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Inequalities of Grunsky–Nehari type for pairs of vector functions

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Abstract. In the paper a new general definition of a pair of vector functions is formulated and inequalities of Grunsky–Nehari type for such pairs are shown and compared. The paper includes an examination of the equality case of these inequalities. Also, a bilinear form of the inequalities under consideration is given. Finally, "strengthenings" of some of the inequalities are proposed.

Introduction. In 1969 Aharonov [1] introduced a definition of a pair (F, G) of functions F and G, $F(0) = G(0) = 0$, which are analytic and univalent in $\Delta = \{z\colon |z| < 1\}$ such that $F(z)G(\zeta) \neq 1$ for all points $z, \zeta \in \Delta$.

This definition can be generalized in a natural way.

For arbitrary non-negative integers m, n such that $m+n \geqslant 2$ let

$$\begin{aligned} a_0 &= \{a_{01}, \ldots, a_{0m}\}, && m \geqslant 1, \\ &= \emptyset, && m = 0, \\ b_0 &= \{b_{0m+1}, \ldots, b_{0m+n}\}, && n \geqslant 1, \\ &= \emptyset, && n = 0, \end{aligned}$$

be two sets of finite elements satisfying the conditions:

$$\begin{aligned} a_{0k} &\neq a_{0j}, \quad k \neq j, \quad k, j = 1, \ldots, m, && m > 1, n \geqslant 0, \\ b_{0k} &\neq b_{0j}, \quad k \neq j, \quad k, j = m+1, \ldots, m+n, && m \geqslant 0, n > 1, \\ a_{0k}b_{0j} &\neq 1, \quad k = 1, \ldots, m,\ j = m+1, \ldots, m+n, && m, n \geqslant 1. \end{aligned}$$

Let C denote the complex plane. Generalizing Aharonov's definition of a pair, we shall say that two mappings F and G, $F = (F_1, \ldots, F_m)\colon \Delta \to C^m$ for $m \geqslant 1$, $G = (G_{m+1}, \ldots, G_{m+n})\colon \Delta \to C^n$ for $n \geqslant 1$ and $F = 0$ or $G = 0$ whenever $m = 0$ or $n = 0$, respectively, *constitute a pair* (F, G)

if $F_1, \ldots, F_m$ and $G_{m+1}, \ldots, G_{m+n}$ are functions of the form:

$$F_k(z) = a_{0k} + a_{1k}z + \ldots, \quad k = 1, \ldots, m, \quad n \geqslant 0,$$
$$G_k(z) = b_{0k} + b_{1k}z + \ldots, \quad k = m+1, \ldots, m+n, \quad m \geqslant 0$$

analytic and univalent in Δ and satisfying the conditions:

$$F_k(z) \neq F_j(\zeta), \quad k \neq j, \; k, j = 1, \ldots, m, \quad m > 1, n \geqslant 0,$$
$$G_k(z) \neq G_j(\zeta), \quad k \neq j, \; k, j = m+1, \ldots, m+n, \quad m \geqslant 0, \; n > 1,$$
$$F_k(z)G_j(\zeta) \neq 1, \quad k = 1, \ldots, m, \; j = m+1, \ldots, m+n, \; m, n \geqslant 1$$

for all points $z, \zeta \in \Delta$.

The pair $(F, 0)$ is identified with the function F. Clearly, $(F, 0) = (0, F)$.

The class of all pairs (F, G) of vector functions F and G as defined above for fixed admissible m, n, a_0, b_0 is denoted by $C_{m,n}(a_0, b_0)$.

The paper deals with inequalities of Grunsky–Nehari type ([11], [19]) for the classes $C_{m,n}(a_0, b_0)$. In Section 1 we introduce denotations. Section 2 contains proofs of three general inequalities for the coefficients of F, G for pairs (F, G) of classes $C_{m,n}(a_0, b_0)$. In Sections 3–5 we establish relations between these inequalities, examine the case of equality and give a bilinear form of the inequalities. The last section includes some "strengthenings".

1. Definitions, denotations and remarks. Let $(F, G) \in C_{m,n}(a_0, b_0)$. Define:

$$A_p^{kj}(z) = \sum_{q=-p}^{\infty} a_{qp}^{kj} z^q, \quad k, j = 1, \ldots, m+n, \quad p = 1, \ldots, \quad z \in \Delta,$$

$$B^{kj}(z, \zeta) = \sum_{q,p=0}^{\infty} b_{qp}^{kj} z^q \zeta^p, \quad k, j = 1, \ldots, m+n, \quad z, \zeta \in \Delta,$$

where

$$A_p^{kj}(z)$$
$$= \left[\frac{1}{F_k(z) - a_{0j}}\right]^p, \quad k, j = 1, \ldots, m, \quad m \geqslant 1, \; n \geqslant 0,$$
$$= \left[\frac{F_k(z)}{1 - F_k(z) b_{0j}}\right]^p, \quad k = 1, \ldots, m, \; j = m+1, \ldots, m+n, \quad m, n \geqslant 1,$$
$$= \left[\frac{1}{G_k(z) - b_{0j}}\right]^p, \quad k, j = m+1, \ldots, m+n, \quad m \geqslant 0, \; n \geqslant 1,$$
$$= \left[\frac{G_k(z)}{1 - G_k(z) a_{0j}}\right]^p, \quad k = m+1, \ldots, m+n, \; j = 1, \ldots, m, \quad m, n \geqslant 1,$$

$$B^{kk}(z,\zeta) = \log\frac{F_k(z)-F_k(\zeta)}{z-\zeta}, \quad k = 1,\dots,m, \quad m \geqslant 1,\ n \geqslant 0,$$
$$= \log\frac{G_k(z)-G_k(\zeta)}{z-\zeta}, \quad k = m+1,\dots,m+n,\ m \geqslant 0, n \geqslant 1,$$

$B^{kj}(z,\zeta)$

$$= \log[F_k(z)-F_j(\zeta)], \quad k \neq j,\ k,j = 1,\dots,m,\ m \geqslant 2,\ n \geqslant 0,$$
$$= \log[G_k(z)-G_j(\zeta)], \quad k \neq j,\ k,j = m+1,\dots,m+n,\ m \geqslant 0,\ n \geqslant 2,$$
$$= \log[1-F_k(z)G_j(\zeta)], \quad k = 1,\dots,m,\ j = m+1,\dots,m+n,\ m,n \geqslant 1,$$
$$= \log[1-G_k(z)F_j(\zeta)], \quad k = m+1,\dots,m+n,\ j = 1,\dots,m,\ m,n \geqslant 1.$$

Moreover, we define:

$$a_{q0}^{kj} = b_{q0}^{kj}, \quad k,j = 1,\dots,m+n,\ q = 0,1,\dots$$

From the definition of $A_p^{kj}(z)$ we obtain immediately the equalities:

$$a_{-qp}^{kj} = 0, \quad p = 1,\dots,k \neq j,\ k,j = 1,\dots,m+n,\ q = 1,\dots,p.$$

If $m, n \geqslant 1$, let x_{kj}^0, $k = 1,\dots,m$, $j = m+1,\dots,m+n$, be arbitrary complex numbers and let

$$x_0 = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0m+n} \end{bmatrix},$$

where

$$x_{0k} = \sum_{j=m+1}^{m+n} x_{kj}^0, \quad k = 1,\dots,m,$$
$$= -\sum_{j=1}^{m} x_{jk}^0, \quad k = m+1,\dots,m+n.$$

If $n = 0$ $(m = 0)$, let x_{0k}, $k = 1,\dots,m$ $(k = 1,\dots,n)$, be arbitrary complex numbers satisfying the condition

$$\sum_{k=1}^{m} x_{0k} = 0 \quad \Big(\sum_{k=1}^{n} x_{0k} = 0\Big),$$

and let

$$x_0 = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0m} \end{bmatrix} \quad \left(x_0 = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0n} \end{bmatrix}\right).$$

For any natural number N, let

$$x_k = \begin{bmatrix} x_{1k} \\ \vdots \\ x_{Nk} \end{bmatrix}, \quad k = 1,\dots,m+n,$$

where x_{qp}, $q = 1, \ldots, N$, $p = 1, \ldots, m+n$, are arbitrary complex numbers.

