ANNALES
POLONICI MATHEMATICI
XX (1968)

On some linear operators defined by double integrals

by Paulina Pych (Poznań)

1. Preliminaries. Let M be the class of all functions $f(x,y)$ 2π-periodic, measurable and bounded in the square $[-\pi,\pi;-\pi,\pi]$.

We shall need the following differences and moduli of smoothness of $f(x,y)\in M$:

$$\Delta_{h,k}f(x,y)=f(x+h,y+k)-f(x,y),$$

$$\Delta_{h,k}^{(1)}f(x,y)=f(x+h,y+k)-f(x-h,y+k)-f(x+h,y-k)+\\+f(x-h,y-k),$$

$$\Delta_{h,k}^{(2)}f(x,y)=f(x+h,y+k)+f(x-h,y+k)+f(x+h,y-k)+\\+f(x-h,y-k)-4f(x,y),$$

$$\omega(s,t)=\omega(s,t;f)=\sup_{|h|\leqslant s,|k|\leqslant t}\{\sup_{|x|,|y|\leqslant\pi}|\Delta_{h,k}f(x,y)|\},$$

$$\omega_1(s,t)=\omega_1(s,t;f)=\sup_{|h|\leqslant s,|k|\leqslant t}\{\sup_{|x|,|y|\leqslant\pi}|\Delta_{h,k}^{(1)}f(x,y)|\},$$

$$\omega_2(s,t)=\omega_2(s,t;f)=\sup_{|h|\leqslant s,|k|\leqslant t}\{\sup_{|x|,|y|\leqslant\pi}|\Delta_{h,k}^{(2)}f(x,y)|\}.$$

Notice that for arbitrary non-negative numbers a,b,s,t,

$$\omega(as,bt)\leqslant(1+a+b)\,\omega(s,t),\tag{1}$$

$$\omega_1(as,bt)\leqslant(1+a)(1+b)\,\omega_1(s,t),\tag{2}$$

$$\omega_2(as,bt)\leqslant\{(1+a)^2+(1+b)^2\}\,\omega_2(s,t).\tag{3}$$

Inequality (1) is known. To prove (2) we observe that, for non-negative integers m, n, the identity

$$\Delta_{mh,nk}^{(1)}f(x,y)=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{h,k}^{(1)}f(x-mh+h+2ih,\,y-nk+k+2jk)$$

implies

$$\omega_1(ms,nt)\leqslant mn\,\omega_1(s,t),$$

whence the desired inequality follows. By

$$\Delta^{(2)}_{mh,nk}f(x,y) = \tfrac{1}{2}\Delta^{(2)}_{mh,0}f(x,y-nk)+\tfrac{1}{2}\Delta^{(2)}_{mh,0}f(x,y+nk)+\Delta^{(2)}_{0,nk}f(x,y)$$

$$= \tfrac{1}{2}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\Delta^{(2)}_{h,0}f(x-mh+h+ih+jh,\, y-nk)+$$

$$+\tfrac{1}{2}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\Delta^{(2)}_{h,0}f(x-mh+h+ih+jh,\, y+nk)+$$

$$+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Delta^{(2)}_{0,k}f(x,\, y-nk+k+ik+jk)\,,$$

we obtain

$$\omega_2(ms,nt) \leqslant m^2\omega_2(s,0)+n^2\omega_2(0,t) \leqslant (m^2+n^2)\,\omega_2(s,t)\,.$$

Thus estimate (3) is now evident.

Denote by $\Pi(a,\beta)$ and $Z(a,\beta)$ the classes of all functions $f(x,y)\in M$ such that

$$\omega_1(s,t;f) \leqslant (2s)^a(2t)^\beta \qquad (0<a,\beta\leqslant 1)\,,$$

$$\omega_2(s,t;f) \leqslant 4(s^a+t^\beta) \qquad (0<a,\beta\leqslant 2)\,,$$

for each $s,t\geqslant 0$, respectively.

It can easily be verified that if

$$\sup_x|\varphi(x+h)-\varphi(x)| \leqslant h^a\,, \qquad \sup_y|\psi(y+k)-\psi(y)| \leqslant k^\beta$$

for $h,k\geqslant 0$, then

$$\varphi(x)\psi(y)\in\Pi(a,\beta);$$

moreover, the inequalities

$$\sup_x|\varphi(x+h)-2\varphi(x)+\varphi(x-h)| \leqslant 2h^a\,,$$

$$\sup_y|\psi(y+k)-2\psi(y)+\psi(y-k)| \leqslant 2k^\beta\,,$$

for $h,k\geqslant 0$, imply

$$\varphi(x)+\psi(y)\in Z(a,\beta)\,.$$

In this paper we present some results concerning the order of convergence of certain linear operators defined in M and give the estimates for their partial derivatives. The symbols $C_1, C_2, \ldots$ signify positive constants.

2. Convergence problems. Let E be a subset of the Euclidean plane, having an accumulation-point (ξ_0,η_0). Suppose that functions

$K_1(s;\xi), K_2(t;\eta)$ are even, 2π-periodic and continuous with respect to the variables s, t, and that they satisfy the conditions

$$\int_{-\pi}^{\pi} K_1(s;\xi)\,ds = 1 = \int_{-\pi}^{\pi} K_2(t;\eta)\,dt ,$$

for every fixed $(\xi,\eta)\in E$.

