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On a biharmonic function in the unit disc

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In this note we examine for a given function $f(t)$ with Fourier coefficients a_0, a_n, b_n the biharmonic function $f(r, \varphi)$ defined in the disc $|re^{i\varphi}| < 1$ by

$$(1) \qquad f(r, \varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[1 + \frac{n}{2}(1-r^2)\right] r^n (a_n \cos n\varphi + b_n \sin n\varphi) .$$

It can easily be verified (cf. [3], pp. 395-400) that

$$(2) \qquad f(r, \varphi) = \frac{(1-r^2)^2}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1 - r\cos(t-\varphi)}{[1+r^2-2r\cos(t-\varphi)]^2} dt$$

for any $f(t)$ Lebesgue-integrable in $\langle -\pi, \pi \rangle$; these functions $f(t)$, with period 2π, are considered below. Our theorems complete the results announced in [2].

For the sake of brevity, we shall write

$$K(r, t) = \frac{(1-r^2)^2(1-r\cos t)}{2\pi[1+r^2-2r\cos t]^2} = \frac{(1-r^2)^2\big(1-r+2r\sin^2(t/2)\big)}{2\pi[(1-r)^2+4r\sin^2(t/2)]^2} .$$

Then, by (2),

$$f(r, \varphi) = \int_{-\pi}^{\pi} f(t) K(r, t-\varphi)\, dt = \int_0^{\pi} [f(\varphi+t) + f(\varphi-t)] K(r, t)\, dt .$$

In particular, the last representation together with (1) gives

$$1 = 2 \int_0^{\pi} K(r, t)\, dt .$$

Hence

$$(3) \qquad f(r, \varphi) - f(\varphi) = \int_0^{\pi} [f(\varphi+t) - 2f(\varphi) + f(\varphi-t)] K(r, t)\, dt .$$

The kernel $K(r,t)$ is positive decreasing in $t \in \langle 0, \pi\rangle$ such that $K(r, \theta) \to 0$ as $r \to 1-$ for arbitrarily small $\theta > 0$. Thus we have

$$\lim_{r \to 1-} f(r, \varphi) = f(\varphi)$$

at almost every φ, and the convergence is uniform over every closed interval of points of continuity.

Let $f(\varphi)$ be bounded in $\langle -\pi, \pi\rangle$ and let $\omega_2(\delta) = \omega_2(\delta; f)$ be the second modulus of smoothness of f, i.e.

$$\omega_2(\delta; f) = \sup_{0 \leqslant t \leqslant \delta} \{\sup_{|\varphi| \leqslant \pi} |f(\varphi + t) - 2f(\varphi) + f(\varphi - t)|\}.$$

We shall denote by Z_a $(0 < a \leqslant 2)$ the class of all these f's for which

$$\omega_2(\delta; f) \leqslant 2\delta^a \quad \text{when} \quad 0 \leqslant \delta \leqslant \pi.$$

Moreover, we shall write

$$U_a(r) = \sup_{f \in Z_a} \{\max_{|\varphi| \leqslant \pi} |f(r, \varphi) - f(\varphi)|\}.$$

THEOREM 1. *The following asymptotic formula*

$$U_a(r) = \frac{2}{\pi} B\left(\frac{1+a}{2}, \frac{3-a}{2}\right)(1-r)^a + \begin{cases} O\{(1-r)^{3a}\} & \textit{for} \quad 0 < a \leqslant 1/2, \\ O\{(1-r)^{1+a}\} & \textit{for} \quad 1/2 < a < 1, \\ O\{(1-r)^2\} & \textit{for} \quad 1 < a < 2 \end{cases}$$

holds as $r \to 1-$.

