ANNALES
POLONICI MATHEMATICI
XXI (1969)

On the oscillatory behavior of the solutions of second order linear differential equations *

by D. WILLETT (Salt Lake City)

1. Introduction. In this paper we obtain oscillation and non-oscillation criteria of the "integral comparison" type for the differential equations $y''+p(t)y=0$ and $\big(r(t)y'\big)'+q(t)y=0$, where p, r and q are real-valued continuous non-trivial functions on an infinite interval $I: T\leqslant t<\infty$. The main strength of our results is that we do not directly restrict the oscillatory behaviour of $p(t)$ and $q(t)$ by assuming they or any of their primitives are positive. A simple non-trivial class of equations which are covered by our results and which illustrate the lack of restriction on the oscillation of $p(t)$ is when $p(t)=\mu t^{\eta}\sin\nu t$, where μ, ν and η are constants. With the exception of the case $|u/\nu|=1/\sqrt{2}$, $\eta=-1$, for which there is probably non-oscillation, our tests completely cover this class of equations (see Section 4).

We call a differential equation *oscillatory* if all its non-trivial solutions vanish infinitely often on I. Otherwise, a differential equation is called *non-oscillatory*. Being linear, the equation

$$y''+p(t)y=0 \tag{1.1}$$

is oscillatory if it has one non-trivial solution that is oscillatory. An equation is said to be *disconjugate* on an interval if each of its non-trivial solutions have at most one zero on that interval.

The literature on oscillation for (1.1) is extensive as can be seen by looking in Cèsari [1]; p. 90. Some evidence of the difficulty of the problem is shown by a relatively recent result of Nehari [10]. Nehari proves that (1.1) is disconjugate if, and only if, for each $T_1>T$ the minimum eigenvalue λ for the boundary value problem

$$u''+\lambda p(t)u=0\,,\qquad u(T)=u'(T_1)=0$$

satisfies $\lambda>1$. In the light of this result and the difficulty of locating eigenvalues in general, it seems unlikely that a simple necessary and sufficient condition for oscillation of (1.1) exists.

* This work was supported in part by NASA Contract No. NGR-45-003-038.

Comparison type oscillation criteria for (1.1) have their basis in the famous Sturm separation theorem. Probably the first such non-trivial criteria was given by Kneser [7] in 1893:

$$\text{(1.2)} \qquad \begin{cases} t^2 p(t) \leqslant 1/4 & \Rightarrow \text{non-oscillation}\,, \\ t^2 p(t) \geqslant (1+\varepsilon)/4 & \Rightarrow \text{oscillation}\,. \end{cases}$$

Later, Fite [3] proved that $p(t) > 0$ and $\int^{\infty} p(s)\,ds = \infty$ imply oscillation. Hence, Hille [6] assumed

$$\text{(1.3)} \qquad p(t) > 0 \quad \text{and} \quad P(t) = \int_t^{\infty} p(s)\,ds < \infty\,,$$

and proved the following generalization of (1.2):

$$\text{(1.4)} \qquad \begin{cases} tP(t) \leqslant 1/4 & \Rightarrow \text{non-oscillation}\,, \\ tP(t) \geqslant (1+\varepsilon)/4 & \Rightarrow \text{oscillation}\,. \end{cases}$$

The condition $p(t) > 0$ in Fite's result was later removed by Wintner [18], and the same condition in the "non-oscillatory part" of Hille's result was replaced by Wintner [19] with the condition $tP(t) \geqslant -3/4$. In the "oscillatory part" of Hille's result, Moore [9] replaced $p(t) > 0$ by the assumption that $tP(t)$ be bounded.

Also assuming (1.3), Wintner [19] proved that

$$P^2(t) \leqslant p(t)/4 \Rightarrow \text{non-oscillation}\,,$$

and this result was later extended to matrix differential equations by Reid [16]. Wintner did not seem aware at that time that

$$P^2(t) \geqslant (1+\varepsilon)p(t)/4 \Rightarrow \text{oscillation}\,.$$

The latter is a direct consequence of a later result of Opial [13]. Assuming that $P(t) = \int_t^{\infty} p(s)\,ds$ exists as an improper integral and $P(t) > 0$, Opial proved that

$$\text{(1.5)} \qquad \begin{cases} \displaystyle\int_t^{\infty} P^2(s)\,ds \leqslant P(t)/4 & \Rightarrow \text{non-oscillation}\,, \\ \displaystyle\int_t^{\infty} P^2(s)\,ds \geqslant (1+\varepsilon)P(t)/4 & \Rightarrow \text{oscillation}\,. \end{cases}$$

Oscillation results generalizing the Fite-Wintner result mentioned above were given by Olech, Opial, Ważewski [12], who proved that

$$(1.6)\qquad \begin{cases} \lim\limits_{t\to\infty} \operatorname{appr} \int\limits_T^t p(s)\,ds = \infty \Rightarrow \text{oscillation}\,, \\ \lim\limits_{t\to\infty} \operatorname{appr\,inf} \int\limits_T^t p(s)\,ds < \lim\limits_{t\to\infty} \operatorname{appr\,sup} \int\limits_T^t p(s)\,ds \Rightarrow \text{oscillation}\,, \end{cases}$$

and Wintner [18], who proved that

$$(1.7)\qquad \lim_{t\to\infty} t^{-1} \int\limits_T^t \Big(\int\limits_T^s p(\tau)\,d\tau\Big)ds = \infty \Rightarrow \text{oscillation}\,.$$

Later, Hartman [5] proved that the non-existence (as a single finite number) of the limit in (1.7) and the condition

$$(1.8)\qquad \liminf_{t\to\infty} t^{-1} \int\limits_T^t \Big(\int\limits_T^s p(\tau)\,d\tau\Big)ds > -\infty$$

imply oscillation.

Recently, Coles [2] obtained extensions of these results of Hartman and Wintner by introducing weighted averages of the type

$$(1.9)\qquad A(t, t_0) = A_f(t, t_0) = \frac{\int\limits_{t_0}^t f(s)\Big(\int\limits_{t_0}^s p(\tau)\,d\tau\Big)ds}{\int\limits_{t_0}^t f(s)\,ds}\,,$$

where $f(t)$ is a non-negative, locally integrable function satisfying

$$(1.10)\qquad F_k(t) = \int^t f(s)\frac{\Big(\int^s f(\tau)\,d\tau\Big)^k}{\int^s f^2(\tau)\,d\tau}\,ds \to \infty \quad \text{as } t\to\infty \text{ for some } k,\ 0 \leqslant k < 1\,.$$

Coles proved that if there exists such a function f such that

$$\liminf A_f(t, T) > -\infty \quad \text{as } t\to\infty$$

and

$$(1.11)\qquad \lim_{t\to\infty} t^{-1} \int^t \Big(\int^s p(\tau)\,d\tau\Big)ds$$

does not exist, then (1.1) is oscillatory.