Let N_k, $k = 1, \ldots, m+n$, be any natural numbers. If $m, n \geqslant 1$, let y^0_{ji}, $j = 1, \ldots, m+n$, $i = 1, \ldots, N_j$, be any complex numbers such that

$$\sum_{i=1}^{N_k} y_{ki} = 0, \quad k = 1, \ldots, m+n,$$

where

$$\begin{aligned} y_{ki} &= \Big(\sum_{j=m+1}^{m+n} \sum_{p=1}^{N_j} y^0_{jp}\Big) y^0_{ki}, \quad k = 1, \ldots, m, \\ &= -\Big(\sum_{j=1}^{m} \sum_{p=1}^{N_j} y^0_{jp}\Big) y^0_{ki}, \quad k = m+1, \ldots, m+n. \end{aligned}$$

If $n = 0$ $(m = 0)$, let y_{ki}, $i = 1, \ldots, N_k$, $k = 1, \ldots, m$ $(k = 1, \ldots, n)$, be arbitrary complex numbers such that

$$\sum_{i=1}^{N_k} y_{ki} = 0, \quad k = 1, \ldots, m \ (k = 1, \ldots, n).$$

Define one-column matrices

$$y_k z_k = \Big(\sum_{i=1}^{N_k} \frac{1}{\sqrt{q}} y_{ki} z^q_{ki}\Big)_{1 \leqslant q}, \quad k = 1, \ldots, m+n,$$

where z_{ji}, $j = 1, \ldots, m+n$, $i = 1, \ldots, N_j$, are arbitrary fixed points of the unit disc Δ.

Denote:

$$\begin{aligned} &\alpha^{kj}_{qp} = \sqrt{\frac{q}{p}}\, a^{kj}_{qp}, && \beta^{kj}_{qp} = \sqrt{qp}\, b^{kj}_{qp}, && q, p = 1, \ldots, \\ &&&&& k, j = 1, \ldots, m+n, \\ &\alpha^{kj}_{0p} = \frac{1}{\sqrt{p}}\, a^{kj}_{0p}, && \beta^{kj}_{0p} = \sqrt{p}\, b^{kj}_{0p}, && p = 1, \ldots, \\ &&&&& k, j = 1, \ldots, m+n, \\ &\alpha^{kj}_{00} = a^{kj}_{00}, && \beta^{kj}_{00} = b^{kj}_{00}, && k, j = 1, \ldots, m+n, \\ &\alpha^{kk}_{-qp} = \sqrt{\frac{q}{p}}\, a^{kk}_{-qp}, && && q, p = 1, \ldots, k = 1, \ldots, m+n. \end{aligned}$$

We define the following matrices:

$$\begin{gathered} A_{00} = (\alpha^{kj}_{00})^{m+n}_{k,j=1}, \quad B_{00} = (\beta^{kj}_{00})^{m+n}_{k,j=1}, \\ A^{kj} = (\alpha^{kj}_{qp})_{1 \leqslant q, 1 \leqslant p}, \quad B^{kj} = (\beta^{kj}_{qp})_{1 \leqslant q, 1 \leqslant p}, \quad k, j = 1, \ldots, m+n, \\ \hat{A}^{kk} = (\alpha^{kk}_{-qp})_{1 \leqslant q, 1 \leqslant p}, \quad k = 1, \ldots, m+n, \end{gathered}$$

one-row matrices:

$$A_{01}^{kj} = (\alpha_{0p}^{kj})_{1\leqslant p}, \quad B_{01}^{kj} = (\beta_{0p}^{kj})_{1\leqslant p}, \quad k, j = 1, \ldots, m+n,$$

$$A_{q}^{kj} = (\alpha_{qp}^{kj})_{1\leqslant p}, \quad B_{q}^{kj} = (\beta_{qp}^{kj})_{1\leqslant p}, \quad q = 1, \ldots, \ k, j = 1, \ldots, m+n,$$

and one-column matrices:

$$A_{10}^{kj} = (\alpha_{q0}^{kj})_{1\leqslant q}, \quad B_{10}^{kj} = (\beta_{q0}^{kj})_{1\leqslant q}, \quad k, j = 1, \ldots, m+n.$$

We apply the following denotations:

$$D(F, G) = \bigcup_{k=1}^{m} F_k(\Delta) \cup \bigcup_{k=m+1}^{m+n} 1/G_k(\Delta), \qquad m, n \geqslant 1,$$

$$= \bigcup_{k=1}^{m} F_k(\Delta) \Big(\bigcup_{k=1}^{n} G_k(\Delta)\Big), \qquad n = 0 \ (m = 0).$$

The transpose of a matrix A is denoted by A'.

The norm $\|A\|$ of a one-column matrix A is defined as follows:

$$\|A\|^2 = A^* A,$$

where A^* stands for the conjugate transpose of A.

2. The basic theorem. In 1972 Hummel [12] obtained inequalities of Grunsky–Nehari type for the class A of Aharonov's pairs. In particular cases he obtained from these inequalities: Aharonov inequalities [1]; Nehari inequalities [19] as well as those of Schiffer and Tammi [22] for bounded functions, i.e., for pairs $(F, \bar{F})$, where $\bar{F}(z) = \overline{F(\bar{z})}$ $(z \in \Delta)$; Hummel and Schiffer inequalities [13] for Bieberbach–Eilenberg functions ([3], [4], [7]), i.e., for pairs (F, F).

The classes $C_{m,n}(a_0, b_0)$ of pairs (F, G) of vector functions F and G play the role analogous to that of Aharonov's class A. On choosing suitable m, n, a_0, b_0 and G we obtain the classes investigated so far by means of area methods and defined in the papers of: Lebedev [15], [16], Jenkins [14], Gromova and Lebedev [8], [9], De Temple [5], [6], Libera [17], Śladkowska [24], [25].

Moreover, owing to the properties of the introduced definition of a pair (F, G) of vector functions F and G and to the properties of Aharonov pairs, area inequalities in all classes $C_{m,n}(a_0, b_0)$ may be given a uniform form. Also, vector-matrix notation greatly contributed to this advantage.

