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Lipschitzian operators of substitution in the space of vector functions of bounded variation

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Abstract. Let X and Y be Banach spaces, let I be an interval and let $h: I\times X \to Y$. The main result of this paper states that if the Nemytskii operator of substitution given by the formula $N\varphi(t) = h(t, \varphi(t))$ $(t \in I)$, mapping the Banach space of bounded variation functions $\mathrm{BV}(I, X)$ into $\mathrm{BV}(I, Y)$, is globally Lipschitzian in the sense of the norms of BV-spaces, then h is, in a sense, a linear function in the second variable.

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces and let I be a set. Let us denote by X^I the space of all functions $\varphi: I \to X$ mapping the set I in the space X. Let $\mathscr{L}(X, Y)$ denote the Banach space of all linear bounded operators mapping X in Y with the standard norm.

Every function $h: I\times X \to Y$ generates the so-called *Nemytskii operator* or *operator of substitution* (e.g. see [1], p. 163)

$$N_h: X^I \to Y^I$$

defined by the formula

$$(N_h\varphi)(t) = h(t, \varphi(t)), \quad t \in I, \ \varphi \in X^I.$$

Let $\mathscr{F}_1 \subset X^I$, $\mathscr{F}_2 \subset Y^I$ and let $(\mathscr{F}_1, \|\cdot\|_1)$, $(\mathscr{F}_2, \|\cdot\|_2)$ be the Banach spaces. Consider the following problem.

Characterize all $h: I\times X \to Y$ for which the operator N_h maps $\mathscr{F}_1$ in $\mathscr{F}_2$ and is Lipschitzian, i.e.

(a) $N_h: \mathscr{F}_1 \to \mathscr{F}_2$,

(b) $\underset{L\geqslant 0}{\exists}\ \underset{\varphi,\psi\in\mathscr{F}_1}{\forall}\ \|N_h\varphi - N_h\psi\|_2 \leqslant L\|\varphi-\psi\|_1$.

Matkowski (see [3]) proved that if I is a convex subset of a normed space and $\mathscr{F}_1$, $\mathscr{F}_2$ are space of Lipschitzian functions with an adequate norms, then conditions (a), (b) imply that h has the form

$$(1) \qquad h(t, x) = G(t)x + H(t) \quad (t \in I, \ x \in X),$$

where $H \in Y^I$ and $G \in \mathscr{L}(X, Y)^I$. Moreover, G, H are Lipschitzian. An analogous fact holds when $\mathscr{F}_1, \mathscr{F}_2$ are spaces of functions fulfilling Hölder's condition, or when $\mathscr{F}_1, \mathscr{F}_2$ are spaces of functions having continuous n-th derivative (see [2], [4]).

In this paper we are going to consider the case of the space of functions of bounded variation.

Let I denote a fixed nondegenerated real interval with endpoints a, b $(-\infty \leqslant a < b \leqslant \infty)$ which need not belong to I.

If $\varphi\colon I \to X$ then the variation of φ is defined by the formula

$$\mathrm{Var}(\varphi) := \sup \sum_{i=1}^{n} \|\varphi(t_i) - \varphi(t_{i-1})\|_X,$$

where supremum is taken over all positive integers n and over all choices $\{t_i\} \subset I$ such that $t_0 < t_1 < \ldots < t_n$. Let $\mathrm{BV}(X)$ be a space of all functions $\varphi\colon I \to X$ such that $\mathrm{Var}(\varphi) < \infty$.

It is easily seen that: functions in $\mathrm{BV}(X)$ are bounded, the space $\mathrm{BV}(X)$ is linear, and the formula

$$\|\varphi\|_{\mathrm{BV}(X)} = \sup_{t \in I} \|\varphi(t)\|_X + \mathrm{Var}(\varphi)$$

gives a norm in $\mathrm{BV}(X)$.

It is possible to show that $(\mathrm{BV}(X), \|\cdot\|_{\mathrm{BV}(X)})$ is a Banach space, and that every function belonging to $\mathrm{BV}(X)$ has left- and right-side limits at each point of I.

We start with the following useful result.