Our main results arise from a combination of Coles' idea of weighted averages and Opial's criteria (1.6). The resulting criteria includes both of their results.

Let $\mathfrak{F}$ be the set of all non-negative locally integrable functions on I that satisfy the condition

$$(1.12)\qquad \limsup_{t\to\infty}\Big(\int^{t} f(s)\,ds\Big)^{1-k}[F_k(\infty)-F_k(t)] > 0 \qquad \text{for some } k,\ 0\leqslant k<1\,.$$

If $F_k(\infty)=\infty$ in (1.12), then we allow $f\in\mathfrak{F}$. Let $\mathfrak{F}_0$ be the set of all non-negative locally integrable functions on I that satisfy

$$(1.13)\qquad \lim_{t\to\infty}\frac{\int^{t} f^2(s)\,ds}{\big[\int^{t} f(s)\,ds\big]^2}=0\,.$$

Members of the classes $\mathfrak{F}$ and $\mathfrak{F}_0$ will be called *weight functions*.

Our main results are the following three theorems:

THEOREM 1.1. *Let $P(t)$ be any continuously differentiable function such that $P'(t)=-p(t)$ on $[T,\infty)$, and let*

$$(1.14)\qquad \overline{P}(t)=\int_t^\infty P^2(s)\exp\Big(2\int_t^s P(\tau)\,d\tau\Big)\,ds\,.$$

Equation (1.1) *is disconjugate on $[T,\infty)$ if*

$$(1.15)\qquad \overline{P}(t)<\infty \quad \text{and} \quad \int_t^\infty \overline{P}^2(s)\exp\Big(2\int_t^s P(\tau)\,d\tau\Big)\,ds\leqslant \overline{P}(t)/4$$

for all $t\geqslant T$.

THEOREM 1.2. *Assume there exists $g\in\mathfrak{F}_0$ such that*

$$(1.16)\qquad c=\lim_{t\to\infty}A_g(t,T)$$

exists, and let

$$(1.17)\qquad P(t)=c-\int_T^t p(s)\,ds\,.$$

If $\overline{P}(t)\not\equiv 0$ and there exists $\varepsilon>0$ such that

$$(1.18)\qquad \overline{P}(t)=\infty \quad \text{or} \quad \int_t^\infty \overline{P}^2(s)\exp\Big(2\int_t^s P(\tau)\,d\tau\Big)\,ds\geqslant (1+\varepsilon)\overline{P}(t)/4$$

for all $t\geqslant T$, then equation (1.1) *is oscillatory.*

THEOREM 1.3. *If there exist $f \in \mathfrak{I}$ and $g \in \mathfrak{I}_0$ such that*

$$\liminf_{t\to\infty} A_f(t, \cdot) > -\infty \tag{1.19}$$

and the limit in (1.16) *does not exist (as a single finite number), then* (1.1) *is oscillatory.*

COROLLARY 1.1. *Assume that*

$$\lim_{\tau\to\infty} \int_T^{\tau} p(s)\,ds$$

exists, and let

$$P(t) = \lim_{\tau\to\infty} \int_t^{\tau} p(s)\,ds\,, \qquad t \geqslant T\,,$$

and $\bar{P}(t)$ be defined by (1.14). *If* (1.15) *holds, then* (1.1) *is disconjugate on $[T, \infty)$. If there exists $\varepsilon > 0$ such that* (1.18) *holds, then* (1.1) *is oscillatory.*

COROLLARY 1.2. *If there exist non-negative, non-integrable, bounded functions h, g on $[T, \infty)$ such that*

$$\lim_{t\to\infty} A_h(t, T) > \lim_{t\to\infty} A_g(t, T)\,,$$

then (1.1) *is oscillatory.*

In order that either (1.12) or (1.13) can be satisfied by a non-negative function f, it is necessary that f be non-integrable on $[T, \infty)$, i.e.,

$$\int^{\infty} f(s)\,ds = \infty\,. \tag{1.20}$$

On the other hand, all bounded non-negative locally integrable functions f satisfying (1.20) belong to $\mathfrak{I}_0$, and $\mathfrak{I}_0 \subset \mathfrak{I}$. Because of (1.20), the values of the lim sup in (1.12) and lim in (1.13) are invariant with respect to the lower limits of integration. If $c = \lim A_g(\tau, T)$ as $\tau\to\infty$ exists for $g \in \mathfrak{I}$, then (1.20) implies $\lim A_g(\tau, t)$ as $\tau\to\infty$ exists for all $t \geqslant T$, and

$$\lim_{\tau\to\infty} A_g(\tau, t) = c - \int_T^t p(s)\,ds\,. \tag{1.21}$$

Hence, Theorems 1.2 and 1.3 state that if there exists $f \in \mathfrak{I}$ such that $\liminf A_f(t, \cdot) > -\infty$ as $t\to\infty$, then either (1.1) is oscillatory or there exist a broad class of primitives of $-p(t)$, namely, $\lim A_g(\tau, t)$ as $\tau\to\infty$ for each $g \in \mathfrak{I}_0$, which can be used for $P(t)$ in (1.18). Hence, Theorems 1.2 and 1.3 are a partial converse to Theorem 1.1.