Proof. By (3),

$$\max_{|\varphi| \leqslant \pi} |f(r, \varphi) - f(\varphi)| \leqslant 2 \int_0^\pi t^a K(r, t)\, dt$$

for any $f \in Z_a$. Therefore,

$$\begin{aligned} (4) \qquad U_a(r) &\leqslant 2^{a+2} \int_0^{\pi/2} t^a K(r, 2t)\, dt \\ &= 2^{a+2}\left[\int_0^{\pi/2} \sin^a t K(r, 2t)\, dt + \int_0^{\pi/2} (t^a - \sin^a t) K(r, 2t)\, dt\right]. \end{aligned}$$

The function

$$f_a(\varphi) = |2\sin(\varphi/2)|^a \qquad (0 < a \leqslant 2)$$

is of class Z_a (see [1]), and

$$f_a(r, 0) - f_a(0) = 2^{a+2} \int_0^{\pi/2} \sin^a t K(r, 2t)\, dt$$

($f_a(r, \varphi)$ is the function (1) for $f_a(\varphi)$); whence

$$U_a(r) \geqslant 2^{a+2} \int_0^{\pi/2} \sin^a t K(r, 2t)\, dt . \tag{5}$$

Inequalities (4) and (5) lead to

$$\begin{aligned} U_a(r) &= 2^{a+2}\Big[\int_0^{\pi/2} \sin^a t K(r, 2t)\, dt + C_a(r) \int_0^{\pi/2} (t^a - \sin^a t) K(r, 2t)\, dt\Big] \\ &= 2^{a+2}[I_1 + C_a(r) I_2], \qquad \text{where } 0 \leqslant C_a(r) \leqslant 1 . \end{aligned}$$

Clearly,

$$\begin{aligned} I_1 &= \int_0^{\pi/4} \sin^a t K(r, 2t)\, dt + O\{(1-r)^2\} \\ &= \int_0^{\pi/4} \sin^a t \cos t K(r, 2t)\, dt + \int_0^{\pi/4} \sin^a t (1-\cos t) K(r, 2t)\, dt + O\{(1-r)^2\} \\ &= Y_1 + Y_2 + O\{(1-r)^2\} . \end{aligned}$$

A change of variable $4r\sin^2 t/(1-r)^2 = x$ gives

$$\begin{aligned} Y_1 &= \frac{(1-r^2)^2}{2\pi} \int_0^{\pi/4} \sin^a t \cos t \frac{1-r+2r\sin^2 t}{[(1-r)^2 + 4r\sin^2 t]^2}\, dt \\ &= \frac{(1+r)^2(1-r)^a}{\pi \cdot 2^{a+4}(\sqrt{r})^{a+1}} \int_0^{2r/(1-r)^2} x^{(a-1)/2} \frac{2+(1-r)x}{(1+x)^2}\, dx \\ &= \frac{(1+r)^2(1-r)^a}{\pi \cdot 2^{a+3}(\sqrt{r})^{a+1}} \int_0^{\infty} \frac{x^{(a-1)/2}}{(1+x)^2}\, dx + \begin{cases} O\{(1-r)^{1+a}\} & \text{for} \quad 0 < a < 1 , \\ O\{(1-r)^2\} & \text{for} \quad 1 < a < 2 . \end{cases} \end{aligned}$$

Since $(1-\cos t) = 2\sin^2(t/2) \leqslant 2\sin t \cos t$ $(0 \leqslant t \leqslant \pi/4)$, we have

$$\begin{aligned} Y_2 &\leqslant \frac{(1-r^2)^2}{\pi} \int_0^{\pi/4} \sin^{(1+a)} t \cos t \frac{1-r+2r\sin^2 t}{[(1-r)^2 + 4r\sin^2 t]^2}\, dt \\ &= \frac{(1+r)^2(1-r)^{1+a}}{\pi \cdot 2^{a+4}(\sqrt{r})^{a+2}} \int_0^{2r/(1-r)^2} x^{a/2} \frac{2+(1-r)x}{(1+x)^2}\, dx , \end{aligned}$$

i.e.

$$Y_2 = \begin{cases} O\{(1-r)^{1+a}\} & \text{for} \quad 0 < a < 1 , \\ O\{(1-r)^2\} & \text{for} \quad 1 < a < 2 . \end{cases}$$

