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Some theorems on second order partial differential inequalities of parabolic type

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This paper is an answer to the problem set by J. Szarski, concerning the construction of some theorems which would permit the estimation of the absolute values of functions by means of other functions, both function systems satisfying some second order partial differential inequalities of parabolic type of the form

$$\frac{\partial u^i}{\partial t} \leqslant f^i\left(t, x_1, \ldots, x_n, u^1, \ldots, u^m, \frac{\partial u^i}{\partial x_1}, \ldots, \frac{\partial u^i}{\partial x_n}, \frac{\partial^2 u^i}{\partial x_1^2}, \ldots, \frac{\partial^2 u^i}{\partial x_j \partial x_k}, \ldots, \frac{\partial^2 u^i}{\partial x_n^2}\right) \quad (i = 1, \ldots, m).$$

The theorems obtained are connected with J. Szarski's works [2], [3], [4] and W. Mlak's work [1].

At the beginning we give the definition of the region Ω, Σ and Σ_a, the notion of parabolicity in J. Szarski's sense and a lemma from work [3].

The fundamental theorems of this work are given in § 2, and they are formulated again in § 3 with generalized boundary inequalities corresponding to a certain extent to the first and the third classical problems of Fourier. In § 3 we also mention theorems containing strong inequalities and we give an example for theorem 2.2.

I want to express my most sincere gratitude to Professor J. Szarski for his valuable advice and help in the preparation of this paper.

§ 1. Definition of the region Ω. We consider the region Ω in the space $(t, x_1, \ldots, x_n)$ satisfying the following conditions:

(a) Ω is situated in the zone $0 < t < T$, where $T \leqslant +\infty$, and for every t_0, $0 \leqslant t_0 < T$, the intersection of Ω and of the zone $0 \leqslant t \leqslant t_0$ is bounded,

(b) for every t_0, $0 \leqslant t_0 < T$, the section of the closure of Ω by the plane $t = t_0$ is not empty; let S_{t_0} denote the projection of this section on the plane $t = 0$,

(c) for every t_0, $0 \leqslant t_0 < T$, and for every point $X_0 \in S_{t_0}$, as well as for an arbitrary sequence of numbers t_ν, $\nu = 1, 2, \ldots$, such that $0 \leqslant t_\nu < T$

and $t_\nu \to t_0$, there is a sequence of points X_ν, $\nu = 1, 2, \dots$, $X_\nu \in S_{t_\nu}$ and $X_\nu \to X_0$.

Definition of the set Σ and Σ_a. We denote by Σ the part of the boundary of Ω situated in the open zone $0 < t < T$. If we have a function $a(t, x_1, \dots, x_n)$ defined in Σ, then Σ_a will denote that part of the set Σ on which $a > 0$.

Definition of the notion of parabolicity. Let the function

$$f(t, x_1, \dots, x_n, z^1, \dots, z^m, p_1, \dots, p_n, q_{11}, q_{12}, \dots, q_{jk}, \dots, q_{nn})$$

or shortly $f(t, X, Z, P, Q)$, where

$$X = (x_1, \dots, x_n), \quad Z = (z^1, \dots, z^m),$$
$$P = (p_1, \dots, p_n), \quad Q = (q_{11}, \dots, q_{jk}, \dots, q_{nn}),$$

be defined for $(t, X) \in \Omega$, $(Z, P) \in D_1$, $Q \in D_2$ (sets D_1, D_2 are arbitrary) and let a sequence of functions $\big(s^1(t, X), \dots, s^m(t, X)\big) = S(t, X)$ of class C^1 in the region Ω such that the point $(S, s^i_X) \in D_1$ for $(t, X) \in \Omega$ $(i = 1, \dots, m)$ be given. We will say that the function f is *elliptic with regard to the sequence* $s^i(t, X)$ $(i = 1, \dots, m)$ if for every couple of points $R = \{r_{jk}\}$, $\bar{R} = \{\bar{r}_{jk}\}$ $(j, k = 1, \dots, n)$, $r_{jk} = r_{kj}$, $\bar{r}_{jk} = \bar{r}_{kj}$, $R \in D_2$, $\bar{R} \in D_2$ such that

$$\sum_{jk=1}^{n} (r_{jk} - \bar{r}_{jk}) \lambda_j \lambda_k \geqslant 0 \quad \text{for arbitrary } \lambda_1, \dots, \lambda_n$$

the inequality

$$f(t, X, S, s^i_X, R) \geqslant f(t, X, S, s^i_X, \bar{R}) \quad \text{for} \quad (t, X) \in \Omega$$

holds true.