Two directions of improvement of the above results immediately come to mind. First, can the size of the two sets $\mathfrak{I}$ and $\mathfrak{I}_0$ of weight functions be made larger? In this regard, it may be of interest to know that all the above results remain valid if instead of (1.12), condition (1.20) and the condition

$$\liminf_{t\to\infty} F_1(t)/\left(\ln \int^t f(s)\,ds\right) > 0$$

are used to define the class $\mathfrak{Z}$. However, (1.20) alone is not a sufficient condition. In this regard, Professor Coles has recently pointed out to me that if Theorem 1.3 holds with (1.20) replacing (1.12), then all solutions of (1.1) oscillate provided just

$$\limsup_{t\to\infty} \int^t p(s)\,ds = \infty, \tag{1.22}$$

which is not true in general as can be shown quite easily by examples. We do note, however, that (1.22) does imply (1.1) is oscillatory provided that $p(t)$ is also bounded on at least one side (cf. [14]). The second direction of improvement for our results is to simplify (1.15) and (1.18). In this regard, we give the following theorems:

THEOREM 1.4. *Let $P(t) \in C[T_1, \infty)$ and $\bar{P}$ be defined by (1.14). Then there exists $T \geqslant T_1$ such that (1.15) holds, if any one of the following conditions holds for all $t \geqslant T_1$:*

(i) $\int_t^\infty P^2(s)\,ds \leqslant P(t)/4$;

(ii) $\bar{P}(t) \leqslant P(t)/2$;

(iii) $\lim_{\tau\to\infty} \int^\tau P(s)\,ds < \infty$, $\int^\infty P^2(s)\,ds < \infty$, *and there exists* $\varepsilon > 0$ *such that*

$$\int_t^\infty \left(\int_s^\infty P^2(\tau)\,d\tau\right)^2 ds \leqslant (1-\varepsilon)\int_t^\infty P^2(s)\,ds/4 .$$

THEOREM 1.5. *Let $P(t) \in C[T_1, \infty)$ and $\bar{P}(t)$ be defined by (1.14). Then there exists $T \geqslant T_1$ such that (1.18) holds, if any one of the following conditions holds for all $t \geqslant T_1$ and some $\varepsilon > 0$:*

(j) $P(t) \geqslant 0$ *and* $\int_t^\infty P^2(s)\,ds \geqslant (1+\varepsilon)P(t)/4$;

(jj) $\bar{P}(t) \geqslant (1+\varepsilon)|P(t)|/2$;

(jjj) $\lim_{\tau\to\infty} \int^\tau P(s)\,ds < \infty$, $\int^\infty P^2(s)\,ds = \infty$ *or* $\int_t^\infty \left(\int_s^\infty P^2(\tau)\,d\tau\right)^2 ds \geqslant (1+\varepsilon)\times$ $\times \int_t^\infty P^2(s)\,ds/4$.

Parts (i) and (j) above imply that Opial's result (1.5) is included in Corollary 1.1. The result of Coles cited above is included in Theorem 1.3. Corollary 1.2 implies the second part of (1.6). By modifying the proof of Lemma 2.1 in the next section, one can also prove that (1.1) is oscillatory provided $\lim A_f(t, \cdot) = \infty$ as $t \to \infty$ for some $f \in \mathfrak{J}$. This generalizes another result of Coles [2] and the first part of (1.6).

Some preliminary lemmas useful in the proofs of Theorems 1.1, 1.3. and 1.3 are given in the next section. The first two of these lemmas are of special interest because they generalize some well-known results of Hartman ([5], p. 391). Proofs of all five theorems will be given in Section 3. Some specific examples are presented in Section 4. A transformation theorem extending Theorems 1.2 and 1.3 to cover problems when $\int^{\infty} p(s)\,ds = -\infty$ is given in Section 5. An extension of oscillation results for (1.1) in general to equations of the form

$$\big(r(t)y'\big)' + q(t)y = 0 ,$$

with $r(t) > 0$, is given in Section 5.

2. Preliminary lemmas. It is well known that there exists a continuously differentiable solution $v(t)$ to the Ricatti equation

$$v'(t) + p(t) + v^2(t) = 0 , \qquad t \geq T , \tag{2.1}$$

if, and only if, there exists a disconjugate solution $y(t)$ to (1.1). In this section we consider some properties of equation (2.1) and its equivalent integral equation formulation

$$v(t) = v(s) - \int_s^t p(\tau)\,d\tau - \int_s^t v^2(\tau)\,d\tau , \qquad t \geq s \geq T . \tag{2.2}$$

LEMMA 2.1. *Assume that* $v(t)$ *satisfies* (2.1). *If there exists* $f \in \mathfrak{J}$ *such that* $\liminf A_f(t, \cdot) > -\infty$ *as* $t \to \infty$, *then* $\int^{\infty} v^2(s)\,ds < \infty$.

LEMMA 2.2. *Assume that* $v(t)$ *satisfies* (2.1). *If* $\int^{\infty} v^2(s)\,ds < \infty$, *then for any* $f \in \mathfrak{J}_0$, $\lim A_f(t, \cdot)$ *as* $t \to \infty$ *exists.*

Hartman [5] has proven that if (1.8) is true, then $\int^{\infty} v^2(s)\,ds < \infty$; and conversely, if $\int^{\infty} v^2(s)\,ds < \infty$, then the limit in (1.11) must exist. The classes $\mathfrak{J}$ and $\mathfrak{J}_0$ have been determined solely for the purpose of obtaining the generalization contained in Lemmas 2.1 and 2.2 of these results of Hartman. We need these lemmas in the proofs of Theorems 1.2 and 1.3.

Thus, if the rather unnatural conditions defining the classes $\mathfrak{J}$ and $\mathfrak{J}_0$ are to be improved, then the above two lemmas constitute the area in which the work must be done.

Proof of Lemma 2.1. Let $A(t,s) = A_f(t,s)$ and assume that $\int^{\infty} v^2(s)\,ds = \infty$. Integrating (2.2), we obtain

$$(2.3)\quad \int_{t_0}^{t} f(s)v(s)\,ds = [v(t_0) - A(t,t_0)]\int_{t_0}^{t} f(s)\,ds - \int_{t_0}^{t} f(s)\Big(\int_{t_0}^{s} v^2(\tau)\,d\tau\Big)ds\,, \qquad t \geqslant t_0 \geqslant T\,.$$

Putting $t = t_0$ and $s = T$ in (2.2), we get next that

$$(2.4)\quad v(t_0) - A(t,t_0) = v(T) - A(t,T) - \int_{T}^{t_0} v^2(s)\,ds + o(1) \qquad \text{as } t\to\infty,\ t_0 \geqslant T\,,$$

since

$$A(t,t_0) = A(t,T) - \int_{T}^{t_0} p(s)\,ds + o(1) \qquad \text{as } t\to\infty\,.$$

Since f satisfies (1.12), there exists a positive number μ such that

$$(2.5)\qquad \mu^{-1} < (1-k)\limsup_{t\to\infty}\Big(\int^{t} f(s)\,ds\Big)^{1-k}[F_k(\infty) - F_k(t)]\,.$$