Thus we get

$$I_1 = \frac{(1+r)^2(1-r)^a}{\pi\cdot 2^{a+3}(\sqrt{r})^{a+1}}\int\limits_0^\infty \frac{x^{(a-1)/2}}{(1+x)^2}dx + \begin{cases} O\{(1-r)^{1+a}\} & \text{if} \quad 0<a<1\,, \\ O\{(1-r)^2\} & \text{if} \quad 1<a<2\,. \end{cases}$$

Consider now the integral I_2. Suppose first that $0<a<1$. Then the inequality

$$t^a-\sin^a t \leqslant d_a \sin^{3a} t \quad (0\leqslant t\leqslant \pi/2)\,, \quad \text{where} \quad d_a = \frac{1}{(3!)^a}\left(\frac{\pi}{2}\right)^{3a},$$

implies

$$I_2 \leqslant d_a \int\limits_0^{\pi/2} \sin^{3a} t K(r,2t)\,dt = d_a \int\limits_0^{\pi/4} \sin^{3a} t K(r,2t)\,dt + O\{(1-r)^2\}\,.$$

Further,

$$I_2 \leqslant \sqrt{2}\, d_a \int\limits_0^{\pi/4} \sin^{3a} t\cos t K(r,2t)\,dt + O\{(1-r)^2\}$$

$$= \frac{\sqrt{2}\, d_a(1+r)^2(1-r)^{3a}}{\pi\cdot 2^{3a+4}(\sqrt{r})^{3a+1}} \int\limits_0^{2r/(1-r)^2} x^{(3a-1)/2}\frac{2+(1-r)x}{(1+x)^2}dx + O\{(1-r)^2\}$$

$$= \frac{\sqrt{2}\, d_a(1+r)^2(1-r)^{3a}}{\pi\cdot 2^{3a+3}(\sqrt{r})^{3a+1}} \int\limits_0^{\infty} \frac{x^{(3a-1)/2}}{(1+x)^2}dx +$$

$$+\begin{cases} O\{(1-r)^{3a+1}\} & \text{for} \quad 0<a\leqslant 1/3\,, \\ O\{(1-r)^2\} & \text{for} \quad 1/3<a<1\,. \end{cases}$$

Consequently,

$$I_2 = \begin{cases} O\{(1-r)^{3a}\} & \text{for} \quad 0<a\leqslant 2/3\,, \\ O\{(1-r)^2\} & \text{for} \quad 2/3<a<1\,. \end{cases}$$

In the case of $1<a<2$,

$$t^a-\sin^a t \leqslant \left(\frac{t^4}{3}\right)^{a/2} \leqslant \frac{1}{3^{a/2}}\left(\frac{\pi}{2}\right)^{2a}\sin^{2a} t \quad (0\leqslant t\leqslant \pi/2)\,.$$

Hence

$$I_2 \leqslant \frac{1}{3^{a/2}}\left(\frac{\pi}{2}\right)^{2a}\frac{(1-r^2)^2}{2\pi}\int\limits_0^{\pi/2} \sin^2 t\frac{1-r+2r\sin^2 t}{[(1-r)^2+4r\sin^2 t]^2}dt$$

$$= \frac{1}{3^{a/2}}\left(\frac{\pi}{2}\right)^{2a}\frac{(2+r)}{16}(1-r)^2\,,$$

i.e.

$$I_2 = \begin{cases} O\{(1-r)^{3a}\} & \text{for} \quad 0<a\leqslant 2/3\,, \\ O\{(1-r)^2\} & \text{for} \quad 2/3<a<1 \text{ or } 1<a<2\,. \end{cases}$$

Putting together the results, we obtain

$$U_a(r) = \frac{(1+r)^2(1-r)^a}{2\pi(\sqrt{r})^{a+1}}\int_0^\infty \frac{x^{(a-1)/2}}{(1+x)^2}dx + \begin{cases} O\{(1-r)^{3a}\} & \text{if} \quad 0 < a \leqslant 1/2\,, \\ O\{(1-r)^{1+a}\} & \text{if} \quad 1/2 < a < 1\,, \\ O\{(1-r)^2\} & \text{if} \quad 1 < a < 2\,, \end{cases}$$

and the desired formula follows immediately.