Write

$$\Phi(s,t;\xi,\eta) = K_1(s;\xi)K_2(t;\eta) , \tag{4}$$

and let us consider the operators

$$f(x,y;\xi,\eta) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s,t)\Phi(s-x,t-y;\xi,\eta)\,ds\,dt \tag{5}$$

linear in M.

Then the following theorem holds:

2.1. *Let*

$$\int_0^{\pi} |K_1(s;\xi)|\,ds \leqslant C_1 , \qquad \int_0^{\pi} s^2|K_1(s;\xi)|\,ds \leqslant C_2\lambda(\xi) ,$$

$$\int_0^{\pi} |K_2(t;\eta)|\,dt \leqslant C_3 , \qquad \int_0^{\pi} t^2|K_2(t;\eta)|\,dt \leqslant C_4\mu(\eta) ,$$

for $(\xi,\eta)\in E$. Then, given any $f(x,y)\in M$ with the modulus of smoothness $\omega_2(s,t)$,

$$\sup_{|x|,\,|y|\leqslant\pi} |f(x,y;\xi,\eta)-f(x,y)| \leqslant C_5\,\omega_2\big(\sqrt{\lambda(\xi)},\sqrt{\mu(\eta)}\big) \qquad \textit{in } E ,$$

where $C_5 = C_1(\sqrt{C_3}+\sqrt{C_4})^2 + C_3(\sqrt{C_1}+\sqrt{C_2})^2$.

Proof. Start with the identity

$$f(x,y;\xi,\eta)-f(x,y) = \int_0^{\pi}\int_0^{\pi} \{\Delta_{s,t}^{(2)} f(x,y)\}\Phi(s,t;\xi,\eta)\,ds\,dt . \tag{6}$$

Applying (3) and our assumptions, we get

$$\begin{aligned}
&|f(x,y;\xi,\eta)-f(x,y)|\\
&\quad\leqslant \int_0^{\pi}\int_0^{\pi} \omega_2(s,t)\,|\Phi(s,t;\xi,\eta)|\,ds\,dt\\
&\quad\leqslant \omega_2\big(\sqrt{\lambda(\xi)},\sqrt{\mu(\eta)}\big)\int_0^{\pi}\int_0^{\pi}\left\{\left(1+\frac{s}{\sqrt{\lambda(\xi)}}\right)^2+\left(1+\frac{t}{\sqrt{\mu(\eta)}}\right)^2\right\}|\Phi(s,t;\xi,\eta)|\,ds\,dt\\
&\quad\leqslant \omega_2\big(\sqrt{\lambda(\xi)},\sqrt{\mu(\eta)}\big)\times\\
&\qquad\times\left\{C_3\int_0^{\pi}\left(1+\frac{s}{\sqrt{\lambda(\xi)}}\right)^2|K_1(s;\xi)|\,ds + C_1\int_0^{\pi}\left(1+\frac{t}{\sqrt{\mu(\eta)}}\right)^2|K_2(t;\eta)|\,dt\right\} .
\end{aligned}$$

By Schwarz' inequality,

$$\int_0^\pi s\,|K_1(s;\,\xi)|\,ds \leqslant \sqrt{C_1C_2\lambda(\xi)}\,, \qquad \int_0^\pi t\,|K_2(t;\,\eta)|\,dt \leqslant \sqrt{C_3C_4\mu(\eta)}\,.$$

Consequently,

$$|f(x,y;\,\xi,\eta)-f(x,y)| \leqslant \omega_2\big(\sqrt{\lambda(\xi)},\sqrt{\mu(\eta)}\big)\big\{C_3\big(C_1+2\sqrt{C_1C_2}+C_2\big)+C_1\big(C_3+2\sqrt{C_3C_4}+C_4\big)\big\}\,,$$

whence the assertion follows.

Suppose that $K_1(s;\,\xi)$, $K_2(t;\,\eta)$ are positive for $0 \leqslant s,t \leqslant \pi$, $(\xi,\eta)\in E$, and write

$$u_\alpha(\xi) = \int_0^\pi \Big(2\sin\frac{s}{2}\Big)^\alpha K_1(s;\,\xi)\,ds\,, \qquad v_\beta(\eta) = \int_0^\pi \Big(2\sin\frac{t}{2}\Big)^\beta K_2(t;\,\eta)\,dt\,,$$

$$U_{\alpha,\beta}(\xi,\eta) = \sup_{f\in Z(\alpha,\beta)}\{\sup_{|x|,|y|\leqslant\pi}|f(x,y;\,\xi,\eta)-f(x,y)|\} \qquad (0<\alpha,\beta\leqslant 2)\,.$$

Under these assumptions we have the following theorem:

2.2. *If*

(i) $$\lim_{\xi\to\xi_0}\{u_{2\alpha}(\xi)/u_\alpha(\xi)\} = 0 = \lim_{\eta\to\eta_0}\{v_{2\beta}(\eta)/v_\beta(\eta)\}\,,$$

then

$$U_{\alpha,\beta}(\xi,\eta) = 2\{u_\alpha(\xi)+v_\beta(\eta)\}\big(1+o(1)\big)$$

as $(\xi,\eta)\to(\xi_0,\eta_0)$.