Using the fact that $A(t,T)$ is bounded away from $-\infty$ by assumption and

$$\int_{T}^{t_0} v^2(s)\,ds \to \infty \qquad \text{as } t_0\to\infty\,,$$

we conclude from (2.4) that there exist numbers t_0 and t_1, $t_1 \geqslant t_0 \geqslant T$, such that

$$(2.6)\qquad v(t_0) - A(t,t_0) \leqslant -\mu \qquad \text{for all } t \geqslant t_1\,.$$

Let

$$(2.7)\qquad z(t) = \int_{t_0}^{t} f(s)v(s)\,ds\,.$$

The Cauchy-Schwartz inequality implies

$$(2.8)\qquad \int_{t_0}^{s} v^2(\tau)\,d\tau \geqslant z^2(s)\Big/\int_{t_0}^{s} f^2(\tau)\,d\tau\,.$$

Putting (2.6), (2.7), and (2.8) into (2.3), we obtain the inequality

$$z(t) \leqslant -\mu\int_{t_0}^{t} f(s)\,ds-\int_{t_0}^{t} f(s)\Big(\int_{t_0}^{s} f^2(\tau)\,d\tau\Big)^{-1}z^2(s)\,ds = -R(t)\,, \qquad t \geqslant t_1\,,$$

where $R(t)$ is defined by this equation.

Hence,

$$R'(t) = \mu f(t)+f(t)z^2(t)/\int_{t_0}^{t} f^2(\tau)\,d\tau \tag{2.9}$$

for almost all $t \geqslant t_1$ (a.e.). The inequalities $z^2(t) \geqslant R^2(t)$, $t \geqslant t_1$, and (2.9) imply

$$R'(t)/R^2(t) \geqslant f(t)/\int_{t_0}^{t} f^2(\tau)\,d\tau\,, \qquad t \geqslant t_1 \quad \text{(a.e.)}\,. \tag{2.10}$$

However, from the definition of $R(t)$, we obtain

$$R(t) \geqslant \mu\int_{t_0}^{t} f(s)\,ds\,, \qquad t \geqslant t_1\,. \tag{2.11}$$

Inequalities (2.10) and (2.11) imply that

$$R'(t)R^{k-2}(t) \geqslant \mu^k f(t)\Big(\int_{t_0}^{t} f(s)\,ds\Big)^k/\int_{t_0}^{t} f^2(s)\,ds\,, \qquad t \geqslant t_1 \quad \text{(a.e.)}\,.$$

Since $k < 1$, subsequent integration from t ($\geqslant t_1$) to ∞ and use of (2.11) to replace $R(t)$ produces

$$\begin{aligned}\mu^{-1} &\geqslant (1-k)\mu^{k-1}R^{1-k}(t)[F_k(\infty)-F_k(t)]\\ &\geqslant (1-k)\Big(\int_{t_0}^{t} f(s)\,ds\Big)^{1-k}[F_k(\infty)-F_k(t)]\,.\end{aligned}$$

Taking $\limsup$ as $t \to \infty$ of the last inequality, we arrive at a contradiction of (2.5). Hence, $\int^{\infty} v^2(s)\,ds < \infty$.

Proof of Lemma 2.2. Equation (2.3) implies

$$A(t, t_0) = v(t_0)-\frac{\int_{t_0}^{t} f(s)v(s)\,ds}{\int_{t_0}^{t} f(s)\,ds}-\frac{\int_{t_0}^{t} f(s)\Big(\int_{t_0}^{s} v^2(\tau)\,d\tau\Big)ds}{\int_{t_0}^{t} f(s)\,ds}\,.$$

Since

$$\lim_{t\to\infty}\frac{\int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} v^2(\tau)\,d\tau\right)ds}{\int_{t_0}^{t} f(s)\,ds} = \int_{t_0}^{\infty} v^2(s)\,ds$$

exists, and (1.13) implies

$$0 \leqslant \lim_{t\to\infty}\frac{\left|\int_{t_0}^{t} f(s)v(s)\,ds\right|}{\int_{t_0}^{t} f(s)\,ds} \leqslant \lim_{t\to\infty}\frac{\left(\int_{t_0}^{t} f^2(s)\,ds\right)^{1/2}}{\int_{t_0}^{t} f(s)\,ds}\left(\int_{t_0}^{\infty} v^2(s)\,ds\right)^{1/2} = 0\,,$$

we conclude $\lim A(t, t_0)$ as $t\to\infty$ exists and

$$\lim_{t\to\infty} A(t, t_0) = v(t_0) - \int_{t_0}^{\infty} v^2(s)\,ds\,, \qquad t_0 \geqslant T\,. \tag{2.12}$$

The next two lemmas are slight generalizations of some results of Opial [12]. We use these stronger versions of Opial's results in the proofs of Theorems 1.1 and 1.2.

LEMMA 2.3. *Assume that $P(t)$ and $Q(t, s)$ are non-negative continuous functions for $T \leqslant t, s < \infty$. If*

$$\int_{t}^{\infty} Q(t, s)P^2(s)\,ds \leqslant P(t)/4\,, \qquad t \geqslant T\,, \tag{2.13}$$

then the equation

$$v(t) = P(t) + \int_{t}^{\infty} Q(t, s)v^2(s)\,ds\,, \qquad t \geqslant T\,, \tag{2.14}$$

has a continuous solution $v(t)$.

LEMMA 2.4. *Assume that $P(t)$ and $Q(t, s)$ are non-negative continuous functions for $T \leqslant t, s < \infty$. If there exists $\varepsilon > 0$ such that*

$$\int_{t}^{\infty} Q(t, s)P^2(s)\,ds \geqslant (1+\varepsilon)P(t)/4 \not\equiv 0\,, \qquad t \geqslant T\,, \tag{2.15}$$

then the inequality

$$v(t) \geqslant P(t) + \int_{t}^{\infty} Q(t, s)v^2(s)\,ds\,, \qquad t \geqslant T\,, \tag{2.16}$$

does not have a continuous solution for $v(t)$.