A system of equations

$$\frac{\partial z^i}{\partial t} = f^i\left(t, x_1, \dots, x_n, z^1, \dots, z^m, \frac{\partial z^i}{\partial x_1}, \dots, \frac{\partial z^i}{\partial x_n}, \frac{\partial^2 z^i}{\partial x_1^2}, \dots, \frac{\partial^2 z^i}{\partial x_j \partial x_k}, \dots, \frac{\partial^2 z^i}{\partial x_n^2}\right) \tag{1.1}$$
$$(i = 1, \dots, m)$$

or shortly

$$z^i_t = f^i(t, X, Z, z^i_X, z^i_{XX}) \quad (i = 1, \dots, m),$$

where

$$z^i_t = \frac{\partial z^i}{\partial t}, \quad z^i_X = \left(\frac{\partial z^i}{\partial x_1}, \dots, \frac{\partial z^i}{\partial x_n}\right), \quad z^i_{XX} = \left(\frac{\partial^2 z^i}{\partial x_1^2}, \dots, \frac{\partial^2 z^i}{\partial x_j \partial x_k}, \dots, \frac{\partial^2 z^i}{\partial x_n^2}\right)$$

is called *parabolic with regard to the sequence* $s^i(t, X)$ $(i = 1, \dots, m)$ if every f^i is elliptic with regard to this sequence.

The solution $z^i(t, X)$ $(i = 1, \dots, m)$ of system (1.1) is called a *regular parabolic solution* of (1.1) in Ω if it is continuous in $\bar{\Omega}$, if the derivatives appearing in system (1.1) are continuous in Ω and if f^i $(i = 1, \dots, m)$ are elliptic functions with regard to this solution.

CONDITION (W_+). *A system of functions* $g^i(r_1, \ldots, r_m, \tau)$ $(i = 1, \ldots, m)$ *satisfies condition* (W_+) *with regard to* $r_1, \ldots, r_m$ *if for every fixed* i $(i = 1, \ldots, m)$ *and for* $\bar{r}_j \leqslant \bar{\bar{r}}_j$, $j \neq i$, $\bar{r}_i = \bar{\bar{r}}_i$ *we have the inequality*

$$g^i(\bar{r}_1, \ldots, \bar{r}_m, \tau) \leqslant g^i(\bar{\bar{r}}_1, \ldots, \bar{\bar{r}}_m, \tau) .$$

We formulate a lemma of J. Szarski [3]:

LEMMA 1.1. *Let* $\sigma^i(t, y_1, \ldots, y_m)$ $(i = 1, \ldots, m)$ *be continuous, non-negative functions for* $0 \leqslant t < T$, $y_i \geqslant 0$ *and satisfy the condition* (W_+) *with regard to* $y_1, \ldots, y_m$. *Assume that the only solution of the system* $\frac{dy_i}{dt} = \sigma^i(t, y_1, \ldots, y_m)$ *starting from point* $(0, 0, \ldots, 0)$ *is* $y_i \equiv 0$.

Let the functions $\varphi_i(t)$ $(i = 1, \ldots, m)$ *be continuous in the interval* $0 \leqslant t < T$ *and*

$$\varphi_i(0) \leqslant 0 \qquad (i = 1, \ldots, m) .$$

Suppose finally that if, for a certain index i_0 $(1 \leqslant i_0 \leqslant m)$ *and a certain* t_0, $0 < t_0 < T$, *the inequality*

$$\text{(I)} \qquad \varphi_{i_0}(t_0) > 0$$

holds true, then

$$\bar{D}_- \varphi_{i_0}(t_0) \leqslant \sigma^{i_0}\big(t_0, \varphi_1(t_0), \ldots, \varphi_m(t_0)\big) .$$

Under the above assumptions the following inequality is true:

$$\varphi_i(t) \leqslant 0 \qquad (i = 1, \ldots, m) \qquad \textit{for} \qquad 0 \leqslant t < T .$$

§ 2. Now we will formulate some theorems giving estimates of absolute values of functions. In connection with this task and in view of the hypothesis of parabolicity we have to make some special assumptions.