The following theorem holds:

THEOREM 1. *If* $(F, G) \in C_{m,n}(a_0, b_0)$, *then for arbitrary admissible matrices* $x_0, x_k, y_k z_k$, $k = 1, \ldots, m+n$,

$$(2.1)\qquad 2\mathrm{Re}\{x_0^* A_{00} x_0\} + 2\mathrm{Re}\Big\{\sum_{k,j=1}^{m+n} \bar{x}_{0k} A_{01}^{kj} x_j\Big\} + \sum_{k=1}^{m+n} \Big\|\sum_{j=1}^{m+n} (A_{10}^{kj} x_{0j} + A^{kj} x_j)\Big\|^2 \leqslant \sum_{k=1}^{m+n} \|\hat{A}^{kk} x_k\|^2,$$

$$(2.2)\qquad 2\mathrm{Re}\{x_0^* B_{00} x_0\} + 2\mathrm{Re}\Big\{\sum_{k,j=1}^{m+n} \bar{x}_{0k} B_{01}^{kj} x_j\Big\} + \sum_{k=1}^{m+n} \Big\|\sum_{j=1}^{m+n} (B_{10}^{kj} x_{0j} + B^{kj} x_j)\Big\|^2 \leqslant \sum_{k=1}^{m+n} \|x_k\|^2$$

and

$$(2.3)\qquad \sum_{k=1}^{m+n} \Big\|\sum_{j=1}^{m+n} B^{kj} y_j z_j\Big\|^2 \leqslant \sum_{k=1}^{m+n} \|y_k z_k\|^2.$$

For any matrices $x_0, x_k, y_k z_k$, $k = 1, \ldots, m+n$, *the equalities are satisfied only if* $\mu[C \setminus D(F, G)] = 0$.

Proof. First consider the case $m, n \geqslant 1$.

Let $r_0 = |z_0|$ when $G_k(z_0) = 0$ for some $k = m+1, \ldots, m+n$ and $z_0 \in \Delta$ or $r_0 = 0$ when $G_k(z) \neq 0$ for each $k = m+1, \ldots, m+n$ and $z \in \Delta$.

For an arbitrary fixed $r \in (r_0; 1)$, let D_r be a region which is $(m+n)$-connected, bounded by curves Γ_k, $k = 1, \ldots, m+n$, has the following parametric representation:

$$\begin{aligned} \Gamma_k\colon\ w &= F_k(z), \qquad k = 1, \ldots, m, \\ &= 1/G_k(z), \qquad k = m+1, \ldots, m+n, \end{aligned}$$

where $z = r\exp(i\varphi)$ $(0 \leqslant \varphi \leqslant 2\pi)$ and the orientation of its boundary ∂D_r is determined by the parameter φ running from 0 to 2π.

Let P be a function analytic in the closure $\bar{D}_r$. Then for arbitrary complex numbers x_{kj}^0, $k = 1, \ldots, m$, $j = m+1, \ldots, m+n$, a function g of the form

$$(2.4)\qquad g(w) = \sum_{k=1}^{m} \sum_{j=m+1}^{m+n} x_{kj}^0 \log \frac{w - a_{0k}}{1 - w b_{0j}} + P(w)$$

is analytic in $\bar{D}_r$.

Moreover, let

$$(2.5)\qquad \lim_{R\to\infty} \int_{|w|=R} \overline{g(w)}\, dg(w) = 0,$$

whenever $\infty \in D_r$.

We define

$$(2.6)\quad \sum_{q=-\infty}^{\infty} c_q^k z^q + x_{0k}\log z = g[F_k(z)], \qquad k=1,\dots,m, \qquad (|z|=r)$$
$$= g[1/G_k(z)], \qquad k=m+1,\dots,m+n$$

where

$$x_{0k} = \sum_{j=m+1}^{m+n} x_{jk}^0, \qquad k=1,\dots,m,$$
$$= -\sum_{j=1}^{m} x_{jk}^0, \qquad k=m+1,\dots,m+n.$$

If $\infty \in D_r$, let $|w| = R$ be a circle negatively oriented with respect to its interior and such that the curves Γ_k, $k=1,\dots,m+n$, are inside the circle.

Denote by D_r^0 a region which is one-connected and bounded by the curves Γ_k, $k=1,\dots,m+n$, the circle $|w|=R$ and by some analytic arcs joining the point R of the circle $|w|=R$ with the points

$$w_k = F_k(r), \qquad k=1,\dots,m,$$
$$= 1/G_k(r), \qquad k=m+1,\dots,m+n.$$

If $\infty \notin D_r$, say, it is inside the curve Γ_{m+n}, then the symbol D_r^0 will denote a region which is one-connected, bounded by the curves Γ_k, $k=1,\dots,m+n$, and by some analytic arcs joining the point $w_{m+n} = 1/G_{m+n}(r)$ with the points

$$w_k = F_k(r), \qquad k=1,\dots,m,$$
$$= 1/G_k(r), \qquad k=m+1,\dots,m+n-1.$$

By applying Green's theorem ([18], p. 241), we obtain

$$0 < \iint_{D_r^0} |g'(w)|^2 d\tau = \frac{1}{2i}\int_{\partial D_r^0} \overline{g(w)}\, dg(w).$$

Thus, making use of (2.5), we have

$$0 > \frac{1}{2\pi i}\sum_{k=1}^{m}\Big\{\int_{\Gamma_k}\overline{\Big[g(w)+2\pi i\sum_{p=1}^{k} x_{0p-1}\Big]}\, dg(w) + 2\pi i\,\bar{x}_{0k}\, g[F_k(r)]\Big\} +$$
$$+\frac{1}{2\pi i}\sum_{k=m+n}^{m+n}\Big\{\int_{\Gamma_k}\overline{\Big[g(w)+2\pi i\sum_{p=1}^{k} x_{0p-1}\Big]}\, dg(w) + 2\pi i\bar{x}_{0k}\, g[1/G_k(r)]\Big\},$$

where $x_{00} = 0$. Hence, by (2.6),

$$0 > \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=1}^{m+n}\Big[\bar{x}_{0k}(\log r - i\varphi) + \sum_{q=-\infty}^{\infty}\overline{c_q^k z^q}\Big]\Big[x_{0k} + \sum_{q=-\infty}^{\infty} q c_q^k z^q\Big]\,d\varphi +$$
$$+\sum_{k=1}^{m+n}\Big\{\bar{x}_{0k}\sum_{q=-\infty}^{\infty} c_q^k r^q + |x_{0k}|^2\log r\Big\}.$$

Thus

$$\sum_{k=1}^{m+n}\Big[2\operatorname{Re}\{\bar{x}_{0k}c_0^k\} + \sum_{q=-\infty}^{\infty} q|c_q^k|^2 r^{2q}\Big] < \sum_{k=1}^{m+n} 2|x_{0k}|^2\log r^{-1}.$$

Consequently, when $r \to 1$,

$$\sum_{k=1}^{m+n}\Big[2\operatorname{Re}\{\bar{x}_{0k}c_0^k\} + \sum_{q=-\infty}^{\infty} q|c_q^k|^2\Big] \leqslant 0. \tag{2.7}$$

We shall show inequality (2.1) in the case considered, i.e. for $m, n \geqslant 1$. To this aim, define

$$P(w) = \sum_{p=1}^{N}\Big\{\sum_{j=1}^{m}\frac{x_{pj}}{\sqrt{p}}\Big[\frac{1}{w-a_{0j}}\Big]^p + \sum_{j=m+1}^{m+n}\frac{x_{pj}}{\sqrt{p}}\Big[\frac{w}{1-wb_{0j}}\Big]^p\Big\}, \tag{2.8}$$

where x_{pj}, $p = 1, \ldots, N$, $j = 1, \ldots, m+n$, are arbitrary complex numbers. Then, on using the notation accepted, (2.4) and (2.6) yield

$$\sum_{p=1}^{N}\Big\{\sum_{j=1}^{m+n}\frac{x_{pj}}{\sqrt{p}}A_p^{kj}(z)\Big\} + \sum_{j=1}^{m+n} x_{0j}B^{kj}(z,0) + x_{0k}\log z$$
$$\begin{aligned} &= g[F_k(z)], && k = 1, \ldots, m \\ &= g[1/G_k(z)], && k = m+1, \ldots, m+n \end{aligned} \qquad (|z| = r),$$

$$\begin{aligned} c_q^k &= \frac{1}{\sqrt{q}}\sum_{j=1}^{m+n}\sum_{p=0}^{N} a_{qp}^{kj}x_{pj}, && q = 1, \ldots \\ &= \sum_{j=1}^{m+n}\sum_{p=0}^{N} a_{0p}^{kj}x_{pj}, && q = 0 \\ c_{-q}^k &= \frac{1}{\sqrt{q}}\sum_{p=1}^{N} a_{-qp}^{kk}x_{pk}, && q = 1, \ldots, N \\ &= 0, && q = N+1, \ldots \end{aligned} \qquad (k = 1, \ldots, m+n), \tag{2.9}$$

