$$

$$\leqslant \max\big(K(\alpha)C_5 l_R+C_4, (K(\alpha)+1)C_5 l_R, \tfrac{1}{2}+\varkappa L_R C_6+C_7\big)\, \|w-\tilde{w}\|_{\mathscr{M}},$$

provided that

$$(38)\qquad \varkappa \leqslant 1/2mL_R.$$

5. An integral equation for the fixed point of $\tilde{T}$. In the next section we shall see that $\tilde{T}$ possesses a fixed point in M, if $\operatorname{diam} G$ is sufficiently small. For sufficiently small domains now it will be proved that each fixed point of $\tilde{T}$ is simultaneously also a solution of the given system (1) of partial differential equations. From (27) for a fixed point w of $\tilde{T}$ one gets

$$(39)\qquad F_{ik} = \sum_{\lambda=0}^{n-1} (-1)^{\lambda} \sum_{j_0\ldots j_\lambda}^{*k} T_{j_0} \ldots T_{j_\lambda} B^{ik}_{j_0\ldots j_\lambda},$$

because $\partial w_i/\partial z_k^* - f_{ik} = F_{ik}$. From the continuity of the derivatives $\partial w_i/\partial z_k^*$ it follows that all F_{ik} are also continuous. Let F the matrix with elements F_{ij}; then we have the estimate

$$(40)\qquad |F_{ik}| \leqslant \left|\frac{\partial w_i}{\partial z_k^*}\right|+|f_{ik}| \leqslant \frac{1}{\varkappa} R+K_R.$$

Let $\mathscr{R}^{\mathscr{F}}$ be the space consisting of all matrices F, where

$$\|F\|_{\mathscr{M}^{\mathscr{F}}} = \max_{i,k} \|F_{ik}\|_{\mathscr{C}(\bar{G})}.$$

Then estimate (40) shows that the matrix F, corresponding to the fixed point w of $\tilde{T}$, belongs to the ball

$$\mathcal{M}^{\mathcal{F}} = \left\{F \in \mathcal{R}^{\mathcal{F}}:\ \|F\| \leqslant \frac{1}{\varkappa} R + K_R\right\}.$$

Now, with the help of the right-hand side of (39), an operator $\tilde{T}^*$ mapping $\mathcal{M}^{\mathcal{F}}$ into $\mathcal{R}^{\mathcal{F}}$ will be defined. Analogously to (18) one gets

$$\|\tilde{T}^* F\|_{\mathcal{R}^{\mathcal{F}}} = \Big\| \sum_{\lambda=0}^{n-1} (-1)^{\lambda} \sum_{j_0 \dots j_\lambda}^{*k} T_{j_0} \dots T_{j_\lambda} B^{ik}_{j_0 \dots j_\lambda} \Big\|_{\mathcal{R}^{\mathcal{F}}} \leqslant C_8, \tag{41}$$

where $C_8 \to 0$ if $mG \to 0$. We note that C_8 depends on $\varkappa$. Moreover, similarly to (32) we obtain

$$\|\tilde{T}^* F - \tilde{T}^* \tilde{F}\|_{\mathcal{R}^{\mathcal{F}}} \leqslant C_9 \|F - \tilde{F}\|_{\mathcal{R}^{\mathcal{F}}}, \tag{42}$$

where $C_9 \to 0$ if $\operatorname{diam} G \to 0$ (this estimate can be proved in the same way as (32); we remark that the coefficients of F_{ik} in $B^{ik}_{j_0 \dots j_\lambda}$ are the functions $f_{\sigma\tau}$ and their derivatives, with w substituted, the fixed-point of $\tilde{T}$ under consideration). Also the coefficient C_9 in (42) depends on $\varkappa$.

The matrix F, corresponding to a given fixed point of $\tilde{T}$, is a fixed point for the so-called associate operator $\tilde{T}^*$. On the other hand, $F = 0$ is a fixed point of $\tilde{T}^*$ (this follows from the fact that $\Lambda^{(\lambda+1)}_{j_0 \dots j_\lambda}$ and $B^{ik}_{j_0 \dots j_\lambda}$ are homogeneous). Because a contractive operator possesses a uniquely determined fixed point, we have the following result:

If it is ensured that $\tilde{T}^*$ is a contractive operator, then it follows $F = 0$, which means that w is a solution of the given system (1).

6. Solution of the boundary value problem. Let $\varkappa$ be a fixed number, for which it holds (see (27), (38)):

$$0 < \varkappa \leqslant \min\left(\tfrac{1}{2} R/K_r, \frac{1}{2mL_R}\right).$$

Since all $C_j, j = 1, \dots, 9$, tend to 0 (if $\operatorname{diam} G \to 0$), one can choose $\operatorname{diam} G$ so small that the following inequalities are fulfilled:

$$\max\left(R_1 + K(\alpha) C_2 + 2C_1,\ R_1 + K(\alpha)(C_1 + C_2) + C_1,\ \tfrac{1}{2} R + \varkappa C_3\right) \leqslant R,$$
$$\max\left(K(\alpha) C_5 l_R + C_4,\ (K(\alpha)+1) C_5 l_R,\ \tfrac{1}{2} + \varkappa L_R C_6 + C_7\right) < 1,$$
$$C_8 \leqslant \frac{1}{\varkappa} R + K_R,\ C_9 < 1.$$

Then (30) shows that $\tilde{T}$ maps $\mathcal{M}$ into itself. Analogously one gets from (41) that $\tilde{T}^*$ maps $\mathcal{M}^{\mathcal{F}}$ into itself. From (37) and (42), respectively, it follows that $\tilde{T}$ and $\tilde{T}^*$ are contractive.

By Banach's fixed point theorem we get the existence of a fixed point w of $\tilde{T}$. This fixed point satisfies the given boundary condition (as it has been proved in section 2). On the other hand, the associate operator $\tilde{T}^*$ is contractive, too. That means that F vanishes identically. Consequently, it follows from section 5 that the fixed point w is a solution of (1).

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