Proof. By (6),

$$\sup_{|x|,|y|\leqslant\pi}|f(x,y;\,\xi,\eta)-f(x,y)| \leqslant 4\int_0^\pi\int_0^\pi (s^\alpha+t^\beta)\Phi(s,t;\,\xi,\eta)\,ds\,dt$$

for any $f\in Z(\alpha,\beta)$. Therefore

$$\begin{aligned}(7)\quad U_{\alpha,\beta}(\xi,\eta) &\leqslant 2\Big\{\int_0^\pi s^\alpha K_1(s;\,\xi)\,ds+\int_0^\pi t^\beta K_2(t;\,\eta)\,dt\Big\}\\ &= 2\Big\{\int_0^\pi \Big(2\sin\frac{s}{2}\Big)^\alpha K_1(s;\,\xi)\,ds+2^\alpha\int_0^\pi\Big[\Big(\frac{s}{2}\Big)^\alpha-\sin^\alpha\frac{s}{2}\Big]K_1(s;\,\xi)\,ds\Big\}+\\ &\quad+2\Big\{\int_0^\pi \Big(2\sin\frac{t}{2}\Big)^\beta K_2(t;\,\eta)\,dt+2^\beta\int_0^\pi\Big[\Big(\frac{t}{2}\Big)^\beta-\sin^\beta\frac{t}{2}\Big]K_2(t;\,\eta)\,dt\Big\}\\ &= 2\{u_\alpha(\xi)+2^\alpha I_1\}+2\{v_\beta(\eta)+2^\beta I_2\}\,.\end{aligned}$$

The function

$$g(x, y) = \left|2\sin\frac{x}{2}\right|^{\alpha} + \left|2\sin\frac{y}{2}\right|^{\beta} \qquad (0 < \alpha, \beta \leqslant 2)$$

is of class $Z(\alpha, \beta)$, and

$$g(0, 0; \xi, \eta) - g(0, 0) = 4\int_0^{\pi}\int_0^{\pi}\left(2^{\alpha}\sin^{\alpha}\frac{s}{2} + 2^{\beta}\sin^{\beta}\frac{t}{2}\right)\Phi(s, t; \xi, \eta)\,ds\,dt$$
$$= 2\{u_{\alpha}(\xi) + v_{\beta}(\eta)\};$$

whence

$$U_{\alpha,\beta}(\xi, \eta) \geqslant 2\{u_{\alpha}(\xi) + v_{\beta}(\eta)\}. \tag{8}$$

Inequalities (7) and (8) lead to

$$U_{\alpha,\beta}(\xi, \eta) = 2\{u_{\alpha}(\xi) + 2^{\alpha}\theta_{\alpha}(\xi)I_1\} + 2\{v_{\beta}(\eta) + 2^{\beta}\theta_{\beta}(\eta)I_2\},$$

where

$$0 \leqslant \theta_{\alpha}(\xi) \leqslant 1, \qquad 0 \leqslant \theta_{\beta}(\eta) \leqslant 1.$$

Observing that

$$t^{\alpha} - \sin^{\alpha} t \leqslant \left(\frac{t^4}{3}\right)^{\alpha/2} \leqslant \frac{1}{3^{\alpha/2}}\left(\frac{\pi}{2}\right)^{2\alpha}\sin^{2\alpha} t \qquad (0 \leqslant t \leqslant \pi/2),$$

and taking into account (i), we get

$$I_1 = o\{u_{\alpha}(\xi)\}, \qquad I_2 = o\{v_{\beta}(\eta)\}.$$

Thus, the proof is completed.

Suppose now that the functions $\bar{K}_1(s; \xi)$, $\bar{K}_2(t; \eta)$ are odd, 2π-periodic and continuous in s, t, for every fixed $(\xi, \eta) \in E$. We shall confine ourselves to $f(x, y) \in \Pi(\alpha, \beta)$; let us define the linear operators and the conjugate function by the formulae

$$\bar{f}(x, y; \xi, \eta) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s, t)\bar{\Phi}(s-x, t-y; \xi, \eta)\,ds\,dt$$
$$= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\{\Delta_{s,t}^{(1)} f(x, y)\}\bar{\Phi}(s, t; \xi, \eta)\,ds\,dt,$$

and

$$\bar{f}(x, y) = \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi}\{\Delta_{s,t}^{(1)} f(x, y)\}\cot\frac{s}{2}\cot\frac{t}{2}\,ds\,dt,$$

with

$$\bar{\Phi}(s, t; \xi, \eta) = \bar{K}_1(s; \xi)\,\bar{K}_2(t; \eta).$$

Write

$$\Psi(s, t; \xi, \eta) = \cot\frac{s}{2}\cot\frac{t}{2} - \bar{\Phi}(s, t; \xi, \eta),$$

$$Q_1(s; \xi) = \cot\frac{s}{2} - \bar{K}_1(s; \xi), \quad Q_2(t; \eta) = \cot\frac{t}{2} - \bar{K}_2(t; \eta).$$

It can easily be observed that

$$\text{(9)} \quad \Psi(s, t; \xi, \eta) = Q_1(s; \xi)Q_2(t; \eta) + \bar{K}_1(s; \xi)Q_2(t; \eta) + \bar{K}_2(t; \eta)Q_1(s; \xi)$$
$$= \cot\frac{s}{2}Q_2(t; \eta) + \cot\frac{t}{2}Q_1(s; \xi) - Q_1(s; \xi)Q_2(t; \eta).$$