Thus, applying (2.7), we obtain

$$\sum_{k=1}^{m+n}\Big[2\mathrm{Re}\Big\{\bar{x}_{0k}\sum_{j=1}^{m+n}\sum_{p=0}^{N}a_{0p}^{kj}x_{pj}\Big\}+\sum_{q=1}^{\infty}\Big|\sum_{j=1}^{m+n}\sum_{p=0}^{N}a_{qp}^{kj}x_{pj}\Big|^2\Big] \leqslant \sum_{k=1}^{m+n}\sum_{q=1}^{\infty}\Big|\sum_{p=1}^{N}a_{-qp}^{kj}x_{pk}\Big|^2. \tag{2.10}$$

Inequality (2.10) may be written in the form (2.1).

The proof of inequality (2.2) in the basic theorem is similar to that of inequality (2.1).

Defining then

$$P(w)=\frac{1}{2\pi i}\sum_{p=1}^{N}\int_{|\zeta|=\varrho}\frac{1}{\sqrt{p}\,\zeta^{p+1}}\Big\{\sum_{j=1}^{m}x_{pj}\log\frac{w-F_j(\zeta)}{w-a_{0j}}+\sum_{j=m+1}^{m+n}x_{pj}\log\frac{1-wG_j(\zeta)}{1-wb_{0j}}\Big\}d\zeta, \tag{2.11}$$

where $r_0<\varrho<r<1$ and x_{pj}, $p=1,\ldots,N$, $j=1,\ldots,m+n$, are arbitrary complex numbers, we analogously conclude that

$$\begin{aligned}\frac{1}{2\pi i}\sum_{p=1}^{N}\int_{|\zeta|=\varrho}\frac{1}{\sqrt{p}\,\zeta^{p+1}}\Big\{x_{pk}\Big[B^{kk}(z,\zeta)-B^{kk}(z,0)+\log\Big(1-\frac{\zeta}{z}\Big)\Big]+\\ +\sum_{\substack{j=1\\ j\neq k}}^{m+n}x_{pj}[B^{kj}(z,\zeta)-B^{kj}(z,0)]\Big\}d\zeta+\sum_{j=1}^{m+n}x_{0j}B^{kj}(z,0)+x_{0k}\log z\\ =g[F_k(z)],\quad k=1,\ldots,m\\ =g[1/G_k(z)],\quad k=m+1,\ldots,m+n\end{aligned}\qquad(|z|=r), \tag{2.12}$$

$$\begin{aligned}c_q^k&=\frac{1}{\sqrt{q}}\sum_{j=1}^{m+n}\sum_{p=0}^{N}\beta_{qp}^{kj}x_{pj},\quad q=1,\ldots\\ &=\sum_{j=1}^{m+n}\sum_{p=0}^{N}\beta_{0p}^{kj}x_{pj},\quad q=0\\ c_{-q}^k&=-\frac{1}{\sqrt{q}}x_{qk},\quad q=1,\ldots,N\\ &=0,\quad q=N+1,\ldots\end{aligned}\qquad(k=1,\ldots,m+n), \tag{2.13}$$

and consequently

$$\sum_{k=1}^{m+n}\Big[2\mathrm{Re}\Big\{\bar{x}_{0k}\sum_{j=1}^{m+n}\sum_{p=0}^{N}\beta_{0p}^{kj}x_{pj}\Big\}+\sum_{q=1}^{\infty}\Big|\sum_{j=1}^{m+n}\sum_{p=0}^{N}\beta_{qp}^{kj}x_{pj}\Big|^2\Big]\leqslant\sum_{k=1}^{m+n}\sum_{q=1}^{N}|x_{qk}|^2. \tag{2.14}$$

Hence, on applying the notation accepted, we have (2.2).

Now we shall show inequality (2.3).

First observe that if z_{ji}, $j=1,\ldots,m+n$, $i=1,\ldots,N_j$, N_j — natural numbers, are arbitrary fixed points of Δ and if y_{ji}^0, $j=1,\ldots,m+n$, $i=1,\ldots,N_j$, are arbitrary complex numbers such that

$$\sum_{i=1}^{N_k} y_{ki}=0, \quad k=1,\ldots,m+n,$$

where

$$\begin{aligned} y_{ki} &= \Big(\sum_{j=m+1}^{m+n}\sum_{p=1}^{N_j} y_{jp}^0\Big) y_{ki}^0, \quad && k=1,\ldots,m,\\ &= -\Big(\sum_{j=1}^{m}\sum_{p=1}^{N_j} y_{jp}^0\Big) y_{ki}^0, \quad && k=m+1,\ldots,m+n,\end{aligned}$$

then a function g of the form

$$g(w)=\sum_{k=1}^{m+n}\sum_{i=1}^{N_k}\sum_{j=m+1}^{m+n}\sum_{p=1}^{N_j} y_{ki}y_{jp}\log\frac{w-F_k(z_{ki})}{1-wG_j(z_{jp})} \tag{2.15}$$

is analytic in $\bar{D}_r$ if

$$\max\{r_0, \max_{\substack{k=1,\ldots,m+n\\ i=1,\ldots,N_k}} |z_{ki}|\}<r<1.$$

Then we define

$$\begin{aligned}\sum_{q=-\infty}^{\infty} c_q^k z^q &= g[F_k(z)], \quad && k=1,\ldots,m \\ &= g[1/G_k(z)], \quad && k=m+1,\ldots,m+n\end{aligned} \qquad (|z|=r).$$

Following the procedure described above we construct the region D and making use of Green's theorem we obtain the following inequality:

$$\sum_{k=1}^{m+n}\Big[\sum_{q=-\infty}^{\infty} q|c_q^k|^2\Big]\leqslant 0.$$

Since

$$\begin{aligned}\sum_{i=1}^{N_k} y_{ki}\Big[B^{kk}(z,z_{ki})+\log\Big(1-\frac{z_{ki}}{z}\Big)\Big]+\sum_{\substack{j=1\\ j\neq k}}^{m+n}\sum_{i=1}^{N_j} y_{ji}B^{kj}(z,z_{ji})\\ = g[F_k(z)], \quad k=1,\ldots,m,\\ = g[1/G_k(z)], \quad k=m+1,\ldots,m+n,\end{aligned}$$

we have

$$\begin{aligned} c_q^k &= \sum_{j=1}^{m+n}\sum_{p=1}^{\infty} b_{qp}^{kj}\Big(\sum_{i=1}^{N_k} y_{ji}z_{ji}^p\Big), \quad && q=0,1,\ldots,\\ c_{-q}^k &= -\frac{1}{q}\sum_{i=1}^{N_k} y_{ki}z_{ki}^q, \quad && q=1,\ldots\end{aligned} \tag{2.16}$$

and consequently

$$\sum_{k=1}^{m+n}\sum_{q=1}^{\infty} q\left|\sum_{j=1}^{m+n}\sum_{p=1}^{\infty} b_{qp}^{kj}\left(\sum_{i=1}^{N_j} y_{ji}z_{ji}^{p}\right)\right|^2 \leqslant \sum_{k=1}^{m+n}\sum_{q=1}^{\infty}\frac{1}{q}\left|\sum_{i=1}^{N_k} y_{ki}z_{ki}^{q}\right|^2.$$

Hence, having applied the notation accepted, we immediately obtain (2.3).