Analogously, for $a = 1$, the asymptotic relation

$$U_1(r) = \frac{2}{\pi}(1-r) + O\{(1-r)^2|\ln(1-r)|\} \quad \text{as} \quad r \to 1-$$

can also easily be deduced (cf. [2], theorem 3).

Notice also that the inequalities

$$I(r) \leqslant U_2(r) \leqslant (\pi/2)^2 I(r)\,,$$

where $I(r) = 2^4 \int_0^{\pi/2} \sin^2 t K(r, 2t)\,dt = (2+r)(1-r)^2$, imply

$$2 \leqslant U_2(r)/(1-r)^2 \leqslant 3(\pi/2)^2\,.$$

THEOREM 2. *Suppose that for certain fixed* φ_0, a $(0 < a < 2)$ *the finite limit*

$$\lim_{t\to 0+} \frac{f(\varphi_0+t) - 2f(\varphi_0) + f(\varphi_0 - t)}{t^a} = l$$

exists, and that $f(\varphi_0+t)+f(\varphi_0-t)$ *remains bounded in* $\langle 0, \pi\rangle$. *Then*

$$f(r, \varphi_0) = f(\varphi_0) + \frac{l}{\pi}B\left(\frac{1+a}{2}, \frac{3-a}{2}\right)(1-r)^a + o\{(1-r)^a\} \quad \text{as} \quad r \to 1-\,.$$

Proof. By hypothesis, we can write

$$f(\varphi_0+t) - 2f(\varphi_0) + f(\varphi_0-t) = [l + \lambda(t)]\big(2\sin(t/2)\big)^a\,, \tag{6}$$

where the function $\lambda(t)$ is bounded in $(0, \pi\rangle$, say $|\lambda(t)| \leqslant M$, and $\lambda(t) \to 0$ as $t \to 0+$.

Applying (3) and (6),

$$\begin{aligned} f(r, \varphi_0) - f(\varphi_0) &= l\int_0^\pi \big(2\sin(t/2)\big)^a K(r, t)\,dt + \int_0^\pi \big(2\sin(t/2)\big)^a \lambda(t) K(r, t)\,dt \\ &= l I_a(r) + Y_a(r)\,. \end{aligned}$$

Given $\varepsilon > 0$, there is a $\delta > 0$ such that $|\lambda(t)| < \varepsilon$ for $0 < t \leqslant \delta$. Then

$$\begin{aligned} \left|\frac{Y_a(r)}{I_a(r)}\right| &\leqslant \frac{1}{I_a(r)}\left[\varepsilon\int_0^\delta \left(2\sin\frac{t}{2}\right)^a K(r, t)\,dt + M\int_\delta^\pi \left(2\sin\frac{t}{2}\right)^a K(r, t)\,dt\right] \\ &\leqslant \varepsilon + \frac{M}{I_a(r)}\int_\delta^\pi \left(2\sin\frac{t}{2}\right)^a K(r, t)\,dt\,. \end{aligned}$$

The last integral is $O\{(1-r)^2\}$. Further,

$$I_a(r) = 2^{a+1}\int\limits_0^{\pi/2} \sin^a t K(r, 2t)\,dt$$
$$= \frac{1}{\pi}B\left(\frac{1+a}{2}, \frac{3-a}{2}\right)(1-r)^a + \begin{cases} O\{(1-r)^{1+a}\} & \text{if} \quad 0 < a < 1\,, \\ O\{(1-r)^2\} & \text{if} \quad 1 < a < 2\,, \end{cases}$$

$$I_1(r) = \frac{1}{\pi}(1-r) + O\{(1-r)^2|\ln(1-r)|\}\,.$$

Consequently

$$I_a(r) = \frac{1}{\pi}B\left(\frac{1+a}{2}, \frac{3-a}{2}\right)(1-r)^a + o\{(1-r)^a\} \quad \text{for} \quad 0 < a < 2\,.$$

Thus we have $Y_a(r) = O\{I_a(r)\}$ as $r \to 1-$, and the proof is completed.