The theorem to follow will deal with the asymptotic behaviour of the quantity

$$V_{\alpha,\beta}(\xi, \eta) = \sup_{f\in \Pi(\alpha,\beta)} \{\sup_{|x|,|y|\leqslant\pi} |\bar{f}(x, y; \xi, \eta) - \bar{f}(x, y)|\} \quad (0 < \alpha, \beta \leqslant 1)$$

as $(\xi, \eta)\to(\xi_0, \eta_0)$. Let us write

$$\gamma_\alpha^{(1)}(\xi) = \int_0^{\pi/2} s^\alpha Q_1(s; \xi)\,ds, \quad \varrho_\alpha^{(1)}(\xi) = \int_{\pi/2}^{\pi} (\pi - s)^\alpha Q_1(s; \xi)\,ds,$$

$$\gamma_\beta^{(2)}(\eta) = \int_0^{\pi/2} t^\beta Q_2(t; \eta)\,dt, \quad \varrho_\beta^{(2)}(\eta) = \int_{\pi/2}^{\pi} (\pi - t)^\beta Q_2(t; \eta)\,dt.$$

2.3. *If*

$$\text{(a)} \quad \Psi(s, t; \xi, \eta) \geqslant 0 \quad \textit{for} \quad 0 < s, t \leqslant \pi, \ (\xi, \eta) \in E,$$

$$\text{(b)} \quad \lim_{\xi\to\xi_0} \gamma_\alpha^{(1)}(\xi) = 0 = \lim_{\eta\to\eta_0} \gamma_\beta^{(2)}(\eta),$$

$$\text{(c)} \quad \lim_{\xi\to\xi_0} \{\varrho_\alpha^{(1)}(\xi)/\gamma_\alpha^{(1)}(\xi)\} = 0 = \lim_{\eta\to\eta_0} \{\varrho_\beta^{(2)}(\eta)/\gamma_\beta^{(2)}(\eta)\},$$

then

$$V_{\alpha,\beta}(\xi, \eta) = \frac{1}{\pi^2}2^{\alpha+\beta-1}\left\{\int_0^{\pi/2} \frac{t^\beta}{\sin t}\,dt\,\gamma_\alpha^{(1)}(\xi) + \int_0^{\pi/2} \frac{s^\alpha}{\sin s}\,ds\,\gamma_\beta^{(2)}(\eta)\right\}\big(1 + o(1)\big)$$

as $(\xi, \eta)\to(\xi_0, \eta_0)$ *on* E.

Proof. Clearly,

$$\bar{f}(x, y) - \bar{f}(x, y; \xi, \eta) = \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi \{\Delta_{s,t}^{(1)} f(x, y)\}\Psi(s, t; \xi, \eta)\,ds\,dt$$

$$= \frac{1}{4\pi^2}\left(\int_0^{\pi/2}\int_0^{\pi/2} \Delta_{s,t}^{(1)} f(x, y)\Psi(s, t; \xi, \eta)\,ds\,dt + \right.$$

$$+\int_{\pi/2}^{\pi}\int_{\pi/2}^{\pi} \Delta^{(1)}_{\pi-s,\pi-t} f(x+\pi, y+\pi)\Psi(s,t;\xi,\eta)\,ds\,dt\Big)-$$

$$-\frac{1}{4\pi^2}\Big(\int_{\pi/2}^{\pi}\int_{0}^{\pi/2} \Delta^{(1)}_{s,\pi-t} f(x, y+\pi)\Psi(s,t;\xi,\eta)\,ds\,dt+$$

$$+\int_{0}^{\pi/2}\int_{\pi/2}^{\pi} \Delta^{(1)}_{\pi-s,t} f(x+\pi, y)\Psi(s,t;\xi,\eta)\,ds\,dt\Big)$$

and, consequently,

$$V_{\alpha,\beta}(\xi,\eta)\leqslant\frac{2^{\alpha+\beta}}{4\pi^2}\Big(\int_{0}^{\pi/2}\int_{0}^{\pi/2} s^\alpha t^\beta\Psi(s,t;\xi,\eta)\,ds\,dt+ \tag{10}$$

$$+\int_{\pi/2}^{\pi}\int_{\pi/2}^{\pi}(\pi-s)^\alpha(\pi-t)^\beta\Psi(s,t;\xi,\eta)\,ds\,dt\Big)+$$

$$+\frac{2^{\alpha+\beta}}{4\pi^2}\Big(\int_{\pi/2}^{\pi}\int_{0}^{\pi/2} s^\alpha(\pi-t)^\beta\Psi(s,t;\xi,\eta)\,ds\,dt+$$

$$+\int_{0}^{\pi/2}\int_{\pi/2}^{\pi}(\pi-s)^\alpha t^\beta\Psi(s,t;\xi,\eta)\,ds\,dt\Big).$$

Let $g(x,y)$ be 2π-periodic, odd in each variable separately, and such that

$$g(x,y)=\begin{cases} 2^{\alpha+\beta-2}x^\alpha y^\beta & \text{for}\quad 0\leqslant x\leqslant\pi/2,\quad 0\leqslant y\leqslant\pi/2,\\ 2^{\alpha+\beta-2}x^\alpha(\pi-y)^\beta & \text{for}\quad 0\leqslant x\leqslant\pi/2,\ \pi/2\leqslant y\leqslant\pi,\\ 2^{\alpha+\beta-2}(\pi-x)^\alpha y^\beta & \text{for}\quad \pi/2\leqslant x\leqslant\pi,\quad 0\leqslant y\leqslant\pi/2,\\ 2^{\alpha+\beta-2}(\pi-x)^\alpha(\pi-y)^\beta & \text{for}\quad \pi/2\leqslant x\leqslant\pi,\quad \pi/2\leqslant y\leqslant\pi.\end{cases}$$