Lemma 1. *If* $G \in \mathrm{BV}(\mathscr{L}(X, Y))$, $\varphi \in \mathrm{BV}(X)$, *then the mapping* $G\varphi\colon I \to Y$ *defined by the formula*

$$(G\varphi)(t) = G(t)\,\varphi(t)$$

belongs to $\mathrm{BV}(Y)$ *and*

$$\|G\varphi\|_{\mathrm{BV}(Y)} \leqslant \|G\|_{\mathrm{BV}(\mathscr{L}(X,Y))} \cdot \|\varphi\|_{\mathrm{BV}(X)}.$$

Proof. For $\{t_i\}_{i=0,\ldots,n} \subset I$, $t_0 < t_1 < \ldots < t_n$ we have

$$\sum_{i=1}^{n} \|(G\varphi)(t_i) - (G\varphi)(t_{i-1})\|_Y = \sum_{i=1}^{n} \|G(t_i)\,\varphi(t_i) - G(t_{i-1})\,\varphi(t_{i-1})\|_Y$$

$$= \sum_{i=1}^{n} \|G(t_i)\,\varphi(t_i) - G(t_i)\,\varphi(t_{i-1}) + G(t_i)\,\varphi(t_{i-1}) - G(t_{i-1})\,\varphi(t_{i-1})\|_Y$$

$$\leqslant \sum_{i=1}^{n} \big\|G(t_i)\big(\varphi(t_i) - \varphi(t_{i-1})\big)\big\|_Y + \sum_{i=1}^{n} \big\|\big(G(t_i) - G(t_{i-1})\big)\,\varphi(t_{i-1})\big\|_Y$$

$$\leqslant \sum_{i=1}^{n} \|G(t_i)\|_{\mathscr{L}(X,Y)} \cdot \|\varphi(t_i) - \varphi(t_{i-1})\|_X +$$

$$+ \sum_{i=1}^{n} \|G(t_i) - G(t_{i-1})\|_{\mathscr{L}(X,Y)} \cdot \|\varphi(t_{i-1})\|_X$$

$$\leqslant \sup_{t \in I} \|G(t)\|_{\mathscr{L}(X,Y)} \cdot \operatorname{Var}(\varphi) + \|\varphi\|_{\mathrm{BV}(X)} \cdot \operatorname{Var}(G).$$

Hence

$$\operatorname{Var}(G\varphi) \leqslant \sup_{t \in I} \|G(t)\|_{\mathscr{L}(X,Y)} \cdot \operatorname{Var}(\varphi) + \|\varphi\|_{\mathrm{BV}(X)} \cdot \operatorname{Var}(G) < \infty,$$

and therefore $G\varphi \in \mathrm{BV}(X)$. Using the last inequality, we obtain

$$\begin{aligned} \|G\varphi\|_{\mathrm{BV}(Y)} &= \sup_{t \in I} \|G(t)\,\varphi(t)\|_Y + \operatorname{Var}(G\varphi) \\ &\leqslant \sup_{t \in I} \|G(t)\|_{\mathscr{L}(X,Y)} \cdot \sup_{t \in I} \|\varphi(t)\|_X + \\ &\qquad + \sup_{t \in I} \|G(t)\|_{\mathscr{L}(X,Y)} \cdot \operatorname{Var}(\varphi) + \|\varphi\|_{\mathrm{BV}(X)} \cdot \operatorname{Var}(G) \\ &= \sup_{t \in I} \|G(t)\|_{\mathscr{L}(X,Y)} \cdot \big(\sup_{t \in I} \|\varphi(t)\|_X + \operatorname{Var}(\varphi)\big) + \|\varphi\|_{\mathrm{BV}(X)} \cdot \operatorname{Var}(G) \\ &= \sup_{t \in I} \|G(t)\|_{\mathscr{L}(X,Y)} \cdot \|\varphi\|_{\mathrm{BV}(X)} + \|\varphi\|_{\mathrm{BV}(X)} \cdot \operatorname{Var}(G) \\ &= \big(\sup_{t \in I} \|G(t)\|_{\mathscr{L}(X,Y)} + \operatorname{Var}(G)\big) \cdot \|\varphi\|_{\mathrm{BV}(X)} = \|G\|_{\mathrm{BV}(\mathscr{L}(X,Y))} \cdot \|\varphi\|_{\mathrm{BV}(X)} \end{aligned}$$

which completes the proof of the lemma.