The details of the proofs of Lemma 2.3 and 2.4 for the case when $Q(t, s) \equiv 1$ and strict equality occurs in (2.16) have been given by Opial [12]. The same proofs apply to this slightly more general situation. In the case of Lemma 2.3, the sequence

$$v_0(t) = P(t)\,, \qquad v_{n+1}(t) = P(t) + \int_t^\infty Q(t, s)v_n^2(s)\,ds\,, \qquad n = 0, 1, \ldots,$$

is monotone increasing and bounded above by $2P(t)$. Hence $v_n(t)$ converges to a function $v(t)$, which is a solution of (2.14) and hence is a continuous function. In the case of Lemma 2.4, assume that $v(t)$ is a continuous function which satisfies (2.15). Then $v(t) \geqslant P(t) \geqslant 0$ implies $v^2(t) \geqslant P^2(t)$, which implies in turn that

$$v(t) \geqslant P(t) + \int_t^\infty Q(t, s)P^2(s)\,ds \geqslant \left(1 + \frac{1+\varepsilon}{4}P(t)\right).$$

Continuing in this manner produces $v(t) \geqslant a_n P(t)$, where

$$1 = a_0 < a_1 < \ldots < a_n < \ldots \uparrow \infty \qquad \text{as } n \uparrow \infty\,,$$

which is a contradiction.

3. Proofs of the theorems. The proofs of Theorems 1.1, 1,2, and 1.3 are based upon the fact that if $v(t)$ is a solution of (2.1), then

$$y(t) = \exp\left(\int_T^t v(s)\,ds\right) \tag{3.1}$$

is a non-vanishing solution in $[T, \infty)$ of (1.1), and if $y(t)$ is a non-vanishing solution of (1.1) in $[T, \infty)$, then

$$v(t) = y'(t)/y(t) \tag{3.2}$$

is a solution of (2.1). A large share of the work has already been done in Section 2.

Proof of Theorem 1.1. Since $\overline{P}(t) < \infty$ by assumption, we can consider the equation

$$u(t) = \overline{P}(t) + \int_t^\infty u^2(s)\exp\left(2\int_t^s P(\tau)\,d\tau\right)ds\,, \qquad t \geqslant T\,, \tag{3.3}$$

for the unknown function $u(t)$. Since (1.15) implies (2.13) in the context of equation (3.3), Lemma (2.3) implies that (3.3) has a continuous solution

$u(t)$ defined for all $t \geqslant T$. Furthermore, since $\bar{P}(t)$ is continuously differentiable, equation (3.3) implies that $u'(t)$ exists for all $t \geqslant T$ and

$$u'(t) = -P^2(t) - 2P(t)u(t) - u^2(t) = -\big(P(t) + u(t)\big)^2 . \tag{3.4}$$

Let

$$v(t) = \sqrt{-u'(t)}\,\mathrm{sgn}\big(P(t) + u(t)\big) , \qquad t \geqslant T . \tag{3.5}$$

Hence, for $T \leqslant t < \infty$, $v(t)$ is continuous and $v^2(t) = -u'(t)$. Furthermore, (3.4) and (3.5) imply that

$$v(t) = P(t) + u(t) ,$$

and so,

$$v'(t) = P'(t) + u'(t) = -p(t) - v^2(t) ,$$

that is, $v(t)$ is a solution of (2.1) on $[T, \infty)$. Hence, (3.1) defines a disconjugate solution of (1.1), and Theorem 1.1 is proven.

Proof of Theorem 1.2. Assume that (1.1) has a non-oscillatory solution $y(t)$ on $[T, \infty)$. Then there exists $T_1 \geqslant T$ such that $y(t) \neq 0$ for all $t \geqslant T_1$, and $v(t)$ defined by (3.2) satisfies equations (2.1) and (2.2). The assumptions of the theorem and (1.21) imply that $\lim A_\sigma(\tau, t)$ as $\tau \to \infty$ exists for all $t \geqslant T$, and

$$P(t) = \lim_{\tau \to \infty} A_\sigma(\tau, t) , \qquad t \geqslant T . \tag{3.6}$$

Hence, Lemma 2.1 implies

$$\int_{T_1}^{\infty} v^2(s)\,ds < \infty ,$$

and so the assumptions of Lemma 2.2 are satisfied in the present situation. Hence, (2.12) and (3.6) imply that $v(t)$ satisfies the equation

$$v(t) = P(t) + \int_t^{\infty} v^2(s)\,ds , \qquad t \geqslant T_1 . \tag{3.7}$$

Let

$$u(t) = \int_t^{\infty} v^2(s)\,ds , \qquad t \geqslant T_1 .$$

Equation (3.7) implies that

$$u'(t) = -v^2(t) = -P^2(t) - 2P(t)u(t) - u^2(t) , \qquad t \geqslant T_1 ,$$

since $v(t)$ is continuously differentiable. Multiplying this equation by an integrating factor for $u'(t)+2P(t)u(t)$ and integrating, we obtain for $t_1 \geqslant t \geqslant T_1$

$$(3.8)\quad u(t) = u(t_1)\exp\Big(2\int_t^{t_1} P(s)\,ds\Big)+\int_t^{t_1} P^2(s)\exp\Big(2\int_t^{s} P(\tau)\,d\tau\Big)ds+$$
$$+\int_t^{t_1} u^2(s)\exp\Big(2\int_t^{s} P(\tau)\,d\tau\Big)ds$$
$$\geqslant \int_t^{t_1} P^2(s)\exp\Big(2\int_t^{s} P(\tau)\,d\tau\Big)ds+\int_t^{t_1} u^2(s)\exp\Big(2\int_t^{s} P(\tau)\,d\tau\Big)ds\,.$$

The right-hand side of (3.8) is a non-decreasing function of t_1 bounded by $u(t)$. Hence, the limit as $t_1\to\infty$ must exist, and we conclude that

$$(3.9)\qquad u(t) \geqslant \bar{P}(t)+\int_t^{\infty} u^2(s)\exp\Big(2\int_t^{s} P(\tau)\,d\tau\Big)ds\,,\qquad t \geqslant T_1\,.$$

However, (1.18) and Lemma 2.4 imply that no continuous function $u(t)$ can satisfy (3.9) for all $t \geqslant T_1$. From this contradiction, we conclude that (1.1) is oscillatory.

Proof of Theorem 1.3. Theorem 1.3 is a direct consequence of Lemmas 2.1 and 2.2.

Proof of Theorems 1.4 and 1.5. The details for just part (i) of Theorem 1.4 will be given, since part (j) of Theorem 1.5 is similar, and the other parts of both theorems offer no difficulty.

