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On the backward heat equation

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Abstract. Let $u(t)$ be a solution of the problem

$$u'(t)-\Delta u(t)=0 \quad \text{in } \Omega,\ t>0,$$
$$u(t)=0 \quad \text{on } \partial\Omega,\ t>0,$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ with a smooth boundary $\partial\Omega$. Suppose

$$\sum_{n=1}^{\infty} \lambda_n^s |(u(0), \varphi_n)|^2 \leqslant E^2 \quad \text{for an } s \geqslant 0$$

and

$$\|g-u(1)\| < \varepsilon,$$

where (φ_n) are the orthonormal eigenfunctions of $-\Delta$ in $H_0^1(\Omega)\cap H^2(\Omega)$ and (λ_n) are the corresponding eigenvalues. Here $\|\cdot\|$ is the L_2-norm. In the paper we construct, by truncated eigenfunction expansion, an approximate solution $u_\varepsilon(t)$, stable with respect to variations in g, such that

$$\|u_\varepsilon(t)-u(t)\| \leqslant (1+2^s)^{1/2} E^{1-t}\varepsilon^t(\log(E/\varepsilon))^{-s(1-t)/2}, \quad 0\leqslant t<1,$$

for small $\varepsilon>0$ if $s>0$ and for all $\varepsilon>0$ if $s=0$. The paper also shows how error estimates can be further improved by strengthening regularity conditions on $u(0)$.

Consider the problem

(1) $$u'(t)-\Delta u(t)=0 \quad \text{in } \Omega,\ t>0,$$

(2) $$u(t)=0 \quad \text{on } \partial\Omega,\ t>0,$$

(3) $$u(1)=g,$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ with a smooth boundary $\partial\Omega$, and g is a given element of $L_2=L_2(\Omega)$. As is well known, this is an ill-posed problem. Let $u(t)$ be a solution of (1)–(2), i.e., a mapping $t\to u(t)$ continuous from $t\geqslant 0$ to L_2, C^1 from $t>0$ to $H_0^1(\Omega)\cap H^2(\Omega)$, satisfying (1). In practice, $u(1)$ is known only approximately:

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(4) $$\|g-u(1)\| < \varepsilon \qquad (\|\cdot\| = L_2\text{-norm}),$$

and the problem is to find an approximate solution to the problem (1), (2), (4) that is stable with respect to variations in g. The problem has been studied extensively in recent years. In this note, we shall construct, by truncated eigenfunction expansion, a stabilized approximate solution $u_\varepsilon(t)$, and, by imposing certain regularity conditions on $u(0)$, we shall derive error estimates for $\|u(t)-u_\varepsilon(t)\|$. Comparisons will be made with results in the current literature.

Let (φ_n) be the L_2-orthonormal set of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)\cap H^2(\Omega)$ with the corresponding set of eigenvalues

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$$

Then we have

THEOREM 1. *Let $u(t)$ be a solution of* (1)–(2). *Let* (4) *hold.*

(i) *Suppose*

(5) $$\sum_{n=1}^{\infty} \lambda_n^s |(u(0), \varphi_n)|^2 \leqslant E^2 \quad \textit{for some } s \geqslant 0.$$

Let

(6) $$u_\varepsilon(t) = \sum_{n=1}^{N(\varepsilon)} (g, \varphi_n)\varphi_n \exp(\lambda_n(1-t)),$$

where

(7) $$N(\varepsilon) = \max\left\{n:\ \lambda_n \leqslant \log\left(\frac{E}{\varepsilon}\left(\log\frac{E}{\varepsilon}\right)^{-s/2}\right)\right\}$$

(*if* $\lambda_1 > \log\left(\frac{E}{\varepsilon}\left(\log\frac{E}{\varepsilon}\right)^{-s/2}\right)$, *$u_\varepsilon(t)$ is understood to be the null function*). *Then for $\varepsilon > 0$ sufficiently small, one has*

(8) $$\|u_\varepsilon(t)-u(t)\| \leqslant (1+2^s)^{1/2} E^{1-t}\varepsilon^t (\log(E/\varepsilon))^{-s(1-t)/2}, \quad 0 \leqslant t \leqslant 1.$$

(ii) *Suppose*

(9) $$\sum_{n=1}^{\infty} |(u(0), \varphi_n)|^2 \exp(2\delta\lambda_n) \leqslant E^2 \quad \textit{for a } \delta \geqslant 0.$$