In the sequel, we use the symbol $\|\cdot\|$ without any index for all norms, because the context excludes a misunderstanding.

We consider the following conditions

(I) $N := N_h \colon \mathrm{BV}(X) \to \mathrm{BV}(Y)$;

(II) $\underset{L \geqslant 0}{\exists} \ \underset{\varphi, \psi \in \mathrm{BV}(X)}{\forall} \ \|N\varphi - N\psi\| \leqslant L\|\varphi - \psi\|$.

One can observe that if for a given h the operator N_h fulfils (I)–(II), then for the function

$$g(t, x) := h(t, x) - h(t, O_X) \qquad (t \in I,\ x \in X)$$

the operator N_g fulfils (I)–(II) too. Conversely, if, for a given g, N_g fulfils (I)–(II), $H \in \mathrm{BV}(Y)$ and

$$h(t, x) := g(t, x) + H(t) \qquad (t \in I,\ x \in X),$$

then N fulfils (I)–(II).

Therefore, without any less of generality, we may assume that

(III) $\underset{t\in I}{\forall} h(t, O_X) = O_Y$.

Some sufficient condition for the Nemytskii operator to be Lipschitzian gives the following theorem.

THEOREM 1. *If* $G \in \mathrm{BV}(\mathscr{L}(X, Y))$, $H \in \mathrm{BV}(Y)$ *and*

$$h(t, x) = G(t)x + H(t) \qquad (t \in I,\ x \in X),$$

then N_h *fulfils conditions* (I)–(II) *with the constant* $L = \|G\|$.

Proof. By Lemma 1, for $\varphi \in \mathrm{BV}(X)$, we have $(G\varphi) \in \mathrm{BV}(Y)$, therefore $G\varphi + H = N\varphi \in \mathrm{BV}(Y)$ too. Hence condition (I) is fulfilled. Now, for $\varphi, \psi \in \mathrm{BV}(X)$, we get by Lemma 1

$$\|N\varphi - N\psi\| = \|(G\varphi + H) - (G\psi + H)\| = \|G(\varphi - \psi)\| \leqslant \|G\| \cdot \|\varphi - \psi\|,$$

which completes the proof.

To give some necessary conditions we start with two lemmas.

LEMMA 2. *If conditions* (I)–(II) *are fulfilled, then*

$$\|h(t, x) - h(t, \bar{x})\| \leqslant L\|x - \bar{x}\| \qquad (t \in I,\ x, \bar{x} \in X).$$

Proof. It suffices to notice that the constant functions $\varphi(t) = x$, $\psi(t) = \bar{x}$ belong $\mathrm{BV}(X)$ and by (II) we get

$$\|h(t, x) - h(t, \bar{x})\| = \|(N\varphi)(t) - (N\varphi)(t)\| = \|(N\varphi - N\psi)(t)\| \leqslant \|N\varphi - N\psi\|$$
$$\leqslant L\|\varphi - \psi\| = L\|x - \bar{x}\|.$$

If condition (I) is fulfilled, then we may define the function $h^*\colon (I - \{a\}) \times X \to Y$ by the formula

$$h^*(t, x) := \lim_{\tau \to t^-} h(\tau, x) \qquad (t \in I - \{a\},\ x \in X).$$

Since there exist one-sided limits of $h(\cdot, x)$ (by (I) belonging to $\mathrm{BV}(Y)$), the function h^* is well defined.

Note also the following simple result.

LEMMA 3. *If condition* (I) *is fulfilled, then for each* $x \in X$ *and* $t \in I - \{a\}$ *there exists* $h^*(t, x) := \lim_{\tau \to t^-} h(\tau, x)$.

Moreover, the function $h^*\colon (I - \{a\}) \times X \to Y$ has the following properties:

(a) h^* is left-continuous with respect to the first variable;

(b) for each $x \in X$ the function $h^*(\cdot, x)$ belongs to $\mathrm{BV}(Y)$;

(c) if condition (II) is fulfilled, then

$$\|h^*(t, x)-h^*(t, \bar{x})\| \leqslant L\|x-\bar{x}\| \quad (t \in I-\{a\},\ x, \bar{x} \in X).$$

Now we can prove the following theorem.