Thus inequalities (2.1)–(2.3) for $m, n \geqslant 1$ have been proved.

By applying the procedure described above to the region D_r bounded by the curves:

$$\Gamma_k\colon\ w = F_k(z), \quad k = 1, \ldots, m \ (|z| = r,\ 0 < r < 1)$$

and to the functions:

$$(2.4') \qquad g(w) = \sum_{k=1}^{m} x_{0k}\log(w-a_{0k}) + P(w),$$

$$(2.8') \qquad P(w) = \sum_{p=1}^{N}\left\{\sum_{j=1}^{m}\frac{x_{pj}}{\sqrt{p}}\left[\frac{1}{w-a_{0j}}\right]^p\right\},$$

$$(2.11') \qquad P(w) = \frac{1}{2\pi i}\sum_{p=1}^{N}\int_{|\zeta|=\varrho}\left\{\sum_{j=1}^{m}\frac{x_{pj}}{\sqrt{p}\,\zeta^{p+1}}\log\frac{w-F_j(\zeta)}{w-a_{0j}}\right\}d\zeta$$

$$(0 < \varrho < r < 1),$$

$$(2.15') \qquad g(w) = \sum_{k=1}^{m}\sum_{i=1}^{N_k} y_{ki}\log[w-F_k(z_{ki})] \quad (\max_{\substack{k=1,\ldots,m\\ i=1,\ldots,N_k}} |z_{ki}| < r < 1)$$

playing the role of the functions (2.4), (2.8), (2.11) and (2.15) in case $m, n \geqslant 1$, we obtain the proof of inequalities (2.1)–(2.3) in case $n = 0$.

The case $m = 0$ can be reduced to the case $n = 0$.

The proof of the final remark of Theorem 1 follows from the fact that

$$g'(w) \not\equiv 0$$

and at the same time

$$\iint_{D_r} |g'(w)|^2 d\tau \to 0 \quad \text{when } r \to 1$$

in the case of equalities (2.1)–(2.3).

Remark 1. Inequalities (2.1) in the subclass $C_{1,1}(\{0\}, \{0\})$ of pairs $(F, \bar{F})$ when $x_0 = \bar{x}_0$ were obtained by Schiffer and Tammi [23]. Hummel [12] proved inequalities (2.1) and (2.2) in the class $C_{1,1}(\{0\}, \{0\})$.

De Temple [6] (introducing Faber's polynomials [21]), Lebedev [15], [16] and Gromova and Lebedev [8], [9] — with additional conditions imposed on x_0 and/or with different combinations of coefficients — got area inequalities of type (2.2) for some classes of functions with disjoint images. In our notation these are the classes: $C_{m,0}(a_0, \emptyset)$; the subclass of $C_{m,m}(a_0, \bar{a}_0)$ of pairs $(F, \bar{F})$, where $|a_{0k}| < 1$ for $k = 1, \ldots, m$; $C_{m,1}(a_0, \{0\})$. Inequalities (2.2) in the class $C_{m,0}(a_0, \emptyset)$ in case $x_0 = \bar{x}_0$ are identical with the result of De Temple [6].

Gromova and Lebedev [9] obtained inequalities of type (2.2) in the subclass of $C_{m,m}(a_0, a_0)$ of pairs (F, F) and in the subclass of $C_{m,m}(a_0, -\bar{a}_0)$ of pairs $(F, -\bar{F})$. The subclasses are generalizations of the classes of Bieberbach–Eilenberg's and of Grunsky–Shah's functions ([10], [26], [24]), respectively.

Inequalities (2.3) in the subclasses of $C_{1,1}(\{0\}, \{0\})$ of pairs (F, F) and $(F, -\bar{F})$ were obtained by Jenkins [14] and in the subclass of $C_{2,2}(a_0, \bar{a}_0)$ of pairs $(F, \bar{F})$, where $|a_{01}|, |a_{02}| < 1$, as well as in the subclass of $C_{1,1}(a_0, \bar{a}_0)$ of pairs $(F, \bar{F})$, where $|a_{01}| < 1$, were given by Libera [17].

3. Comparison of the inequalities. In this section we shall examine relations between inequalities (2.1), (2.2) and (2.3).

We claim that inequality (2.2) is a particular case of the power inequality given in (2.1).

Let

$$P_p^j(w) = \frac{1}{2\pi i} \int\limits_{|\zeta|=\varrho} \frac{1}{\zeta^{p+1}} \log \frac{w - F_j(\zeta)}{w - a_{0j}}\, d\zeta = \sum_{q=1}^{p} \frac{1}{\sqrt{qp}}\, \gamma_{qp}^j \left[\frac{1}{w - a_{0j}}\right]^q,$$
$$j = 1, \ldots, m,$$

$$= \frac{1}{2\pi i} \int\limits_{|\zeta|=\varrho} \frac{1}{\zeta^{p+1}} \log \frac{1 - wG_j(\zeta)}{1 - w b_{0j}}\, d\zeta = \sum_{q=1}^{p} \frac{1}{\sqrt{qp}}\, \gamma_{qp}^j \left[\frac{w}{1 - w b_{0j}}\right]^q,$$
$$j = m+1, \ldots, m+n$$

and let

$$\gamma_p^j = (\gamma_{pl}^j)_{1 \leqslant l}, \quad j = 1, \ldots, m+n, \; p = 1, \ldots,$$

be one-row matrices. Then

$$\sum_{l=1}^{p} \frac{1}{\sqrt{pl}}\, \gamma_{lp}^k \sum_{q=-l}^{\infty} a_{ql}^{kk} z^q = -\frac{1}{pz^p} + \sum_{q=0}^{\infty} b_{qp}^{kk} z^q = P_p^k[F_k(z)],$$
$$k = 1, \ldots, m,$$
$$= P_p^k[1/G_k(z)], \quad k = m+1, \ldots, m+n,$$

$$\sum_{l=1}^{p} \frac{1}{\sqrt{pl}}\, \gamma_{lp}^k \sum_{q=0}^{\infty} a_{ql}^{kj} z^q = \sum_{q=0}^{\infty} b_{qp}^{kj} z^q = P_p^j[F_k(z)], \quad k \neq j, \; k = 1, \ldots, m,$$
$$j = 1, \ldots, m+n,$$
$$= P_p^j[1/G_k(z)], \quad k \neq j, \; k = m+1, \ldots, m+n, \; j = 1, \ldots, m+n,$$

where $r_0 < \varrho < r < 1$, $|z| = r$. Hence

$$\sum_{l=1}^{p} \frac{1}{\sqrt{pl}} \gamma_{lp}^{k} a_{-ql}^{kk} = -\frac{\delta_{pq}}{p}, \quad k = 1, \ldots, m+n, \ q \geqslant 1,$$

$$\sum_{l=1}^{p} \frac{1}{\sqrt{pl}} \gamma_{lp}^{j} a_{ql}^{kj} = b_{qp}^{kj}, \qquad k, j = 1, \ldots, m+n, \ k \neq j, \ q \geqslant 0,$$

where δ_{pq} is the Kronecker delta.