This function is continuous everywhere, $g(x,y)\in\Pi(\alpha,\beta)$, and

(11) $\bar{g}(0,0)-\bar{g}(0,0;\xi,\eta)$

$$=\frac{1}{\pi^2}2^{\alpha+\beta-2}\Big(\int_{0}^{\pi/2}\int_{0}^{\pi/2} s^\alpha t^\beta\Psi(s,t;\xi,\eta)\,ds\,dt+\int_{\pi/2}^{\pi}\int_{\pi/2}^{\pi}(\pi-s)^\alpha(\pi-t)^\beta\Psi(s,t;\xi,\eta)\,ds\,dt\Big)+$$

$$+\frac{1}{\pi^2}2^{\alpha+\beta-2}\Big(\int_{\pi/2}^{\pi}\int_{0}^{\pi/2} s^\alpha(\pi-t)^\beta\Psi(s,t;\xi,\eta)\,ds\,dt+\int_{0}^{\pi/2}\int_{\pi/2}^{\pi}(\pi-s)^\alpha t^\beta\Psi(s,t;\xi,\eta)\,ds\,dt\Big).$$

By (10), (11) and (9),

$$V_{\alpha,\beta}(\xi,\eta) = \bar{g}(0,0) - \bar{g}(0,0;\xi,\eta)$$

$$= \frac{2^{\alpha+\beta}}{4\pi^2}\Big\{2\int_0^{\pi/2} \frac{t^\beta}{\sin t} dt\big(\gamma_\alpha^{(1)}(\xi)+\varrho_\alpha^{(1)}(\xi)\big) + 2\int_0^{\pi/2} \frac{s^\alpha}{\sin s} ds\big(\gamma_\beta^{(2)}(\eta)+\varrho_\beta^{(2)}(\eta)\big) -$$

$$-\big(\gamma_\alpha^{(1)}(\xi)+\varrho_\alpha^{(1)}(\xi)\big)\big(\gamma_\beta^{(2)}(\eta)+\varrho_\beta^{(2)}(\eta)\big)\Big\}.$$

Applying (b), (c), we complete the proof.

Remark. The first identity of (9) shows that condition (a) is fulfilled if $\bar{K}_1(s;\xi)$, $\bar{K}_2(t;\eta)$, $Q_1(s;\eta)$ and $Q_2(t;\eta)$ are non-negative for $0 \leqslant s, t \leqslant \pi$, $(\xi,\eta) \in E$.

3. Estimates for the derivatives. Retain the symbol E used in § 2. Suppose that $K_1(s;\xi)$, $K_2(t;\eta)$ are even, 2π-periodic and continuous in s, t, for every fixed $(\xi,\eta) \in E$. Denote by $\omega(s,t)$, $\omega_1(s,t)$, $\omega_2(s,t)$ the moduli of smoothness of $f(x,y) \in M$. Consider again operators (5), with Φ defined by (4).

3.1. *Let $\partial K_1(s;\xi)/\partial s$ be continuous with respect to s and let*

$$\int_0^\pi \left|\frac{\partial K_1(s;\xi)}{\partial s}\right| ds \leqslant \frac{C_6}{\delta_1(\xi)}, \quad \int_0^\pi s\left|\frac{\partial K_1(s;\xi)}{\partial s}\right| ds \leqslant C_7, \quad \int_0^\pi |K_2(t;\eta)|\,dt \leqslant C_8,$$

for $(\xi,\eta) \in E$. Then

$$\sup_{|x|,|y|\leqslant\pi} \left|\frac{\partial f(x,y;\xi,\eta)}{\partial x}\right| \leqslant C_9 \frac{\omega(\delta_1(\xi),0)}{\delta_1(\xi)} \quad in\ E,$$

where $C_9 = 4(C_6+C_7)C_8$.

Proof. Clearly,

$$\frac{\partial f(x,y;\xi,\eta)}{\partial x} = -\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+s,y+t)\frac{\partial K_1(s;\xi)}{\partial s}K_2(t;\eta)\,ds\,dt$$

$$= -\int_0^\pi\int_0^\pi \{f(x+s,y+t)-f(x-s,y+t)+f(x+s,y-t)-$$

$$-f(x-s,y-t)\}\frac{\partial K_1(s;\xi)}{\partial s}K_2(t;\eta)\,ds\,dt.$$

Applying (1), we get

$$|f(x+s,y+t)-f(x-s,y+t)+f(x+s,y-t)-f(x-s,y-t)|$$

$$\leqslant |\Delta_{2s,0}f(x-s,y+t)| + |\Delta_{2s,0}f(x-s,y-t)| \leqslant 2\sup_{|x|,|y|\leqslant\pi}|\Delta_{2s,0}f(x,y)|$$

$$\leqslant 2\omega(2s,0) \leqslant 4\omega(s,0) \leqslant 4\Big(1+\frac{s}{\delta_1(\xi)}\Big)\omega\big(\delta_1(\xi),0\big).$$

Therefore, the derivative $\partial f(x, y;\ \xi, \eta)/\partial x$ does not exceed

$$4\omega\big(\delta_1(\xi), 0\big) \int_0^\pi \left(1+\frac{s}{\delta_1(\xi)}\right)\left|\frac{\partial K_1(s;\ \xi)}{\partial s}\right| ds \cdot \int_0^\pi |K_2(t;\ \eta)|\, dt$$

in absolute value.