THEOREM 2. *Let assumptions* (I)–(III) *be fulfilled. If X is a separable Banach space, then the set*

$$D_h := D := \{t \in I\colon\ h_t \text{ is not linear}\} \quad (h_t := h(t, \cdot))$$

is countable.

Proof. By Lemma 2 the function $h_t\colon X \to Y$ is continuous. Hence, if h_t is not linear, then it is not additive and therefore there exist $u, v \in X$ such that

$$\|h_t(u+v)-h_t(u)-h_t(v)\| > 0.$$

Let A be a countable and dense set in X. By the continuity of h_t, we may assume that $u, v \in A$. Hence we obtain

$$D = \bigcup_{(u,v)\in A\times A} \ \bigcup_{\substack{\alpha\in Q\\ \alpha>0}} C(u, v, \alpha),$$

where

$$C(u, v, \alpha) := \{t \in I\colon\ \|h_t(u+v)-h_t(u)-h_t(v)\| \geqslant \alpha\}.$$

Now we are going to show that the set $C(u, v, \alpha)$ is finite. For "an indirect proof", let us assume that $C(u, v, \alpha)$ is infinite. Then for arbitrary $n \in N$ there exists a sequence $t_1, t_2, \ldots, t_{2n} \in C(u, v, \alpha)$ such that $t_1 < t_2 < \ldots < t_{2n}$. Define the functions φ, ψ as follows:

$$\psi(t) = \begin{cases} u, & t \in \{t_2, t_4, \ldots, t_{2n}\}, \\ 0, & t \notin \{t_2, t_4, \ldots, t_{2n}\}, \end{cases} \qquad \varphi(t) = \psi(t)+v.$$

Of course, $\varphi, \psi \in \mathrm{BV}(X)$ and $\|\varphi-\psi\| = \|v\|$. Since $t_i \in C(u, v, \alpha)$, therefore

$$n\alpha \leqslant \sum_{i=1}^{n} \|h_{t_{2i}}(u+v)-h_{t_{2i}}(u)-h_{t_{2i}}(v)\|. \tag{2}$$

On the other hand, by assumption (III), we get

$$\begin{aligned} &\sum_{i=1}^{n} \|h_{t_{2i}}(u+v)-h_{t_{2i}}(u)-h_{t_{2i}}(v)\| \\ &\qquad = \sum_{i=1}^{n} \|h(t_{2i}, u+v)-h(t_{2i}, u)-h(t_{2i}, v)+h(t_{2i-1}, O_X)\| \end{aligned} \tag{3}$$

$$\leqslant \sum_{i=1}^{n} \|h(t_{2i}, u+v)-h(t_{2i}, u)-h(t_{2i-1}, v)+h(t_{2i-1}, O_X)\|+$$

$$+\sum_{i=1}^{n} \|h(t_{2i}, v)-h(t_{2i-1}, v)\|.$$

By assumptions (I), (II) we have

$$(4) \qquad \sum_{i=1}^{n} \|h(t_{2i}, u+v)-h(t_{2i}, u)-h(t_{2i-1}, v)+h(t_{2i-1}, O_X)\|$$

$$= \sum_{i=1}^{n} \left\|h\big(t_{2i}, \varphi(t_{2i})\big)-h\big(t_{2i}, \psi(t_{2i})\big)-h\big(t_{2i-1}, \varphi(t_{2i-1})\big)+h\big(t_{2i-1}, \psi(t_{2i-1})\big)\right\|$$

$$= \sum_{i=1}^{n} \|(N\varphi-N\psi)(t_{2i})-(N\varphi-N\psi)(t_{2i-1})\| \leqslant \mathrm{Var}(N\varphi-N\psi)$$

$$\leqslant \|N\varphi-N\psi\| \leqslant L\|\varphi-\psi\|,$$

and, obviously,

$$(5) \qquad \sum_{i=1}^{n} \|h(t_{2i}, v)-h(t_{2i-1}, v)\| \leqslant \mathrm{Var}\big(h(\cdot, v)\big).$$

Now, inequalities (2)–(5) and definitions of functions φ, ψ yield the inequality

$$n\alpha \leqslant L\|\varphi-\psi\|+\mathrm{Var}(h(\cdot, v)) = L\|v\|+\mathrm{Var}(h(\cdot, v)) \qquad (n\in N)$$

which implies that $\mathrm{Var}\big(h(\cdot, v)\big) = \infty$. This is a contradiction, because, by (I), $\mathrm{Var}\big(h(\cdot, v)\big) < \infty$.