Multiplying inequality (i) by $P(t)/\int_t^{\infty} P^2(s)\,ds$, which is non-negative by assumption, and integrating in an appropriate manner, we obtain

$$\exp\Big(2\int_s^{r} P(\tau)\,d\tau\Big) \leqslant \Big(\int_s^{\infty} P^2(\tau)\,d\tau\Big/\int_r^{\infty} P^2(\tau)\,d\tau\Big)^{1/2},\qquad r \geqslant s \geqslant T\,.$$

Hence, $\bar{P}(s)$ exists for $s \geqslant T$ and

$$\bar{P}(s) \leqslant \int_t^{\infty} P^2(r)\Big(\int_s^{\infty} P^2(\tau)\,d\tau\Big/\int_r^{\infty} P^2(\tau)\,d\tau\Big)^{1/2}dr \leqslant 2\int_s^{\infty} P^2(\tau)\,d\tau \leqslant P(s)/2\,,$$

from which (i) follows from (ii).

Proof of Corollary 1.2. Let α and β be numbers satisfying

$$\lim_{t\to\infty} A_h(t, T) > \beta > \alpha > \lim_{t\to\infty} A_g(t, T)\,.$$

Let $f(t) = h(t)$ for $T \leqslant t < t_1$, where t_1 is determined so that $A_h(t_1, T) \geqslant \beta$ and $\int_T^{t_1} h(s)ds \geqslant 1$. Let $f(t) = g(t)$ for $t_1 \leqslant t < t_2$, where t_2 is determined so that $A_f(t_2, T) \leqslant \alpha$ and $\int_T^{t_2} f(s)ds \geqslant 2$. This is possible because

$$A_f(t_2, T) = A_g(t_2, T)\big(1+o(1)\big)+o(1) \qquad \text{as } t_2 \to \infty .$$

Continuing in this manner, we obtain a non-negative, non-integrable, bounded function $f(t)$ defined on $[T, \infty)$ such that

$$\limsup_{t\to\infty} A_f(t, T) \geqslant \beta > \alpha \geqslant \liminf_{t\to\infty} A^f(t, T) .$$

Hence, by Theorem 1.3, equation (1.1) is oscillatory.

4. Examples. Theorems 1.1 and 1.2 are sharp for the Euler equation

$$y''+\mu t^{-2}y = 0 \qquad (\mu \text{ constant}) .$$

Corollary 1.1 in conjunction with (iii) and (jjj) of Theorems 1.4 and 1.5 implies that

$$y''+(\mu t^{\eta}\sin \nu t)y = 0 \qquad (\mu \neq 0, \nu \neq 0, \eta \text{ constants}) \tag{4.1}$$

is oscillatory when $-1 < \eta < 0$ and non-oscillatory when $\eta < -1$. When $\eta = -1$, Corollary 1.1 implies (4.1) is oscillatory when $|\mu/\nu| > 1/\sqrt{2}$ and non-oscillatory when $|\mu/\nu| < 1/\sqrt{2}$. The oscillation properties of (4.1) cannot be determined to this extent by any of the previous criteria mentioned in the introduction. In particular, Coles' criteria for oscillation cannot be used when $\eta < 0$, because $\int^{\infty} p(s)ds < \infty$. When $\eta \geqslant 0$, one can prove oscillation for all values of μ, ν by using Theorem 1.3 with

$$f(t) = g(t) = \begin{cases} 1 & \text{when } \mu\nu\cos \nu t \geqslant 0 , \\ 0 & \text{otherwise} . \end{cases}$$

For the equation

$$y''+(\mu t^{-1}\sin \nu t+\lambda t^{-2})y = 0 \qquad (\mu, \lambda, \nu \neq 0 \text{ constants}) , \tag{4.2}$$

we obtain

$$P(t) = \int_t^{\infty} p(s)ds = \lambda t^{-1}+\mu\nu^{-1}t^{-1}\cos \nu t+O(t^{-2}) \qquad \text{as } t \to \infty .$$

Hence, $\int^{\infty} P(s)ds$ does not converge, and so the computational advantages of Theorems 1.4 and 1.5 cannot be utilized. However, no difficult problems arise in computing directly the asymptotic nature of $\bar{P}(t)$. Theorems 1.1 and 1.2 imply that (4.2) is oscillatory if

$$\lambda > \tfrac{1}{4} - \tfrac{1}{2}(\mu/\nu)^2$$

and non-oscillatory if

$$\lambda < \tfrac{1}{4} - \tfrac{1}{2}(\mu/\nu)^2 .$$

The equation

$$y'' + (\mu t \cos \nu t^2) y = 0 \qquad (\mu \neq 0, \nu \neq 0 \text{ constants}) \tag{4.3}$$

offers a non-trivial application of Theorem 1.3, which illustrates the advantage of unbounded weight functions. Here, we have

$$\int_0^t \mu s \cos \nu s^2 ds = \mu \sin \nu t^2 / 2\nu . \tag{4.4}$$

Theorem 1.3 implies that (4.3) oscillates for all values of μ, ν if there exists a function $f \in \mathfrak{I}_0$ such that

$$\lim_{t\to\infty} \frac{\int_0^t f(s) \sin \nu s^2 ds}{\int_0^t f(s) ds}$$

does not exist.

We now show the existence of such a function.

Let $k(n)$ be any integer valued function such that for $n = 1, 2, \ldots$, $0 \leqslant k(n) \leqslant n$, $k(n)$ is non-decreasing, and

$$\limsup_{n\to\infty} k(n)/n > \liminf_{n\to\infty} k(n)/n .$$

Let

$$I_+^n = \{0 \leqslant t \leqslant \sqrt{2\pi n/\nu} : \sin \nu t^2 \geqslant 0\}$$

and

$$I_-^n = \{0 \leqslant t \leqslant \sqrt{2\pi n/\nu} : \sin \nu t^2 < 0\} .$$

Then I_+^n and I_-^n each have n components. Define the function $f(t)$ inductively so that $f(t) = t$ for t in I_+^n and $k(n)$ components of I_-^n. In the remaining $n - k(n)$ components of I_-^n, let $f(t) = 0$. It is an easy calculation to prove that $f \in \mathfrak{I}_0$ and that

$$\lim_{n\to\infty} \frac{\int_0^{\sqrt{2\pi n/\nu}} f(s) \sin \nu s^2 ds}{\int_0^{\sqrt{2\pi n/\nu}} f(s) ds} = \lim_{n\to\infty} \frac{2}{\pi} \frac{1 - k(n) n^{-1}}{1 + k(n) n^{-1}}$$

does not exist.

