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Koebe domains for univalent functions with real coefficients under Montel's normalization

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Abstract. Let $\bar{S}_c(r_0)$, $\bar{S}^*(r_0)$ and $\bar{S}^0(r_0)$ be the classes of all analytic and univalent functions $f(z)$, $z \in K_1$, $K_1 = [z\colon |z| < 1]$ with real coefficients, which are convex, starlike and convex in the direction of the imaginary axis respectively and such that $f(0) = 0$, $f(r_0) = r_0$, $0 \leqslant r_0 < 1$, $z \in K_1$.

In this note the Koebe domains for the classes $\bar{S}_c(r_0)$, $\bar{S}^*(r_0)$ and $\bar{S}^0(r_0)$ are given.

Let $\bar{S}(r_0)$ denote the set of all functions $f(z)$ analytic and univalent in the disc $K_1 = \{z\colon |z| < 1\}$, with real coefficients, and such that

$$f(0) = 0, \quad f(r_0) = r_0, \quad 0 \leqslant r_0 < 1. \tag{1}$$

The subclasses of functions, starlike functions with respect to the origin, and convex functions in the direction of the imaginary axis of the class $\bar{S}(r_0)$ will denoted by $\bar{S}_c(r_0)$, $\bar{S}^*(r_0)$ and $\bar{S}^0(r_0)$, respectively.

In this note we are going to determine the Koebe domains for the classes $\bar{S}_c(r_0)$, $\bar{S}^*(r_0)$, and $\bar{S}^0(r_0)$, i.e. the sets of the form $\bigcap f(K_1)$, where f runs through one of the classes. Let us observe that the domains $f(K_1)$ are symmetrical with respect to the real axis.

We now prove

THEOREM 1. *The set* $\bigcap_{f \in \bar{S}_c(r_0)} f(K_1) = \mathscr{K}[\bar{S}_c(r_0)]$ *is a convex set symmetrical with respect to the real axis and bounded by the curve given by the following parametric equations:*

$$\begin{aligned} u(a) &= \frac{r_0(1-r_0)^a}{(1+r_0)^a-(1-r_0)^a}\left[\frac{1}{\pi}\left(\ln\frac{1+r_0}{1-r_0}\right)\frac{(1+r_0)^a \sin a\pi}{(1+r_0)^a-(1-r_0)^a}-1\right], \\ v(a) &= \pm\frac{1}{\pi}(1-\cos a\pi)\frac{r_0(1-r_0^2)^a}{[(1+r_0)^a-(1-r_0)^a]^2}\ln\frac{1+r_0}{1-r_0}, \quad a \in [0,1], \end{aligned} \tag{2}$$

and

$$u(a) = \frac{r_0(1+r_0)^a}{(1+r_0)^a-(1-r_0)^a}\left[-\frac{1}{\pi}\left(\ln\frac{1+r_0}{1-r_0}\right)\frac{(1-r_0)^a\sin a\pi}{(1+r_0)^a-(1-r_0)^a}+1\right],$$

$$v(a) = \pm\frac{1}{\pi}(1-\cos a\pi)\frac{r_0(1-r_0)^a}{[(1+r_0)^a-(1-r_0)^a]^2}\ln\frac{1+r_0}{1-r_0}, \qquad a\in[0,1],$$

together with the straight line segments

$$u = -\frac{1-r_0}{2}, \qquad -\frac{1-r_0^2}{\pi}\ln\frac{1+r_0}{1-r_0}\leqslant v\leqslant\frac{1-r_0^2}{\pi}\ln\frac{1+r_0}{1-r_0}$$

and

$$u = \frac{1+r_0}{2}, \qquad -\frac{1-r_0^2}{\pi}\ln\frac{1+r_0}{1-r_0}\leqslant v\leqslant\frac{1-r_0^2}{\pi}\ln\frac{1+r_0}{1-r_0}.$$

The above-mentioned domain is also symmetrical with respect to the line $u = r_0/2$. Moreover, we have

$$u(0) = r_0/2, \qquad v(0) = \frac{\pm\pi r_0}{2\ln\dfrac{1+r_0}{1-r_0}}.$$

COROLLARY 1. *With $r_0\to 0$, the class $\bar{S}_c(r_0)$ reduces to the class $\bar{S}_c(0)$ of convex functions with real coefficients, such that $f(0) = 0$ and $f'(0) = 1$, whereas the domain $\mathscr{K}[\bar{S}_c(r_0)]$ reduces to the one* (1).

Proof. First, we observe that any domain $f(K_1)$ convex and symmetrical with respect to the real axis, where $f\in\bar{S}_c(r_0)$, can be represented as the intersection of a collection of convex "angle-domains" which are also symmetrical with respect to the real axis and are such that each of them contains the points 0 and r_0, i.e. domains which are convex angles containing the points 0, r_0 and the real axis as their the bisector.

Let D_a, $0<a\leqslant 1$, denote one of those angle-domains of angle $a\pi$ and let the function $w = F(z)$ be analytic and univalent and such that $F(0) = 0$ and $F(K_1) = D_a$. Since $f(K_1)\subset F(K_1)$, i.e. $f(z) = F(\omega(z))$, where the function $\omega(z)$ is univalent and satisfies the conditions of the Schwarz Lemma. From the principle of subordination it follows that $r_0 = f(r_0) = F(\omega(r_0)) < F(r_0)$, i.e. $r_0/F(r_0) < 1$, while $\omega(z)\neq z$.