The conclusion is now evident.

3.2. *Suppose that $\partial K_1(s;\ \xi)/\partial s$, $\partial K_2(t;\ \eta)/\partial t$ are continuous in s, t and that the first two inequalities of* 3.1 *together with*

$$\int_0^\pi \left|\frac{\partial K_2(t;\ \eta)}{\partial t}\right| dt \leqslant \frac{C_{10}}{\delta_2(\eta)}, \qquad \int_0^\pi t\left|\frac{\partial K_2(t;\ \eta)}{\partial t}\right| dt \leqslant C_{11},$$

hold for every $(\xi, \eta) \in E$. Then

$$\sup_{|x|,|y|\leqslant\pi} \left|\frac{\partial^2 f(x, y;\ \xi, \eta)}{\partial x \partial y}\right| \leqslant C_{12} \frac{\omega_1\big(\delta_1(\xi), \delta_2(\eta)\big)}{\delta_1(\xi)\,\delta_2(\eta)} \quad \text{in } E,$$

where $C_{12} = (C_6 + C_7)(C_{10} + C_{11})$.

Proof. Observing that

$$\frac{\partial^2 f(x, y;\ \xi, \eta)}{\partial x \partial y} = \int_0^\pi\int_0^\pi \{\Delta_{s,t}^{(1)} f(x, y)\} \frac{\partial K_1(s;\ \xi)}{\partial s} \cdot \frac{\partial K_2(t;\ \eta)}{\partial t}\, ds\, dt,$$

and using (2), we obtain

$$\left|\frac{\partial^2 f(x, y;\ \xi, \eta)}{\partial x \partial y}\right| \leqslant \omega_1\big(\delta_1(\xi), \delta_2(\eta)\big) \int_0^\pi \left(1+\frac{s}{\delta_1(\xi)}\right)\left|\frac{\partial K_1(s;\ \xi)}{\partial s}\right| ds \int_0^\pi \left(1+\frac{t}{\delta_2(\eta)}\right)\left|\frac{\partial K_2(t;\ \eta)}{\partial t}\right| dt;$$

whence our assertion follows immediately.

3.3. *Let the derivatives $\partial^\nu K_1(s;\ \xi)/\partial s^\nu$ $(\nu = 1, 2)$ be continuous in s and let $\left.\dfrac{\partial K_1(s;\ \xi)}{\partial s}\right|_{s=\pi} = 0$. Suppose that*

$$\int_0^\pi \left|\frac{\partial^2 K_1(s;\ \xi)}{\partial s^2}\right| ds \leqslant \frac{C_{13}}{\sigma(\xi)}, \qquad \int_0^\pi s^2\left|\frac{\partial^2 K_1(s;\ \xi)}{\partial s^2}\right| ds \leqslant C_{14},$$

$$\int_0^\pi |K_2(t;\ \eta)|\, dt \leqslant C_{15}, \qquad \int_0^\pi t^2 |K_2(t;\ \eta)|\, dt \leqslant C_{16}\tau(\eta),$$

for $(\xi, \eta) \in E$. *Then*

$$\sup_{|x|,|y|\leqslant\pi}\left|\frac{\partial^2 f(x,y;\xi,\eta)}{\partial x^2}\right| \leqslant C_{17}\frac{\omega_2\big(\sqrt{\sigma(\xi)},\sqrt{\tau(\eta)}\big)}{\sigma(\xi)} \quad \textit{in } E,$$

where $C_{17} = C_{15}\big(\sqrt{C_{13}}+\sqrt{C_{14}}\big)^2 + C_{13}\big(\sqrt{C_{15}}+\sqrt{C_{16}}\big)^2$.

Proof. Since

$$\int_{-\pi}^{\pi}\frac{\partial^2 K_1(s;\xi)}{\partial s^2}\,ds = 0,$$

we have

$$\frac{\partial^2 f(x,y;\xi,\eta)}{\partial x^2} = \int_0^{\pi}\int_0^{\pi}\{\Delta_{s,t}^{(2)} f(x,y)\}\frac{\partial^2 K_1(s;\xi)}{\partial s^2}K_2(t;\eta)\,ds\,dt.$$

Next, estimate (3) and the assumptions give

$$\left|\frac{\partial^2 f(x,y;\xi,\eta)}{\partial x^2}\right|$$

$$\leqslant \int_0^{\pi}\int_0^{\pi}\omega_2(s,t)\left|\frac{\partial^2 K_1(s;\xi)}{\partial s^2}\right||K_2(t;\eta)|\,ds\,dt$$

$$\leqslant \omega_2\big(\sqrt{\sigma(\xi)},\sqrt{\tau(\eta)}\big)\times$$

$$\times\int_0^{\pi}\int_0^{\pi}\left\{\left(1+\frac{s}{\sqrt{\sigma(\xi)}}\right)^2+\left(1+\frac{t}{\sqrt{\tau(\eta)}}\right)^2\right\}\left|\frac{\partial^2 K_1(s;\xi)}{\partial s^2}\right||K_2(t;\eta)|\,ds\,dt$$

$$\leqslant \omega_2\big(\sqrt{\sigma(\xi)},\sqrt{\tau(\eta)}\big)\times$$

$$\times\left\{C_{16}\int_0^{\pi}\left(1+\frac{s}{\sqrt{\sigma(\xi)}}\right)^2\left|\frac{\partial^2 K_1(s;\xi)}{\partial s^2}\right|ds+\frac{C_{13}}{\sigma(\xi)}\int_0^{\pi}\left(1+\frac{t}{\sqrt{\tau(\eta)}}\right)^2|K_2(t;\eta)|\,dt\right\}.$$

Observing that, by Schwarz' inequality,

$$\int_0^{\pi}s\left|\frac{\partial^2 K_1(s;\xi)}{\partial s^2}\right|ds \leqslant \frac{\sqrt{C_{13}C_{14}}}{\sqrt{\sigma(\xi)}}, \qquad \int_0^{\pi}t|K_2(t;\eta)|\,dt \leqslant \sqrt{C_{15}C_{16}}\sqrt{\tau(\eta)},$$

we get the desired result.