Since the matrix $(\gamma_{qp}^{j})_{1 \leqslant q, 1 \leqslant p}$ is triangular,

$$\sum_{j=1}^{m+n} \sum_{p=1}^{N} \beta_{qp}^{kj} x_{pj} = \sum_{j=1}^{m+n} \sum_{p=1}^{N} a_{qp}^{kj} \sum_{l=1}^{N} \gamma_{pl}^{j} x_{lj}, \qquad q \geqslant 0,$$

Taking (2.9)–(2.14) into account we conclude that on substituting in (2.1) x_{pj} by $\gamma_p^j x_j$, $p = 1, \ldots, j = 1, \ldots, m+n$ and $\hat{A}^{kk} x_k$ by x_k, $k = 1, \ldots$ $\ldots, m+n$, we obtain (2.2).

It is clear that inequality (2.3) is a particular case of inequality (2.2). Indeed, if in (2.2) we assume $x_0 = 0$, $N \to \infty$ and $x_k = y_k z_k$, $k = 1, \ldots$ $\ldots, m+n$, we immediately get (2.3).

4. The case of equality. The main result of this section consists in the statement that if $(F, G) \in C_{m,n}(a_0, b_0)$ and $\mu[C \setminus D(F, G)] = 0$ (we wish to recall that then for all admissible matrices x_0, x_k, y_k, z_k, $k = 1, \ldots$ $\ldots, m+n$, the equality in (2.1), (2.2), (2.3), respectively, holds), then the matrices $x_0, A_{00}, B_{00}, A_{10}^{kj}, B_{10}^{kj}, A_{01}^{kj}, B_{01}^{kj}, A^{kj}$ and B^{kj}, $k, j = 1, \ldots, m+n$, satisfy the equality in (4.3), (4.4), (4.7), (4.8), (4.12), (4.14) and (4.15), respectively.

The result will be proved on examining the equality in (2.1)–(2.3).

Consider first the equality case in (2.1) and (2.2).

After easy transformations of the equality case in (2.1) and (2.2) we conclude that it suffices to examine the following equalities:

$$(4.1) \qquad 2\mathrm{Re}\{x_0^* A_{00} x_0\} + 2\,\mathrm{Re}\Big\{\sum_{k,j=1}^{m+n} \bar{x}_{0k} A_{01}^{kj} x_j\Big\} + \\ + 2\,\mathrm{Re}\Big\{\sum_{k=1}^{m+n} \Big(\sum_{p=1}^{m+n} A_{10}^{kp} \bar{x}_{0p}\Big)\Big(\sum_{l=1}^{m+n} A^{kl} x_l\Big)\Big\} + \sum_{k=1}^{m+n} \Big\|\sum_{j=1}^{m+n} A_{10}^{kj} x_{0j}\Big\|^2 + \\ + \sum_{k=1}^{m+n} \Big\|\sum_{j=1}^{m+n} A^{kj} x_j\Big\|^2 = \sum_{k=1}^{m+n} \|\hat{A}^{kk} x_k\|^2,$$

$$(4.2)\qquad 2\operatorname{Re}\{x_0^* B_{00} x_0\} + 2\operatorname{Re}\Big\{\sum_{k,j=1}^{m+n} \bar{x}_{0k} B_{01}^{kj} x_j\Big\} + \\ + 2\operatorname{Re}\Big\{\sum_{k=1}^{m+n}\Big(\sum_{p=1}^{m+n} B_{10}^{kp}\bar{x}_{0p}\Big)\Big(\sum_{l=1}^{m+n} B^{kl} x_l\Big)\Big\} + \sum_{k=1}^{m+n}\Big\|\sum_{j=1}^{m+n} B_{10}^{kj} x_{0j}\Big\|^2 + \\ + \sum_{k=1}^{m+n}\Big\|\sum_{j=1}^{m+n} B^{kj} x_j\Big\|^2 = \sum_{k=1}^{m+n}\|x_k\|^2.$$

Assume first that $x_k = 0$, $k = 1, \ldots, m+n$. Then

$$(4.3)\qquad 2\operatorname{Re}\{x_0^* A_{00} x_0\} + \sum_{k=1}^{m+n}\Big\|\sum_{j=1}^{m+n} A_{10}^{kj} x_{0j}\Big\|^2 = 0,$$

$$(4.4)\qquad 2\operatorname{Re}\{x_0^* B_{00} x_0\} + \sum_{k=1}^{m+n}\Big\|\sum_{j=1}^{m+n} B_{10}^{kj} x_{0j}\Big\|^2 = 0$$

for each admissible x_0.

On the other hand, if $x_0 = 0$, then

$$(4.5)\qquad \sum_{k=1}^{m+n}\Big\|\sum_{j=1}^{m+n} A^{kj} x_j\Big\|^2 = \sum_{k=1}^{m+n}\|\hat{A}^{kk} x_k\|^2,$$

$$(4.6)\qquad \sum_{k=1}^{m+n}\Big\|\sum_{j=1}^{m+n} B^{kj} x_j\Big\|^2 = \sum_{k=1}^{m+n}\|x_k\|^2.$$

From (4.1) and (4.2), by (4.3)–(4.6), we obtain

$$\operatorname{Re}\Big\{\sum_{k,j=1}^{m+n} \bar{x}_{0k} A_{01}^{kj} x_j + \sum_{k=1}^{m+n}\Big(\sum_{p=1}^{m+n} A_{10}^{kp*}\bar{x}_{0p}\Big)\Big(\sum_{l=1}^{m+n} A^{kl} x_l\Big)\Big\} = 0,$$

$$\operatorname{Re}\Big\{\sum_{k,j=1}^{m+n} \bar{x}_{0k} B_{01}^{kj} x_j + \sum_{k=1}^{m+n}\Big(\sum_{p=1}^{m+n} B_{10}^{kp*}\bar{x}_{0p}\Big)\Big(\sum_{l=1}^{m+n} B^{kl} x_l\Big)\Big\} = 0.$$

Thus

$$(4.7)\qquad \sum_{k=1}^{m+n} A^{kj*}\Big(\sum_{p=1}^{m+n} A_{10}^{kp} x_{0p}\Big) = -\sum_{k=1}^{m+n} A_{01}^{kj*} x_{0k}, \quad j = 1, \ldots, m+n,$$

$$(4.8)\qquad \sum_{k=1}^{m+n} B^{kj*}\Big(\sum_{p=1}^{m+n} B_{10}^{kp} x_{0p}\Big) = -\sum_{k=1}^{m+n} B_{01}^{kj*} x_{0k}, \quad j = 1, \ldots, m+n$$

for each admissible x_0.