By letting

$$f(t) = \begin{cases} 1 & \text{when } \sin \nu t^2 \geqslant 0\,, \\ 0 & \text{when } \sin \nu t^2 < 0 \end{cases}$$

and putting $P(t)$ equal the weighted average of (4.4) with respect to this function $f(t)$ in the usual manner, we can also obtain the oscillation of (4.3) from Theorem 1.2.

5. Extensions of the results. If

$$\int^{\infty} p(s)\,ds = -\infty\,, \tag{5.1}$$

then none of the "oscillation" theorems in the previous sections or the "oscillation" results cited in the introduction can be applied directly to $y''+p(t)y = 0$. In the case of Theorems 1.1 and 1.3, the problem is that $\liminf A_f(t, \cdot) = -\infty$ as $t\to\infty$ for all $f \in \mathfrak{F}$. Fortunately, this situation can be eliminated for most practical cases by transforming the equation $y''+p(t)y = 0$ into a similar equation without the property (5.1). Of course, $\liminf \int^t p(s)\,ds > -\infty$ as $t\to\infty$ implies $\liminf A_f(t, \cdot) > -\infty$ as $t\to\infty$ for all $f \in \mathfrak{F}$.

THEOREM 5.1. *Let $\gamma(t)$ be any function from the class $C^2[T, \infty)$ satisfying*

$$\gamma(t) > 0\,, \qquad \int^{\infty} \gamma^{-2}(s)\,ds = \infty\,. \tag{5.2}$$

Then $y''+p(t)y = 0$ and $y''+q(t)y = 0$, where

$$p(t) = \frac{q\left(\int^t \gamma^{-2}(s)\,ds\right)}{\gamma^4(t)} - \frac{\gamma''(t)}{\gamma(t)}\,, \tag{5.3}$$

have the same oscillatory behavior.

Proof. The substitutions

$$\tau = \int^t \gamma^{-2}(s)\,ds\,, \qquad y(\tau) = \gamma^{-1}(t)\,x(t)$$

transform

$$\frac{d^2y}{d\tau^2} + q(\tau)y = 0\,, \qquad T_1 \leqslant \tau < \infty\,,$$

into

$$\frac{d^2x}{dt^2} + p(t)x = 0\,, \qquad T \leqslant t < \infty\,,$$

where $p(t)$ is given by (5.3). Oscillation is invariant with respect to this transformation.

Interesting choices for $\gamma(t)$ are $\sqrt{t}$, $\sqrt{t\ln t}$, etc. For example, suppose

$$q_-(t) = \inf\big(0, q(t)\big) = O(e^{e^t}) \quad \text{as } t\to\infty .$$

Then, by letting $\gamma(t) = \sqrt{t\ln t}$, we obtain

$$\frac{|q_-(\ln\ln t)|}{\gamma^4(t)} \leqslant \frac{Ae^{e^{\ln\ln t}}}{t^2\ln^2 t} = \frac{A}{t\ln^2 t}, \qquad t \geqslant T .$$

Hence,

$$\liminf_{t\to\infty} \int^t p(s)\,ds > -\infty ,$$

and there is a chance that the oscillation criteria of Theorems 1.2 and 1.3 can be applied to $x''+p(t)x = 0$.

Theorem 5.1 is also useful as a computational tool. For example, by letting $\gamma(t) = t^{1/4}$, one can show that the behavior of

$$y''+(\mu\sin\nu t^2)y = 0 \qquad (\mu, \nu \text{ constants} \neq 0) \tag{5.4}$$

with regard to oscillation is the same as the behavior of a subclass of equations of the type (4.2), whose oscillation behavior has already been determined. Equation (5.4) is oscillatory if $|\mu/\nu| > \sqrt{2}$ and non-oscillatory if $|\mu/\nu| < \sqrt{2}$.

Some of the oscillation results which have appeared in the literature (see, e.g., [4], [8], [11], [14] and [20]) are for the most part simple consequences of a transformation, which fits in the framework of Theorem 5.1, of a previously known oscillation result. For example, Opial [14] proves that if there exists a function $w(t)$ such that $w \in C^1[T, \infty)$, $w > 0$, w' is bounded, and

$$\int^\infty w(s)\left[q(s)-\frac{1}{4}\left(\frac{w'(s)}{w(s)}\right)^2\right]ds = \infty ,$$

then $y''+q(t)y = 0$ is oscillatory. This test follows from Theorem 5.1 and the well-known Fite-Wintner test,

$$\int^\infty p(s)\,ds = \infty \Rightarrow y''+p(t)y = 0 \tag{5.5}$$

is oscillatory, by taking

$$t = \int_T^{\varphi(t)} w^{-1}(s)\,ds , \qquad \varphi(t) = \int^t \gamma^{-2}(s)\,ds .$$

Another example is the so-called *logarithmic sequence* of tests, which can traced back to at least Riemann-Weber ([15]; 1912, pp. 60-62). In the context of Theorem 5.1, the logarithmic sequence of tests are derived from a given oscillation criterium, such as given in (5.5), by taking

$$\gamma(t) = \sqrt{t}\,, \quad \sqrt{t\ln t}\,, \quad \text{etc.}$$

We end with a theorem which relates the oscillatory behavior of the equation

$$(5.6) \qquad [r(t)y']' + q(t)y = 0\,, \quad T_1 \leqslant t < \infty\,,$$

to the oscillatory behavior of an equation of the form (1.1). Transformations which preserve the infinite length of the interval for the case when $r^{-1}(t)$ is integrable on $[T, \infty)$ do not seem to be well known.

THEOREM 5.2. *Assume* $r(t) > 0$.

(i) *If* $\int^{\infty} r^{-1}(t)\,dt < \infty$, *then the transformation*

$$\tau = \Big(\int_t^{\infty} r^{-1}(s)\,ds\Big)^{-1}, \quad y(t) = \tau^{-1}x(\tau)\,,$$

transforms (5.6) *into*

$$\frac{d^2x}{d\tau^2} + r(t)q(t)\tau^{-4}x = 0\,, \quad T \leqslant \tau < \infty\,,$$

and leaves oscillation invariant.