Normalizing $F(z)$ so as to obtain $\Phi(z) = [r_0F(z)]/F(r_0)$, we see that the function $\Phi(z)$ maps K_1 onto the contracted and normalized angle-domain $\bar{D}_a$, where $\bar{D}_a$ is a normalizing angle-domain of angle $a\pi$, $0<a\leqslant 1$.

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(1) M. T. McGregor, *On the three classes of univalent functions with real coefficients*, J. London Math. Soc. 39 (1964), p. 43–50.

Taking into account that for every function $f \in \bar{S}_c(r_0)$ there exists a maximal set $I_f \subset (0, 1]$ such that $f(K_1) = \bigcap_{a \in I_f} D_a$, we have

$$\bigcap_{f \in S_c(r_0)} f(K_1) = \bigcap_{a} D_a \supseteq \bigcap_{a} \bar{D}_a, \tag{3}$$

where $\bigcup_f I_f = (0, 1]$.

But any $\bar{D}_a$ is the image of K_1 by a function $f \in \bar{S}_c(r_0)$, so

$$\bigcap_{a} \bar{D}_a \supseteq \bigcap_{f \in \bar{S}_c(r_0)} f(K_1). \tag{4}$$

Relations (3) and (4) allow us to write

$$\mathscr{K}[\bar{S}_c(r_0)] = \bigcap_{f \in \bar{S}_c(r_0)} f(K_1) = \bigcap_{a} \bar{D}_a. \tag{5}$$

Therefore, to determine the Koebe domain for $\bar{S}_c(r_0)$ it is enough to establish the intersection $\bigcap_{a \in (0,1]} \bar{D}_a$ of the normalized angle-domains.

Every function $F(z)$ which maps K_1 onto a convex angle $a\pi$ and is symmetrical with respect to the real axis has the form

$$F(z) = a\left[\left(\frac{1 \pm z}{1 \mp z}\right)^a - 1\right], \quad a \in (0, 1], \tag{6}$$

where $a > 0$.

For the function $F(z)$ which is a majorant of a function $f(z) \in \bar{S}(r_0)$ the following condition must hold

$$r_0 < a\left[\left(\frac{1 \pm r_0}{1 \mp r_0}\right)^a - 1\right]. \tag{7}$$

By (6), every function $\Phi(z)$ such that $\Phi(K_1) = \bar{D}_a$, $\Phi(0) = 0$ and $\Phi(r_0) = r_0$ has the form

$$w = \Phi(z) = r_0 \frac{a\left[\left(\frac{1 \pm z}{1 \mp z}\right)^a - 1\right]}{a\left[\left(\frac{1 \pm r_0}{1 \mp r_0}\right)^a - 1\right]} = r_0 \frac{\left[\left(\frac{1 \pm z}{1 \mp z}\right)^a - 1\right]}{\left[\left(\frac{1 \pm r_0}{1 \mp r_0}\right)^a - 1\right]}. \tag{8}$$

Now we are going to determine $\bigcap_a \bar{D}_a$. This is equivalent to finding the envelope of the boundary lines of all such domains. Let $W = u + iv$.

First we find the part of the envelope of the boundary $\bar{D}_a$ which lies in the upper half-plane, then we shall obtain the other part by the reflection of the previous part in the real axis. For the angle-domains $\bar{D}_a$

having the corners on the left of 0 we find the envelope of the lines $w = \Phi(e^{i\theta})$, $\theta \in [0, 2\pi]$, taking the upper sings in (8) and $a \in (0, 1]$. Similarly, for the angle-domains $\bar{D}_a$ which have corners on the right of r_0 we find the envelope of the lines $w = \Phi(e^{i\theta})$, $0 \leqslant \theta \leqslant 2\pi$, taking the lower sings in (8) and $a \in (0, 1]$.

The equations of those lines are the following:

(9)

$$v = \tan\frac{a\pi}{2}\left[u + \frac{r_0}{\left(\frac{1+r_0}{1-r_0}\right)^a - 1}\right], \quad v = -\tan\frac{a\pi}{2}\left[u + \frac{r_0}{\left(\frac{1-r_0}{1+r_0}\right)^a - 1}\right].$$

Hence, by the well-known procedure, we obtain (2).

Analogously one can prove the following theorems:

THEOREM 2. *The boundary of the set* $\mathscr{K}[\bar{S}^*(r_0)] = \bigcap_{f \in \bar{S}^*(r_0)} f(K_1)$ *is the curve given by the equations*

$$u(a) = -\tfrac{1}{4}(1+r_0)^{2a}(1-r_0)^{2(1-a)}a^{-a}(1-a)^{-(1-a)}\cos a\pi,$$
$$v(a) = \pm\tfrac{1}{4}(1+r_0)^{2a}(1-r_0)^{2(1-a)}a^{-a}(1-a)\sin a\pi, \quad a \in [0, 1],$$

where

$$u(0) = -\tfrac{1}{4}(1-r_0)^2, \quad v(0) = 0,$$

and

$$u(1) = \tfrac{1}{4}(1+r_0)^2, \quad v(1) = 0.$$

THEOREM 3. *The boundary of the set* $\mathscr{K}[\bar{S}^0(r_0)] = \bigcap_{f \in \bar{S}^0(r_0)} f(K_1)$ *is the ellipse*

$$\frac{(u - \frac{1}{2}r_0)^2}{(\frac{1}{2})^2} + \frac{v^2}{\left(\frac{\sqrt{1-r_0^2}}{2}\right)^2} = 1.$$

Putting $r_0 \to 0$, *we obtain the Koebe domains for the class* $\bar{S}^*(0)$ *and* $\bar{S}^0(0)$ (1).

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