THEOREM 3. *Let $f(\varphi)$ be bounded in $\langle -\pi, \pi\rangle$ and let $\omega_2(\delta)$ signify the second modulus of smoothness of this function. Then*

$$\max_{|\varphi|\leqslant\pi}\left|\frac{\partial^2 f(r, \varphi)}{\partial\varphi^2}\right| \leqslant C\frac{r\omega_2(1-r)}{(1-r)^2} \quad \textit{for} \quad 0 \leqslant r < 1\,,$$

with an absolute constant C.

Proof. Starting with the identity

$$\frac{\partial^2 f(r, \varphi)}{\partial\varphi^2} = \int\limits_{-\pi}^{\pi} f(\varphi + t)\frac{\partial^2 K(r, t)}{\partial t^2}\,dt\,,$$

and observing that, by (1), (2), $\partial^2 K(r, t)/\partial t^2$ is even and its integral extended over the interval $\langle -\pi, \pi\rangle$ vanishes, we obtain

$$\frac{\partial^2 f(r, \varphi)}{\partial\varphi^2} = \int\limits_0^{\pi} [f(\varphi + t) - 2f(\varphi) + f(\varphi - t)]\frac{\partial^2 K(r, t)}{\partial t^2}\,dt\,.$$

Direct computation gives

$$\frac{\partial^2 K(r, t)}{\partial t^2} = \frac{r(1-r^2)^2}{2\pi}\left[\frac{2r\cos 2t - (3-r^2)\cos t}{(1+r^2-2r\cos t)^3} + \frac{6r\sin^2 t(3-r^2-2r\cos t)}{(1+r^2-2r\cos t)^4}\right].$$

Hence

$$\left|\frac{\partial^2 f(r, \varphi)}{\partial\varphi^2}\right| \leqslant \frac{r(1-r^2)^2}{2\pi}\int\limits_0^{\pi} \omega_2(t)[Q_1(r, t) + Q_2(r, t)]\,dt\,,$$

where

$$Q_1(r, t) = \frac{2(1-r)(3+r) + t^2(3+8r-r^2)}{2[(1-r)^2 + 4rt^2/\pi^2]^3}\,,$$

$$Q_2(r, t) = \frac{6rt^2[(1-r)(3+r) + rt^2]}{[(1-r)^2 + 4rt^2/\pi^2]^4}\,.$$

Now, it is easily seen that if $0 \leqslant r \leqslant 1/2$,

$$\left|\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right| \leqslant \frac{16r}{\pi}\int\limits_0^\pi \omega_2(t)(7+91t^2+12t^4)\,dt\,,$$

and by the inequality

$$\omega_2(t) \leqslant \{(1-r+t)^2\omega_2(1-r)\}/(1-r)^2\,,$$

we have

$$\max_{|\varphi|\leqslant\pi}\left|\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right| \leqslant C_1\frac{r\omega_2(1-r)}{(1-r)^2} \qquad (C_1 = \text{const})\,.$$

In the case of $1/2 < r < 1$,

$$\left|\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right| \leqslant \frac{r(1+r)^2(1-r)^3}{4\sqrt{r}}\int\limits_0^{2\sqrt{r}/(1-r)} \omega_2\left(\frac{\pi(1-r)}{2\sqrt{r}}x\right)[Q_1^*(r,x)+Q_2^*(r,x)]\,dx\,,$$

where $Q_\nu^*(r,x) = Q_\nu(r,\pi(1-r)x/2\sqrt{r})$. Applying the estimate

$$\omega_2\left(\frac{\pi(1-r)}{2\sqrt{r}}x\right) \leqslant \left(1+\frac{\pi}{2\sqrt{r}}x\right)^2\omega_2(1-r)\,,$$

we get

$$\left|\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right| \leqslant \frac{r\omega_2(1-r)}{2(1-r)^2}\int\limits_0^\infty \left(1+\frac{\pi}{2\sqrt{r}}x\right)^2\left[\frac{32+6\pi^2x^2}{(1+x^2)^3}+\frac{3\pi^2x^2(32+\pi^2x^2)}{2(1+x^2)^4}\right]dx\,.$$

Consequently

$$\max_{|\varphi|\leqslant\pi}\left|\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right| \leqslant C_2\frac{r\omega_2(1-r)}{(1-r)^2} \qquad (C_2 = \text{const})\,.$$

The proof is thus completed.