Let $u_\varepsilon(t)$ be defined by (6) *with $N(\varepsilon)$ now given by*

(10) $$N(\varepsilon) = \max\left\{n:\ \lambda_n \leqslant \frac{1}{\delta+1}\log\frac{E}{\varepsilon}\right\}.$$

Then

(11) $$\|u_\varepsilon(t)-u(t)\| \leqslant \sqrt{2}E^{(1-t)/(1+\delta)}\varepsilon^{(t+\delta)/(\delta+1)}, \quad 0 \leqslant t < 1.$$

Proof. We prove (i) only. We have

(12) $$u(t) = \sum_{n=1}^{\infty} (u(0),\, \varphi_n)\varphi_n \exp(-\lambda_n t)$$
$$= \sum_{n=1}^{N(\varepsilon)} (u(1),\, \varphi_n)\varphi_n \exp(\lambda_n(1-t)) + \sum_{n=N(\varepsilon)+1}^{\infty} (u(0),\, \varphi_n)\varphi_n \exp(-\lambda_n t).$$

Hence, in view of (4)–(7) and Bessel's inequality,

$$\|u_\varepsilon(t)-u(t)\|^2 = \sum_{n=1}^{N(\varepsilon)} |(g-u(1),\, \varphi_n)|^2 \exp(2\lambda_n(1-t))$$
$$+ \sum_{n=N(\varepsilon)+1}^{\infty} |(u(0),\, \varphi_n)|^2 \exp(-2\lambda_n t)$$
$$\leqslant \|g-u(1)\|^2 \exp(2\lambda_{N(\varepsilon)}(1-t))$$
$$+ \lambda_{N(\varepsilon)+1}^{-s} \exp(-2\lambda_{N(\varepsilon)+1} t) \sum_{n=1}^{\infty} \lambda_n^s |(u(0),\, \varphi_n)|^2$$
$$\leqslant \varepsilon^2 \exp(2\lambda_{N(\varepsilon)}(1-t)) + E^2 \lambda_{N(\varepsilon)+1}^{-s} \exp(-2\lambda_{N(\varepsilon)+1} t).$$

By (7)

$$\exp(2\lambda_{N(\varepsilon)}(1-t)) \leqslant (E/\varepsilon)^{2(1-t)} (\log(E/\varepsilon))^{-s(1-t)},$$
$$\exp(-2\lambda_{N(\varepsilon)+1} t) \leqslant (\varepsilon/E)^{2t} (\log(E/\varepsilon))^{st}$$

and $\lambda_{N(\varepsilon)+1} > \frac{1}{2}\log(E/\varepsilon)$ for $\varepsilon > 0$ sufficiently small. Hence

$$\|u_\varepsilon(t)-u(t)\|^2 \leqslant (1+2^s) E^{2(1-t)} \varepsilon^{2t} (\log(E/\varepsilon))^{-s(1-t)}$$

for $\varepsilon > 0$ sufficiently small. This proves (i). The proof of (ii), which follows similar lines, is omitted.

Several remarks are in order, regarding Theorem 1 above. We take the case $s = 0$ in part (i), in which (5) reduces to

(5′) $$\|u(0)\| \leqslant E.$$

Under conditions (4) and (5′), Miller [7] using his modified quasireversibility method gave a stabilized approximate solution $u_\beta(t)$ satisfying the error estimate

(13) $$\|u_\beta(t)-u(t)\| \leqslant 2\varepsilon^t E^{1-t}, \quad 0 \leqslant t < 1.$$

Ewing, using the Sobolev equation, gave an approximate solution w_β with the error estimate [3]

(14) $$\|u(t)-w_\beta(t)\| \leqslant \frac{4(1-t)E}{t^2 \log(E/\varepsilon)} + \varepsilon^t E^{1-t}, \quad 0 < t < 1.$$

Ang [1(a)] converted the problem into one involving an integral equation of the first kind, and, by an appropriate regularization, gave a stabilized approximate solution $u_\beta(t)$ with the error estimate

$$\|u_\beta(t)-u(t)\| \leqslant 2\alpha^t \varepsilon^t E^{1-t}, \quad 0 \leqslant t < 1, \tag{15}$$

where $0 < \alpha(\varepsilon) < 1$ is such that $\alpha(\varepsilon) \to 1$ as $\varepsilon \to 0$.

The backward heat equation was considered in Lattes–Lions [6], using the quasireversibility method, but the problem of approximation on $0 \leqslant t < 1$ was not considered there. Other treatments of the backward heat equation can be found in Colton and Wimp [2], Franklin [4], Gajewski and Zaccharias [5], Payne [8]. In [1(b)], the nonlinear case was also considered.

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