Let $x_j \neq 0$ for an arbitrary fixed $j = 1, \ldots, m+n$, and $x_k = 0$ for $k \neq j$, $k = 1, \ldots, m+n$. Then (4.5) and (4.6) yield

$$(4.9)\qquad \sum_{k=1}^{m+n}\|A^{kj} x_j\|^2 = \|\hat{A}^{jj} x_j\|^2, \quad j = 1, \ldots, m+n,$$

$$(4.10)\qquad \sum_{k=1}^{m+n}\|B^{kj} x_j\|^2 = \|x_j\|^2, \quad j = 1, \ldots, m+n.$$

Reshaping (4.5) and taking (4.9) into account we obtain

$$\operatorname{Re}\Big\{\sum_{k=1}^{m+n}\Big[\sum_{j=1}^{m+n} x_j^* A^{kj*}\Big(\sum_{p=j+1}^{m+n} A^{kp}x_p\Big)\Big]\Big\} = 0. \tag{4.11}$$

Let

$$I_r = (\delta_{lr})_{1\leqslant l}, \quad r = 1, \ldots,$$

be one-column matrices and let $x_j = I_r$, $x_p = \exp(\theta i)I_s$ and $x_l = 0$ whenever $j \neq p$, $j \neq l$, $p \neq l$ and $j, p, l = 1, \ldots, m+n$. Here θ stands for a real number. Moreover, denote

$$A^{kj}I_r = A^{kj}(r).$$

Then from (4.11) it follows that

$$\sum_{k=1}^{m+n} A^{kj*}(r)A^{kp}(s) = 0, \quad j \neq p, \; j, p = 1, \ldots, m+n,$$

for any $r, s = 1, \ldots$ Consequently

$$\sum_{k=1}^{m+n} A^{kj*}A^{kp} = 0, \quad j \neq p, \; j, p = 1, \ldots, m+n. \tag{4.12}$$

Observe that if $x_j = I_p$, then, in accordance with our notational convention, (4.9) yields

$$\sum_{k=1}^{m+n}\sum_{q=1}^{\infty} q|a_{qp}^{kj}|^2 = \sum_{q=1}^{p} q|a_{-qp}^{jj}|^2, \quad p = 1, \ldots, \quad j = 1, \ldots, m+n.$$

When $x_j = I_r$, $j = 1, \ldots, m+n$, from (4.10) it follows that

$$\sum_{k=1}^{m+n} \|B^{kj}(r)\|^2 = 1, \quad j = 1, \ldots, m+n \tag{4.13}$$

for any $r = 1, \ldots$ Assuming in (4.10) that $x_j = I_r + \lambda I_s$, $r \neq s$ and $x_l = 0$ for $l \neq j$, $l, j = 1, \ldots, m+n$, where λ is an arbitrary complex number, we obtain

$$\sum_{k=1}^{m+n} \|B^{kj}(r) + \lambda B^{kj}(s)\|^2 = 1 + |\lambda|^2, \quad j = 1, \ldots, m+n$$

and by (4.13) we have

$$\sum_{k=1}^{m+n} B^{kj*}(r)B^{kj}(s) = 0, \quad r \neq s, \; j = 1, \ldots, m+n.$$

Hence, taking (4.13) into account once more, we get

$$\sum_{k=1}^{m+n} B^{kj*}B^{kj} = I, \quad j = 1, \ldots, m+n, \tag{4.14}$$

where

$$I = (\delta_{rs})_{1\leqslant r, 1\leqslant s}.$$

On the other hand, if we assume in (4.10) that $x_j = I_r$, $x_l = \lambda I_s$, $j \neq l$, $j, l = 1, \ldots, m+n$, r, s, λ — arbitrary and $x_p = 0$ for $p \neq j$, $p \neq l$, $p = 1, \ldots, m+n$, we obtain

$$\sum_{k=1}^{m+n} \|B^{kj}(r) + \lambda B^{kl}(s)\|^2 = 1 + |\lambda|^2.$$

Hence, by (4.13), we have

$$\sum_{k=1}^{m+n} B^{kj*}(r) B^{kl}(s) = 0, \quad j \neq l, \; j, l = 1, \ldots, m+n.$$

Thus

$$\sum_{k=1}^{m+n} B^{kj*} B^{kl} = 0, \quad j \neq l, \; j, l = 1, \ldots, m+n. \tag{4.15}$$

Analogously one can prove that (4.14) and (4.15) are simple consequences of the equality case in (2.3).

Remark 2. In Aharonov class [1], [2] analogous investigation of equalities of type (2.1) and (2.2) has been carried out by Hummel [12].

5. Bilinear form of the basic theorem. In this section we shall formulate and prove a bilinear version of Theorem 1.

THEOREM 2. *If $(F, G) \in C_{m,n}(a_0, b_0)$ and $x_0, x_k, y_k z_k$, $k = 1, \ldots, m+n$, are arbitrary admissible matrices, then:*

$$\operatorname{Re}\Big\{x_0^* A_{00} x_0 + \sum_{k,j=1}^{m+n} \Big[\bar{x}_{0k} A_{01}^{kj} x_j + \sum_{q=1}^{N} (a_{q0}^{kj} x_{0j} + A_q^{kj} x_j) A_{-q}^{kk} x_k\Big]\Big\} \leqslant \sum_{k=1}^{m+n} \|\hat{A}^{kk} x_k\|^2 \tag{5.1}$$

and the equality holds if and only if

$$\operatorname{Re}\Big\{x_0^* A_{00} x_0 + \sum_{k,j=1}^{m+n} \bar{x}_{0k} A_{01}^{kj} x_j\Big\} = 0, \tag{5.2}$$

$$\sum_{j=1}^{m+n} [a_{q0}^{kj} x_{0j} + A_q^{kj} x_j] = x_k^* A_{-q}^{kk*}, \quad q = 1, \ldots, N, \; k = 1, \ldots, m+n, \tag{5.3}$$

$$= 0, \qquad q = N+1, \ldots, \; k = 1, \ldots, m+n;$$

$$\operatorname{Re}\Big\{x_0^* B_{00} x_0 + \sum_{k,j=1}^{m+n} \Big[\bar{x}_{0k} B_{01}^{kj} x_j + \sum_{q=1}^{N} (\beta_{q0}^{kj} x_{0j} + B_q^{kj} x_j) x_{qk}\Big]\Big\} \leqslant \sum_{k=1}^{m+n} \|x_k\|^2 \tag{5.4}$$

and the equality holds if and only if

$$\operatorname{Re}\Big\{x_0^* B_{00} x_0 + \sum_{k,j=1}^{m+n} \bar{x}_{0k} B_{01}^{kj} x_j\Big\} = 0, \tag{5.5}$$

$$\sum_{j=1}^{m+n} [\beta_{q0}^{kj} x_{0j} + B_q^{kj} x_j] = \bar{x}_{qk}, \qquad q = 1, \ldots, N,\ k = 1, \ldots, m+n, \tag{5.6}$$

$$= 0, \qquad q = N+1, \ldots,\ k = 1, \ldots, m+n;$$

$$\operatorname{Re}\Big\{\sum_{k,j=1}^{m+n} \sum_{q=1}^{\infty} \Big(\sum_{i=1}^{N_k} \frac{1}{\sqrt{q}} y_{ki} z_{ki}^q\Big) B_q^{kj} y_j z_j\Big\} \leqslant \sum_{k=1}^{m+n} \|y_k z_k\|^2 \tag{5.7}$$

and the equality holds if and only if

$$\sum_{j=1}^{m+n} B_q^{kj} y_j z_j = \frac{1}{\sqrt{q}} \sum_{i=1}^{N_k} \overline{y_{ki} z_{ki}^q}, \qquad q = 1, \ldots,\ k = 1, \ldots, m+n. \tag{5.8}$$

Proof. We have already shown that inequality (2.7) holds. Thus let

$$U_0 = \operatorname{Re}\Big\{\sum_{k=1}^{m+n} \bar{x}_{0k} c_0^k\Big\},$$

$$U_{kq} = \sqrt{q} c_q^k, \qquad V_{kq} = \sqrt{q} c_{-q}^k, \qquad k = 1, \ldots, m+n,\ q = 1, \ldots$$

Then, by Schwarz inequality, we have

$$U_0 + \operatorname{Re}\Big\{\sum_{k=1}^{\infty} U_k V_k\Big\} \leqslant U_0 + \Big[\Big(\sum_{k=1}^{\infty} |V_k|^2 - 2U_0\Big) \sum_{k=1}^{\infty} |V_k|^2\Big]^{1/2} \leqslant \sum_{k=1}^{\infty} |V_k|^2.$$

Thus

$$\operatorname{Re}\Big\{\sum_{k=1}^{m+n} \Big[\bar{x}_{0k} c_0^k + \sum_{q=1}^{\infty} q c_q^k c_{-q}^k\Big]\Big\} \leqslant \sum_{k=1}^{m+n} \sum_{q=1}^{\infty} q |c_{-q}^k|^2. \tag{5.9}$$

Observe that the equality holds if and only if

$$U_0 = 0, \qquad U_k = \overline{V_k}, \qquad k = 1, \ldots,$$

i.e. if and only if

$$\operatorname{Re}\Big\{\sum_{k=1}^{m+n} \bar{x}_{0k} c_0^k\Big\} = 0, \qquad c_q^k = \overline{c_{-q}^k}, \qquad k = 1, \ldots, m+n,\ q = 1, \ldots \tag{5.10}$$

Assuming in (5.9) and (5.10) the values given in (2.9), (2.13) and (2.16) and applying the notation accepted we obtain the required result.