Thus for arbitrary $u, v\in A$ and $\alpha\in Q\cap(R_+ - \{0\})$ the set $C(u, v, \alpha)$ is finite, and, consequently, the set D as a countable sum of the sets $C(u, v, \alpha)$ is countable.

Remark 1. In the same way one can prove a more general theorem which states that the cardinal number of the set D is not greater than the density of the space X.

Remark 2. In paper [5] it is shown an example in which the set D is essentially infinite but the function h is not continuous with respect to the first variable.

The following theorem was proved for the real case $X = Y = R$ in paper [5].

THEOREM 3. *If conditions* (I)–(III) *are fulfilled, then there exists a function* $G\colon I \to \mathscr{L}(X, Y)$ *such that*

$$h^*(t, x) = G(t)x \qquad (t\in I-\{a\},\ x\in X).$$

Proof. Let us fix $t \in I - \{a\}$, $n \in N$ and take $\{t_1, t_2, \ldots, t_{2n}\} \subset I$ such that $t_1 < t_2 < \ldots < t_{2n} < t$. Choose arbitrary $u, v \in X$ and define φ, ψ by the formulae

$$\psi(\tau) = \begin{cases} u, & \tau \in \{t_2, t_4, \ldots, t_{2n}\}, \\ O, & \tau \notin \{t_2, t_4, \ldots, t_{2n}\}, \end{cases} \qquad \varphi(\tau) = \psi(\tau) + v.$$

Of course, $\varphi, \psi \in \mathrm{BV}(X)$ and $\|\varphi - \psi\| = \|v\|$.

Now, by virtue of the definition of the norm in $\mathrm{BV}(Y)$, we get the following inequality:

$$\sum_{i=1}^{n} \|N\varphi(t_{2i}) - N\psi(t_{2i}) - N\varphi(t_{2i-1}) + N\psi(t_{2i-1})\|$$
$$\leqslant \mathrm{Var}(N\varphi - N\psi) \leqslant \|N\varphi - N\psi\|.$$

Hence, according to assumption (II), we get

$$\sum_{i=1}^{n} \left\|h\big(t_{2i}, \varphi(t_{2i})\big) - h\big(t_{2i}, \psi(t_{2i})\big) - h\big(t_{2i-1}, \varphi(t_{2i-1})\big) + h\big(t_{2i-1}, \psi(t_{2i-1})\big)\right\|$$
$$\leqslant L\|\varphi - \psi\|.$$

Putting here the values of functions φ, ψ and making use of assumption (III), we get

$$\sum_{i=1}^{n} \|h(t_{2i}, u+v) - h(t_{2i}, u) - h(t_{2i-1}, v)\| \leqslant L\|v\|.$$

Hence, letting $t_1 \to t$, we get the inequality

$$\sum_{i=1}^{n} \|h^*(t, u+v) - h^*(t, u) - h^*(t, v)\| \leqslant L\|v\|$$

which me may write in the form

$$\|h^*(t, u+v) - h^*(t, u) - h^*(t, v)\| \leqslant \frac{1}{n} L\|v\|$$

for arbitrary $n \in N$ and $u, v \in X$. Hence, as $n \to \infty$, we get

$$h^*(t, u+v) - h^*(t, u) - h^*(t, v) = O \qquad (t \in I - \{a\}). \tag{6}$$

Let us define $G \colon I \to Y^X$ by the formulae

$$\begin{aligned} G(t)x &= h^*(t, x), && (t \in I - \{a\}), \\ G(a)x &= 0, && (\text{iff } a \in I). \end{aligned}$$

Equality (6) means that, for each $t \in I$, $G(t)$ is additive. Since, by Lemma 2, it is also continuous for a fixed $t \in I$, this completes the proof.