THEOREM 4. *If a finite derivative $f^{(s)}(\varphi_0)$ (s is 1 or 2) exists, then*

$$\lim_{(r,\varphi)}\frac{\partial^s f(r,\varphi)}{\partial\varphi^s} = f^{(s)}(\varphi_0)$$

as (r,φ) approaches $(1,\varphi_0)$ in such a way that $(\varphi-\varphi_0)/(1-r)$ remains bounded (cf. [4], pp. 100-101).

Proof. Let us confine ourselves to the case of $-\pi < \varphi_0 < \pi$, $s = 2$. The function

$$f_*(t) = f(\varphi_0)+f'(\varphi_0)\sin(t-\varphi_0)+\tfrac{1}{2}f''(\varphi_0)\sin^2(t-\varphi_0)$$

is 2π-periodic infinite-differentiable in $(-\infty,\infty)$,

$$f_*^{(\nu)}(\varphi_0) = f^{(\nu)}(\varphi_0) \quad \text{for} \quad \nu = 0,1,2\,.$$

Moreover,

$$\eta(t) = \big(f(t)-f_*(t)\big)/\sin^2(t-\varphi_0) \quad \text{tends to zero as } t\to\varphi_0 .$$

As for the Poisson integral,

$$\frac{\partial^2 f_*(r,\varphi)}{\partial\varphi^2} = \int_{-\pi}^{\pi} f_*(t)\,\frac{\partial^2 K(r,t-\varphi)}{\partial t^2}\,dt = \int_{-\pi}^{\pi} f_*''(t)K(r,t-\varphi)\,dt$$

and

$$\lim_{\substack{r\to 1-\\ \varphi\to\varphi_0}} \int_{-\pi}^{\pi} f_*''(t)K(r,t-\varphi)\,dt = f_*''(\varphi_0) = f''(\varphi_0) .$$

Writing $h(t) = f(t)-f_*(t)$, we have

$$\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2} = \frac{\partial^2 f_*(r,\varphi)}{\partial\varphi^2} + \frac{\partial^2 h(r,\varphi)}{\partial\varphi^2} ,$$

and

$$\frac{\partial^2 h(r,\varphi)}{\partial\varphi^2} = \int_{-\pi}^{\pi} \eta(t)\sin^2(t-\varphi_0)\,\frac{\partial^2 K(r,t-\varphi)}{\partial t^2}\,dt .$$

Given any $\varepsilon > 0$, let δ $(0<\delta\leqslant \min(\pi-\varphi_0, \pi+\varphi_0))$ be such that $|\eta(t)|\leqslant\varepsilon$ for $|t-\varphi_0|\leqslant\delta$. Then,

$$\left|\frac{\partial^2 h(r,\varphi)}{\partial\varphi^2}\right| \leqslant \varepsilon \int_{|t-\varphi_0|\leqslant\delta} \sin^2(t-\varphi_0)\left|\frac{\partial^2 K(r,t-\varphi)}{\partial t^2}\right| dt + |I_\delta(r,\varphi)| ,$$

where

$$I_\delta(r,\varphi) = \int_{|t-\varphi_0|>\delta} \eta(t)\sin^2(t-\varphi_0)\,\frac{\partial^2 K(r,t-\varphi)}{\partial t^2}\,dt .$$