Remark 3. Inequality (5.5) and conditions (5.6), (5.7) under the assumption that $x_0 = \bar{x}_0$ in the class $C_{m,0}(a_0, \emptyset)$ have been given by De Temple [6].

6. Inequalities for pairs of vector functions. In the sequel let u_0, $u_k, v_k\zeta_k$, $k=1,\ldots,m+n$, denote matrices standing in place of the respective matrices x_0, x_k, y_kz_k, $k=1,\ldots,m+n$, discussed before.

Recall inequalities (5.1), (5.4) and (5.7); they can be rewritten in the form

$$(6.1)\qquad \operatorname{Re}\Big\{x_0^*A_{00}x_0+\sum_{k,j=1}^{m+n}[\bar{x}_{0k}A_{01}^{kj}x_j+x_k'\hat{A}^{kk'}A_{10}^{kj}x_{0j}+x_k'\hat{A}^{kk'}A^{kj}x_j]\Big\} \leqslant \sum_{k=1}^{m+n}\|\hat{A}^{kk}x_k\|^2,$$

$$(6.2)\qquad \operatorname{Re}\Big\{x_0^*B_{00}x_0+\sum_{k,j=1}^{m+n}[\bar{x}_{0k}B_{01}^{kj}x_j+x_k'B_{10}^{kj}x_{0j}+x_k'B^{kj}x_j]\Big\}\leqslant\sum_{k=1}^{m+n}\|x_k\|^2$$

and

$$(6.3)\qquad \operatorname{Re}\Big\{\sum_{k,j=1}^{m+n}y_kz_k'B^{kj}y_jz_j\Big\}\leqslant\sum_{k=1}^{m+n}\|y_kz_k\|^2.$$

We shall show that one can introduce to these inequalities additional parameters u_0, u_k, $v_k\zeta_k$, $k=1,\ldots,m+n$, $u_0=x_0=\bar{x}_0$. In this way the following "strengthening" of the inequalities is obtained:

THEOREM 3. *If* $(F,G)\in C_{m,n}(a_0,b_0)$ *and* $x_0, u_0, x_k, u_k, y_kz_k, v_k\zeta_k$, $k=1,\ldots,m+n$, *are arbitrary admissible matrices and* $u_0=x_0=\bar{x}_0$, *then*

$$(6.4)\qquad \operatorname{Re}\Big\{u_0'A_{00}x_0+\tfrac{1}{2}\sum_{k,j=1}^{m+n}[u_{0k}A_{01}^{kj}x_j+u_k'A_{10}^{kj}x_{0j}+u_{0k}A_{01}^{kj}\hat{A}^{jj}x_j+ u_k'\hat{A}^{kk'}A_{10}^{kj}x_{0j}+u_k'\hat{A}^{kk'}A^{kj}x_j+u_k'A^{kj}\hat{A}^{jj}x_j]\Big\}\leqslant\tfrac{1}{2}\sum_{k=1}^{m+n}(\|\hat{A}^{kk}x_k\|^2+\|\hat{A}^{kk}u_k\|^2),$$

$$(6.5)\qquad \operatorname{Re}\Big\{u_0'B_{00}x_0+\sum_{k,j=1}^{m+n}[u_{0k}B_{01}^{kj}x_j+u_k'B_{10}^{kj}x_{0j}+u_k'B^{kj}x_j]\Big\} \leqslant\tfrac{1}{2}\sum_{k=1}^{m+n}(\|x_k\|^2+\|u_k\|^2)$$

and

$$(6.6)\qquad \operatorname{Re}\Big\{\sum_{k,j=1}^{m+n}v_k\zeta_k'B^{kj}y_jz_j\Big\}\leqslant\tfrac{1}{2}\sum_{k=1}^{m+n}(\|y_kz_k\|^2+\|v_k\zeta_k\|^2).$$

Proof. Let $\hat{c}_0^k$ stand for the quantities (2.9), (2.13) and (2.16) in which x_0, x_k, y_kz_k, $k=1,\ldots,m+n$, have been substituted by $u_0, u_k, v_k\zeta_k$, $k=1,\ldots,m+n$, respectively. We also assume that $u_0=x_0=\bar{x}_0$. Then by (2.7)

$$(6.7)\qquad \sum_{k=1}^{m+n}\Big[2\operatorname{Re}\{u_{0k}\hat{c}_0^k\}+\sum_{q=-\infty}^{\infty}q|\hat{c}_q^k|^2\Big]\leqslant 0.$$

On adding the corresponding sides of inequalities (2.7) and (6.7) we obtain

$$(6.8)\qquad \sum_{k=1}^{m+n}\Big[2\operatorname{Re}\{u_{0k}\hat{c}_0^k+x_{0k}c_0^k\}+\sum_{q=1}^{\infty}q(|\hat{c}_q^k|^2+|c_q^k|^2)\Big] \leqslant \sum_{k=1}^{m+n}\sum_{q=1}^{\infty}q(|\hat{c}_{-q}^k|^2+|c_{-q}^k|^2).$$

Similarly to the procedure used in the proof of Theorem 2, on applying the bilinearization method to inequality (6.8) we arrive at the following inequality:

$$(6.9)\qquad \operatorname{Re}\Big\{\sum_{k=1}^{m+n}\Big[u_{0k}\hat{c}_0^k+x_{0k}c_0^k+\sum_{q=1}^{\infty}q(c_q^k\hat{c}_{-q}^k+\hat{c}_q^kc_{-q}^k)\Big]\Big\} \leqslant \sum_{k=1}^{m+n}\sum_{q=1}^{\infty}q(|c_{-q}^k|^2+|\hat{c}_{-q}^k|^2).$$

Assuming now in (6.9) the values c_q^k and $\hat{c}_q^k$ (see (2.9), (2.13) and (2.16)) we immediately obtain (6.4)–(6.6).

Remark 4. Using Pommerenke's method [20], Hummel [12] obtained inequalities (6.5) in the class $C_{1,1}(\{0\},\{0\})$.

Libera [17] obtained inequalities of type (6.6) in the subclass of $C_{2,2}(a_0,\bar{a}_0)$ of pairs $(F,\bar{F})$, where $|a_{01}|$, $|a_{02}|<1$ and in the subclass of $C_{1,1}(a_0,\bar{a}_0)$ of pairs $(F,\bar{F})$, where $|a_{01}|<1$.

As known, the inequalities of the type under investigation obtained earlier by other authors have various applications. A study concerning application of the inequalities obtained here will be the subject of a subsequent paper.

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