COROLLARY 1. *If* (I), (II) *are fulfilled, then there exist the function* $G\colon I \to \mathscr{L}(X, Y)$ *and the left-continuous function* $H^* \in \mathrm{BV}(Y)$ *such that*

$$h^*(t, x) = G(t)x + H^*(t) \qquad (t \in I - \{a\},\ x \in X).$$

Proof. It suffices to define

$$H^*(t) := \lim_{\tau \to t^-} h(\tau, O_X) \qquad (t \in I - \{a\}).$$

If X is a finite-dimensional space, then $G \in \mathrm{BV}(\mathscr{L}(X, Y))$ (cf. [5]). It turns out that in general G need not to be an element of $\mathrm{BV}(\mathscr{L}(X, Y))$. For a construction of an example we quote without a proof the following lemma.

LEMMA 4. ([6]) *Let* l^2 *be the Hilbert space of all real sequences* (x_k) *with the norm*

$$\|(x_k)\| = \sqrt{\sum_{k=1}^{\infty} x_k^2}.$$

If $\varphi\colon I \to l^2$ (*i.e.* $\varphi = (\varphi_1, \varphi_2, \ldots)$) *and* $\varphi \in \mathrm{BV}(l^2)$, *then* $\varphi_k \in \mathrm{BV}(\boldsymbol{R})$ *for each* $k \in N$ *and*

$$\sum_{k=1}^{\infty} (\mathrm{Var}(\varphi_k))^2 \leqslant (\mathrm{Var}(\varphi))^2.$$

EXAMPLE. Let $X = l^2$, $Y = R$ and $I = (0, \infty)$. For $t \in (k-1, k]$ put

$$G(t)x = \frac{1}{k} x_k \qquad (x = (x_1, x_2, \ldots) \in X,\ k \in \boldsymbol{N}).$$

Of course, $G\colon I \to \mathscr{L}(l^2, R)$ and G is left-continuous. Setting $h(t, x) := G(t)x$, we evidently have $h = h^*$.

We are going to show conditions (I)–(III) are fulfilled and $G \notin \mathrm{BV}(\mathscr{L}(l^2, R))$.

Since $h(t, \boldsymbol{O}) = G(t)(0, 0, \ldots) = \frac{1}{k} \cdot 0 = 0$, condition (III) is fulfilled.

Now take $\varphi = (\varphi_1, \varphi_2, \ldots) \in \mathrm{BV}(l^2)$. By Lemma 4, $\varphi_k \in \mathrm{BV}(R)$ $(k = 1, 2, \ldots)$ and

$$\sum_{k=1}^{\infty} (\mathrm{Var}(\varphi_k))^2 \leqslant (\mathrm{Var}(\varphi))^2. \tag{7}$$

Take an arbitrary increasing sequence $\{t_i\}_{i=0,1,\ldots,n} \subset I = (0, \infty)$ and write it in the form

$$t_0^1 < \ldots < t_{n_1}^1 < t_0^2 < \ldots < t_{n_2}^2 < \ldots < t_0^m < \ldots < t_{n_m}^m$$