Clearly,

$$\begin{aligned}
&\int_{|t-\varphi_0|\leqslant\delta} \sin^2(t-\varphi_0)\left|\frac{\partial^2 K(r,t-\varphi)}{\partial t^2}\right| dt \\
&\qquad\leqslant \int_{-\pi}^{\pi} \sin^2(t-\varphi_0)\left|\frac{\partial^2 K(r,t-\varphi)}{\partial t^2}\right| dt \\
&\qquad= \int_{-\pi}^{\pi} \sin^2 t\left|\frac{\partial^2 K(r,t)}{\partial t^2}\right| dt + \int_{-\pi}^{\pi} [\sin^2(t+\varphi-\varphi_0)-\sin^2 t]\times \\
&\qquad\quad\times\left|\frac{\partial^2 K(r,t)}{\partial t^2}\right| dt = A+B .
\end{aligned}$$

The integral A is uniformly bounded in r $(0 \leqslant r < 1)$,

$$|B| \leqslant |\varphi-\varphi_0| \int_{-\pi}^{\pi} |\sin(t+\varphi-\varphi_0)+\sin t| \left|\frac{\partial^2 K(r,t)}{\partial t^2}\right| dt$$

$$\leqslant 2|\varphi-\varphi_0| \left\{2\int_0^{\pi} \sin t \left|\frac{\partial^2 K(r,t)}{\partial t^2}\right| dt + |\varphi-\varphi_0| \int_0^{\pi} \left|\frac{\partial^2 K(r,t)}{\partial t^2}\right| dt\right\}$$

$$\leqslant M_1|\varphi-\varphi_0|/(1-r)+M_2(\varphi-\varphi_0)^2/(1-r)^2 \qquad (M_\nu = \mathrm{const}) .$$

Also, it can easily be observed that

$$\lim I_\delta(r,\varphi) = 0 \quad \text{as} \quad (r,\varphi)\to(1,\varphi_0) .$$

Putting together the results, we get the above-mentioned assertion.

THEOREM 5. *For any bounded function $f(\varphi)$ with modulus $\omega_2(\delta)$, we have*

$$\max_{|\varphi|\leqslant\pi} \left|\frac{\partial f(r,\varphi)}{\partial r}\right| \leqslant M\frac{\omega_2(1-r)}{1-r} \quad \textit{for} \quad 0 < r_0 \leqslant r < 1 ,$$

where M is a constant depending only on r_0. In particular, if $\omega_2(\delta) = o(\delta)$, the relation

$$\lim_{r\to 1-} \frac{\partial f(r,\varphi)}{\partial r} = 0 \tag{7}$$

holds uniformly in φ.

Proof. Let $A_0 = a_0/2$, $A_n(\varphi) = a_n\cos n\varphi + b_n\sin n\varphi$, where a_0, a_n, b_n are the Fourier coefficients of f. In view of (1),

$$\frac{\partial f(r,\varphi)}{\partial r} = (1-r^2)\sum_{n=1}^{\infty}\left(n+\frac{n^2}{2}\right) r^{n-1}A_n(\varphi) .$$

Denote by $f^*(r,\varphi)$ the Abel-Poisson means of the Fourier series of f, i.e.

$$f^*(r,\varphi) = A_0 + \sum_{n=1}^{\infty} r^n A_n(\varphi) ,$$

and write

$$S_r(\varphi) = \sum_{n=1}^{\infty} n r^n A_n(\varphi) , \qquad T_r(\varphi) = \sum_{n=1}^{\infty} n^2 r^n A_n(\varphi) .$$

Then, by (1),

$$f(r,\varphi) = f^*(r,\varphi) + \tfrac{1}{2}(1-r^2)S_r(\varphi) .$$

Applying Theorem 1 of [2], and observing that

$$|f^*(r,\varphi)-f(\varphi)| \leqslant C_3\frac{\omega_2(1-r)}{1-r} \quad \text{for} \quad 0 \leqslant r < 1 ,$$

we get

$$|f(r,\varphi)-f^*(r,\varphi)| \leqslant C_4\frac{\omega_2(1-r)}{1-r} \qquad (C_{3,4}=\text{const});$$

whence

$$(1-r^2)\,|S_r(\varphi)| \leqslant 2C_4\frac{\omega_2(1-r)}{1-r} \quad \text{for} \quad 0\leqslant r<1\,.$$