4. Examples. Let $\sum_{m,n=0}^{\infty}A_{m,n}(x,y)$ be the double Fourier series of the function $f(x,y)$ and let

$$f(x,y;r,R) = \sum_{m,n=0}^{\infty}\left\{1+\frac{m}{2}(1-r^2)\right\}\left\{1+\frac{n}{2}(1-R^2)\right\}r^m R^n A_{m,n}(x,y),$$

for $0 < r,\ R < 1$. This operator can be written in the form

$$f(x, y; r, R) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s, t)K(s-x; r)K(t-y; R)\,ds\,dt\,,$$

where

$$K(s; r) = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{m=1}^{\infty}\left\{1+\frac{m}{2}(1-r^2)\right\}r^m\cos ms = \frac{(1-r^2)^2(1-r\cos s)}{2\pi(1-2r\cos s+r^2)^2}\,.$$

The kernel $K(s; r)$ is evidently 2π-periodic, non-negative, even, continuous and

$$\int_{-\pi}^{\pi} K(s; r)\,ds = 1\ .$$

Moreover,

$$\int_{0}^{\pi} s^2K(s; r)\,ds \leqslant \frac{\pi^2}{2}\int_{0}^{\pi}(1-\cos s)K(s; r)\,ds \leqslant \frac{3\pi^2}{8}(1-r)^2\,.$$

Denote by $\omega(s, t)$, $\omega_1(s, t)$, $\omega_2(s, t)$ the moduli of $f(x, y) \in M$. From 2.1 it follows at once that:

4.1. *Given any* $f(x, y) \in M$,

$$\sup_{|x|,|y|\leqslant\pi} |f(x, y; r, R)-f(x, y)| \leqslant 7\,\omega_2(1-r, 1-R)$$

whenever $0 < r,\ R < 1$.

It is known [2] that

$$u_a(r) = \int_{0}^{\pi}\left(2\sin\frac{s}{2}\right)^a K(s; r)\,ds = 2^{a+1}\int_{0}^{\pi/2}\sin^a s\,K(2s; r)\,ds$$

$$= \frac{1}{\pi}B\left(\frac{1+a}{2}, \frac{3-a}{2}\right)(1-r)^a\big(1+o(1)\big) \qquad \text{as } r\to 1-\,,$$

and an analogous relation is valid for $v_\beta(R) = u_\beta(R)$ if $0 < a,\ \beta < 2$. Also, it can easily be observed that $u_{2a}(r) = o\{u_a(r)\}$ as $r\to 1-$ $(0 < a < 2)$. Therefore, by 2.2, the following asymptotic formula for the quantity

$$U_{a,\beta}(r, R) = \sup_{f\in Z(a,\beta)}\{\sup_{|x|,|y|\leqslant\pi}|f(x, y; r, R)-f(x, y)|\}$$

holds.

4.2. *If* $0 < \alpha,\ \beta < 2$, *then*

$$U_{\alpha,\beta}(r, R) = \frac{2}{\pi}\left\{B\left(\frac{1+\alpha}{2}, \frac{3-\alpha}{2}\right)(1-r)^{\alpha} + B\left(\frac{1+\beta}{2}, \frac{3-\beta}{2}\right)(1-R)^{\beta}\right\}\left(1+o(1)\right)$$

as $(r, R) \to (1-, 1-)$.

By 3.1, 3.2 and 3.3,

4.3. *We have*

$$\left|\frac{\partial f(x, y; r, R)}{\partial x}\right| \leqslant C_{18}\frac{\omega(1-r, 0)}{1-r}, \qquad \left|\frac{\partial f(x, y; r, R)}{\partial y}\right| \leqslant C_{18}\frac{\omega(0, 1-R)}{1-R},$$

$$\left|\frac{\partial^2 f(x, y; r, R)}{\partial x \partial y}\right| \leqslant C_{19}\frac{\omega_1(1-r, 1-R)}{(1-r)(1-R)},$$

$$\left|\frac{\partial^2 f(x, y; r, R)}{\partial x^2}\right| \leqslant C_{20}\frac{\omega_2(1-r, 1-R)}{(1-r)^2},$$

for each x, y *and* $0 < r_0 \leqslant r < 1$, $0 < R_0 \leqslant R < 1$.