such that

$$t_j^k \in (k-1, k] \quad (k = 1, 2, \ldots, m,\ j = 0, 1, \ldots, n_k).$$

Now, applying (7) and Schwarz's inequality, we get the estimation

$$\sum_{i=0}^{n} |N\varphi(t_i) - N\varphi(t_{i-1})| = \sum_{k=1}^{m} \sum_{j=1}^{n_k} |G(t_j^k)\varphi(t_j^k) - G(t_{j-1}^k)\varphi(t_{j-1}^k)| +$$
$$+ \sum_{k=1}^{m-1} |G(t_0^{k+1})\varphi(t_0^{k+1}) - G(t_{n_k}^k)\varphi(t_{n_k}^k)|$$
$$\leqslant \sum_{k=1}^{m} \sum_{j=1}^{n_k} \left| \frac{1}{k}\varphi_k(t_j^k) - \frac{1}{k}\varphi_k(t_{j-1}^k) \right| + \sum_{k=1}^{m-1} \frac{1}{k+1} |\varphi_{k+1}(t_0^{k+1})| + \sum_{k=1}^{m-1} \frac{1}{k} |\varphi_k(t_{n_k}^k)|$$
$$\leqslant \sum_{k=1}^{m} \frac{1}{k} \sum_{j=1}^{n_k} |\varphi_k(t_j^k) - \varphi_k(t_{j-1}^k)| + \sum_{k=1}^{m-1} \frac{1}{k+1} (|\varphi_{k+1}(t_0^1)| +$$
$$+ |\varphi_{k+1}(t_0^{k+1}) - \varphi_{k+1}(t_0^1)|) + \sum_{k=1}^{m-1} \frac{1}{k} (|\varphi_k(t_0^1)| + |\varphi_k(t_{n_k}^k) - \varphi_k(t_0^1)|)$$
$$\leqslant \sum_{k=1}^{m} \frac{1}{k} \operatorname{Var}(\varphi_k) + \sum_{k=1}^{m-1} \frac{1}{k+1} |\varphi_{k+1}(t_0^1)| +$$
$$+ \sum_{k=1}^{m-1} \frac{1}{k+1} \operatorname{Var}(\varphi_{k+1}) + \sum_{k=1}^{m-1} \frac{1}{k} |\varphi_k(t_0^1)| + \sum_{k=1}^{m-1} \frac{1}{k} \operatorname{Var}(\varphi_k)$$
$$\leqslant 3 \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \cdot \sqrt{\sum_{k=1}^{\infty} (\operatorname{Var}(\varphi_k))^2} + 2 \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \cdot \sqrt{\sum_{k=1}^{\infty} (\varphi_k(t_0^1))^2}$$
$$\leqslant 3 \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \cdot \operatorname{Var}(\varphi) + 2 \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \cdot \|\varphi(t_0^1)\|.$$

Hence condition (I) is fulfilled and

$$\operatorname{Var}(N\varphi) \leqslant 3 \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \cdot \|\varphi\|.$$

If $\varphi, \psi \in \mathrm{BV}(l^2)$ then $(\varphi - \psi) \in \mathrm{BV}(l^2)$ and, in view of the last inequality we have

$$\operatorname{Var}(N\varphi - N\psi) = \operatorname{Var}(N(\varphi - \psi)) \leqslant 3 \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \cdot \|\varphi - \psi\|. \tag{8}$$

Since $\|G(t)\|_{\mathscr{L}(X,Y)} \leqslant 1$ $(t \in I)$, we obtain

$$\sup_{t \in I} |N\varphi(t) - N\psi(t)| = \sup_{t \in I} |G(t)(\varphi - \psi)(t)| \leqslant \sup_{t \in I} \|G(t)\|_{\mathscr{L}(X,Y)} \|(\varphi - \psi)(t)\|_{l^2}$$
$$\leqslant \sup_{t \in I} \|(\varphi - \psi)(t)\|_{l^2} \leqslant \|\varphi - \psi\|.$$

By inequality (8) we get

$$\|N\varphi - N\psi\| = \sup_{t \in I} |N\varphi(t) - N\psi(t)| + \operatorname{Var}(N\varphi - N\psi)$$
$$\leqslant \Big(3\sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} + 1\Big)\|\varphi - \psi\|,$$

which means that condition (II) is fulfilled. Thus we have shown that the Nemytskii operator generated by $h = h^*$ fulfils conditions (I), (II), (III).

For the increasing sequence $t_i = i+1$ $(i = 0, 1, \ldots, n)$, we have

$$\sum_{i=1}^{n} \|G(t_i) - G(t_{i-1})\| = \sum_{i=1}^{n} \|G(i+1) - G(i)\| \geqslant \sum_{i=1}^{n} |G(i+1)e_i - G(i)e_i| = \sum_{i=1}^{n} \frac{1}{i}$$

(where e_i is the i-th element of standard base of l^2) and, therefore, $\operatorname{Var}(G) = \infty$.

It is seen that this G in the example is not continuous. In this connection we have the following.

PROBLEM. Let $h(t, x) = G(t)x$ $(t \in I,\ x \in X,\ G\colon I \to \mathscr{L}(X, Y))$. Let G be continuous and let conditions (I), (II) be fulfilled.

Does G belong to $\mathrm{BV}(\mathscr{L}(X, Y))$?

References

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