It is easily seen (cf. [4], pp. 108-109) that if $0\leqslant r<1$,

$$|T_r(\varphi)| = \left|\frac{\partial^2 f^*(r,\varphi)}{\partial\varphi^2}\right| \leqslant C_5\frac{r\omega_2(1-r)}{(1-r)^2} \qquad (C_5=\text{const})\,.$$

Thus

$$\left|\frac{\partial f(r,\varphi)}{\partial r}\right| = \frac{1-r^2}{r}\left|S_r(\varphi)+\frac{1}{2}T_r(\varphi)\right|$$

$$\leqslant\left(\frac{2C_4}{r}+\frac{(1+r)C_5}{2}\right)\frac{\omega_2(1-r)}{1-r} \leqslant M\frac{\omega_2(1-r)}{1-r} \quad \text{as} \quad r\geqslant r_0\,.$$

For the whole class Z_a the estimate in the last theorem cannot be improved. Indeed, the function

$$f(\varphi) = K_a\sum_{n=1}^{\infty}\frac{\cos n\varphi}{n^{1+a}} \qquad (0<a<2)\,,$$

with a suitable constant K_a, is of class Z_a. Its Abel-Poisson means possesses the derivative

$$\frac{\partial^2 f^*(r,\varphi)}{\partial\varphi^2} = -K_a\sum_{n=1}^{\infty} n^{1-a}r^n\cos n\varphi \qquad (0\leqslant r<1)\,.$$

Since

$$\sum_{n=1}^{\infty} n^{1-a}r^n = \Gamma(a)/(1-r)^{2-a}+\varrho_r(a)\,,$$

where

$$\varrho_r(a) = \begin{cases} O\{(1-r)^{a-1}\} & \text{for} \quad 0<a<1\,,\\ O\{|\ln(1-r)|\} & \text{for} \quad a=1\,,\\ O(1) & \text{for} \quad 1<a<2\end{cases}$$

as $r\to 1-$ (see [4], pp. 76-77), we have

$$\left|\left.\frac{\partial^2 f^*(r,\varphi)}{\partial\varphi^2}\right|_{\varphi=0}\right| \geqslant \frac{C_6}{(1-r)^{2-a}} \quad \text{for} \quad 0<r_1\leqslant r<1\,,$$

where C_6 is a positive constant depending only on a and r_1. Now, it can easily be observed that there is a positive constant C_7 such that

$$\max_{|\varphi|\leqslant\pi}\left|\frac{\partial f(r,\varphi)}{\partial r}\right| \geqslant \frac{C_7}{(1-r)^{1-a}} \quad \text{for} \quad 0<r_1\leqslant r<1\,.$$

The biharmonic function generated by

$$f(\varphi) = 4\sin^2(\varphi/2) = 2(1-\cos\varphi)$$

belonging to Z_2 is of the form:

$$f(r,\varphi) = 2-(3-r^2)r\cos\varphi\,.$$

Hence,

$$\max_{|\varphi|\leqslant\pi}\left|\frac{\partial f(r,\varphi)}{\partial r}\right| = 3(1-r^2) \geqslant 3(1-r) \qquad \text{for} \qquad 0\leqslant r<1\,.$$

If $f(\varphi) = |2\sin(\varphi/2)|$,

$$\lim_{r\to 1-}\frac{\partial f(r,0)}{\partial r} = -\frac{2}{\pi} < 0\,.$$

Thus, there exist functions $f(\varphi)$ satisfying the Lipschitz condition for which relation (7) is false at some φ.

Evidently, if $f(\varphi)$ possesses a finite second derivative in a set E, the relation

$$\lim_{r\to 1-}\frac{\partial^2 f^*(r,\varphi)}{\partial\varphi^2} = f''(\varphi)$$

implies (7) for any $\varphi \in E$.

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