Finally, we shall show that:

4.4. *Under the restrictions of* 4.3,

$$\left|\frac{\partial f(x, y; r, R)}{\partial r}\right| \leqslant C_{21}\frac{\omega_2(1-r, 1-R)}{1-r}.$$

Proof. Observing that

$$\int_{-\pi}^{\pi} \frac{\partial K(s; r)}{\partial r} ds = 0,$$

we obtain

$$\frac{\partial f(x, y; r, R)}{\partial r} = \int_0^{\pi}\int_0^{\pi} \{\Delta_{s,t}^{(2)} f(x, y)\} \frac{\partial K(s; r)}{\partial r} K(t; R)\, ds\, dt;$$

whence

$$\left|\frac{\partial f(x, y; r, R)}{\partial r}\right| \leqslant \omega_2(1-r, 1-R) \times$$

$$\times \int_0^{\pi}\int_0^{\pi} \left\{\left(1+\frac{s}{1-r}\right)^2 + \left(1+\frac{t}{1-R}\right)^2\right\} \left|\frac{\partial K(s; r)}{\partial r}\right| K(t; R)\, ds\, dt$$

$$\leqslant \omega_2(1-r, 1-R)\left\{\frac{1}{2}\int_0^{\pi}\left(1+\frac{s}{1-r}\right)^2\left|\frac{\partial K(s; r)}{\partial r}\right| ds + \right.$$

$$\left. + \int_0^{\pi}\left|\frac{\partial K(s; r)}{\partial r}\right| ds \int_0^{\pi}\left(1+\frac{t}{1-R}\right)^2 K(t; R)\, dt\right\}.$$

By simple evaluations of the integrals (see [2]), the assertion follows.

Notice also that 4.3 remains true for the Abel-Poisson means:

$$f(x, y; r, R) = \sum_{m,n=0}^{\infty} r^m R^n A_{m,n}(x, y)$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) \frac{1-r^2}{1-2r\cos(s-x)+r^2} \times$$

$$\times \frac{1-R^2}{1-2R\cos(t-y)+R^2} ds\, dt\,.$$

Considering the Jackson double integrals

$$f(x, y; m, n) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) I(s-x; m) I(t-y; n) ds\, dt\,,$$

where

$$I(s; m) = \frac{3}{2\pi m(2m^2+1)} \left(\frac{\sin(ms/2)}{\sin(s/2)}\right)^4,$$

similar results can be stated. E.g.

4.5. *Given any* $f(x, y) \in M$,

(i) $$\sup_{|x|,|y|\leqslant\pi} |f(x, y; m, n) - f(x, y)| \leqslant 7\omega_2\left(\frac{1}{m}, \frac{1}{n}\right) \qquad (m, n \geqslant 1)\,,$$

(ii) *if* $0 < a, \beta \leqslant 2$,

$$\sup_{f\in Z(a,\beta)} \{ \sup_{|x|,|y|\leqslant\pi} |f(x, y; m, n) - f(x, y)| \}$$

$$= \frac{3}{\pi}\left\{\frac{2^a}{m^a} \int_0^{\infty} \frac{\sin^4 z}{z^{4-a}} dz + \frac{2^\beta}{n^\beta} \int_0^{\infty} \frac{\sin^4 z}{z^{4-\beta}} dz\right\} \big(1 + o(1)\big) \qquad as\ (m, n) \to (\infty, \infty)\,.$$

Theorem 2.3 enables us to get the asymptotic relation for the conjugate Abel-Poisson operators

$$\bar{f}(x, y; r, R) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) \bar{P}(s-x; r) \bar{P}(t-y; R) ds\, dt\,,$$

where

$$\bar{P}(s; r) = \frac{2r\sin s}{1-2r\cos s + r^2} \qquad (0 < r, R < 1)\,.$$

In this case

$$Q(s; r) = \cot\frac{s}{2} - \bar{P}(s; r) = \frac{1-r^2}{1-2r\cos s + r^2} \cot\frac{s}{2}$$

and (see [1])

$$\gamma_a(r) = \int_0^{\pi/2} s^a Q(s;\, r)\, ds = \frac{\pi}{\sin(\pi a/2)}(1-r)^a + O\{(1-r)^{a+1}\} \quad \text{as} \quad r \to 1-\,,$$

$$\varrho_a(r) = \int_{\pi/2}^{\pi} (\pi - s)^a Q(s;\, r)\, ds = O\{(1-r)^2\} \quad \text{as} \quad r \to 1-\,,$$

whenever $0 < a \leqslant 1$. Moreover, $\overline{P}(s;\, r) \geqslant 0$ and $Q(s;\, r) \geqslant 0$ for $0 < s \leqslant \pi$, $0 < r < 1$. Thus

4.6. *If* $0 < a, \beta \leqslant 1$, *we have*

$$\sup_{f \in \Pi(a,\beta)} \{ \sup_{|x|,|y| \leqslant \pi} |\bar{f}(x, y;\, r, R) - \bar{f}(x, y)| \}$$

$$= \frac{2^{a+\beta-1}}{\pi} \left\{ \frac{1}{\sin(\pi a/2)} \int_0^{\pi/2} \frac{t^\beta}{\sin t} dt (1-r)^a + \frac{1}{\sin(\pi\beta/2)} \int_0^{\pi/2} \frac{t^a}{\sin t} dt (1-R)^\beta \right\} \big(1 + o(1)\big)$$

as $(r, R) \to (1-, 1-)$.

References

[1] B. Sz.-Nagy, *Sur l'ordre de l'approximation d'une fonction par son intégrale de Poisson*, Acta Math. Acad. Sci. Hungaricae 1 (1950), pp. 183-188.

[2] P. Pych, *On a biharmonic function in the disc*, Ann. Polon. Math. (ce volume), pp. 203-213.

Reçu par la Rédaction le 29. 10. 1965