(ii) *If* $\int^{\infty} r^{-1}(t)\,dt = \infty$, *then the transformation*

$$\tau = \int_T^{t} r^{-1}(s)\,ds\,, \quad y(t) = x(\tau)\,,$$

transforms (5.6) *into*

$$\frac{d^2x}{d\tau^2} + r(t)q(t)x = 0\,, \quad T \leqslant \tau < \infty\,,$$

and leaves oscillation invariant.

Theorem 5.2, which is not the only theorem if its type, implies the existence of a one-to-one correspondence between oscillation results for (1.1) and (5.6). The Fite-Wintner test (5.5) becomes:

$$\text{(i)} \qquad \int^{\infty} r^{-1}(t)\,dt = \infty \quad \text{and} \quad \int^{\infty} q(t)\,dt = \infty\,,$$

or

$$\text{(ii)} \qquad \int^{\infty} r^{-1}(t)\,dt < \infty \quad \text{and} \quad \int^{\infty} q(s)\Big(\int_s^{\infty} r^{-1}(\tau)\,d\tau\Big)^2 ds = \infty$$

imply (5.6) is oscillatory. On the other hand, the well-known result (see Moore [9])

$$\int^{\infty} r^{-1}(t)\,dt < \infty \quad \text{and} \quad \int^{\infty} |q(t)|\,dt < \infty \Rightarrow \text{non-oscillation}$$

for (5.6) is equivalent by Theorem 5.2 to

$$\int^{\infty} t^2|p(t)|\,dt < \infty \Rightarrow \text{non-oscillation}$$

for (1.1). One can cite many other examples of equivalent oscillation results for (1.1) and (5.6), some of which have appeared at different places in the literature and some of which may be new. We end with a corollary that gives the results for (5.6) that are equivalent to Theorems 1.1 thru 1.3 for (1.1).

COROLLARY 5.1. (i) *If* $\int^{\infty} r^{-1}(t)\,dt = \infty$, *then Theorems* 1.1, 1.2, 1.3 *with all differentials* du *replaced by* $r^{-1}(u)\,du$ *and* $p(t)$ *replaced by* $q(t)r(t)$ *are valid for equation* (5.6).

(ii) *If* $\int^{\infty} r^{-1}(t)\,dt < \infty$, *then Theorems* 1.1, 1.2, 1.3 *with all differentials* du *replaced by* $r^{-1}(u)\Big(\int_u^{\infty} r^{-1}(t)\,dt\Big)^{-2} du$ *and* $p(t)$ *replaced by* $q(t)r(t)\Big(\int_t^{\infty} r^{-1}(s)\,ds\Big)^4$ *are valid for equation* (5.6).

Corollary 5.1 states that the criteria given in Theorems 1.1 thru 1.3 apply to (5.6) if the appropriate changes are made in all the equations occurring in the statements of these theorems. This means that even the equations defining the sets $\mathfrak{I}$ and $\mathfrak{I}_0$ have to be changed, e.g., in the case (i), $f \in \mathfrak{I}_0$ if

$$\lim_{t\to\infty} \frac{\int^{t} f^2(s)\,r^{-1}(s)\,ds}{\Big(\int^{t} f(s)\,r^{-1}(s)\,ds\Big)^2} = 0\,.$$

References

[1] L. Cèsari, *Asymptotic behavior and stability problems in ordinary differential equations*, Berlin 1959.

[2] W. J. Coles, *An oscillation criterion for second-order differential equations*, Proc. Amer. Math. Soc. 19 (1968), pp. 755-759.

[3] W. B. Fite, *Concerning the zeros of the solutions of certain differential equations*, Trans. Amer. Math. Soc. 19 (1918), pp. 341-352.

[4] E. Gagliardo, *Sui criteri di oscillazione per gli integrali di un'equazione differenziale lineare del secondo ordine*, Boll. Un. Mat. Ital. 9 (1954), pp. 177-189.

[5] P. Hartman, *On non-oscillatory linear differential equations of second order*, Amer. Journ. Math. 74 (1952) p. 389-400.

[6] E. Hille, *Non-oscillation theorems*, Trans. Amer. Math. Soc. 64 (1948), pp. 234-252.

[7] A. Kneser, *Untersuchungen über die reellen Nullstellen der Integrale linear Differentialgleichungen*, Math. Ann. 42 (1893), pp. 409-435.

[8] J. C. P. Miller, *On a criterion for oscillatory solutions of a linear differential equation of the second order*, Proc. Cambridge Philos. Soc. 36 (1940), pp. 283-287.

[9] R. A. Moore, *The behavior of solutions of a linear differential equation of second order*, Pacific Journ. Math. 5 (1955), pp. 125-145.

[10] Z. Nehari, *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc. 85 (1957), pp. 428-445.

[11] L. D. Nikolenko, *On a sufficient condition for non-oscillation character of solutions of the equation* $y''+f(x)y = 0$, Dokl. Akad. Nauk SSSR (N. S.) 110 (1956), pp. 929-931.

[12] C. Olech, Z. Opial and T. Ważewski, *Sur le problème d'oscillation des intégrales de l'équation* $y''+g(t)y = 0$, Bull. Acad. Polon. Sci. 5 (1957), pp. 621-626.

[13] Z. Opial, *Sur les intégrales oscillantes de l'équation différentielle* $u''+f(t)u = 0$, Ann. Polon. Math. 4 (1958), pp. 308-313.

[14] — *Sur un critère d'oscillation des intégrales de l'équation différentielle* $(Q(t)x')' +f(t)x = 0$, ibidem 6 (1959-1960), pp. 99-104.

[15] B. Riemann and H. Weber, *Die Partiellen Differentialgleichungen der Mathematischen Physik*, II, 5th ed., Braunschweig (1912), p. 575.

[16] W. Reid, *Oscillation criteria for linear differential systems with complex coefficients*, Pacific Journ. Math. 6 (1956), pp. 733-751.

[17] C. Sturm, *Sur les équations différentielles linéaires du second ordre*, Journ. Math. Pures Appl. 1 (1836), pp. 106-186.

[18] A. Winter, *A criterion of oscillatory stability*, Quart. Appl. Math. 7 (1949), pp. 115-117.

[19] — *On the non-existence of conjugate points*, Amer. Journ. Math. 73 (1951), pp. 368-380.

[20] M. Zlámal, *Oscillation criterions*, Časopis Pěst. Mat. Fis. 75 (1950), pp. 213-217.

Reçu par la Rédaction le 28. 10. 1967