THEOREM 1.2. *Let us assume that*

1° *the functions* $f^i(t, X, Z, P, Q)$ $(i = 1, \ldots, m)$ *defined for* $(t, x) \in \Omega$, $Z \geqslant 0$, $P \geqslant 0$ *and arbitrary* Q *satisfy the condition* (W_+) *with respect to* Z *as well as*

$$(1.2) \qquad f^i(t, X, Z, P, Q) - f^i(t, X, \bar{Z}, P, Q) \leqslant \sigma^i(t, z^1 - \bar{z}^1, \ldots, z^m - \bar{z}^m)$$

for $z_j \geqslant \bar{z}_j$ $(j = 1, \ldots, m)$, *where the functions* $\sigma^i(t, y_1, \ldots, y_m)$ *satisfy the assumptions of lemma* 1.1,

2° *the functions* $u^i(t, X)$, $v^i(t, X)$ $(i = 1, \ldots, m)$ *are continuous for* $(t, X) \in \bar{\Omega}$, $v^i(t, X) \geqslant 0$, $v^i_X(t, X) \geqslant 0$ *for* $(t, X) \in \Omega$ *and the functions* $f^i(t, X, Z, P, Q)$ *are elliptic with regard to the sequence* $v^i(t, X)$ $(i = 1, \ldots, m)$,

$$(2.2) \qquad |u^i(t, X)| \leqslant v^i(t, X) \qquad \textit{for} \qquad (t, X) \in S_0 + \Sigma ,$$

3° *if for a certain point* $(t_0, X_0) \in \Omega$ *and a certain index* i_0 $(1 \leqslant i_0 \leqslant m)$

$$|u^{i_0}(t_0, X_0)| > v^{i_0}(t_0, X_0)$$

is true, then the functions u^{i_0}, v^{i_0} *have in the neighbourhood of this point continuous derivatives* $\frac{\partial}{\partial t}, \frac{\partial}{\partial x_j}, \frac{\partial^2}{\partial x_j \partial x_k}$ *and satisfy the differential inequalities*

$$(3.2)\qquad |u_t^{i_0}(t_0, X_0)| \leqslant f^{i_0}\big(t_0, X_0, |U(t_0, X_0)|, |u_X^{i_0}(t_0, X_0)|, |u^{i_0}(t_0, X_0)|_{XX}\big)\,,$$

$$(4.2)\qquad v_t^{i_0}(t_0, X_0) \geqslant f^{i_0}\big(t_0, X_0, V(t_0, X_0), v_X^{i_0}(t_0, X_0), v_{XX}^{i_0}(t_0, X_0)\big)\,.$$

Under the above assumptions the following inequality is true:

$$|u^i(t, X)| \leqslant v^i(t, X) \quad \textit{for} \quad (t, X) \in \Omega \quad (i = 1, \ldots, m)\,.$$

Remark 1. The assumption of ellipticity of the functions $f^i(t, X, Z, P, Q)$ is to be understood to mean that they are elliptic at points (t, X) where the functions $v^i(t, X)$ have continuous derivatives of first order. In view of the implicative assumption, the corresponding functions will be elliptic at exceptional points.

Proof. Write

$$\text{(i)}\quad M^i(t) = \max\big(0, \tilde{M}^i(t)\big)\,, \quad \text{where} \quad \tilde{M}^i(t) = \max_{X \in S_t}\big(|u^i(t, X)| - v^i(t, X)\big) \quad (i = 1, \ldots, m)\,.$$

To show that the theorem is true, it is sufficient to prove $M^i(t) \leqslant 0$ for $0 \leqslant t < T$. To do it we make use of lemma 1.1 directly. By virtue of the lemma in [3] the functions $M^i(t)$ are continuous for $0 \leqslant t < T$ and in view of (2.2) we have $M^i(0) \leqslant 0$. It is still to be checked that the implication (I) holds true (with $\varphi_i(t) = M^i(t)$). Suppose that for a certain index i_0 $(1 \leqslant i_0 \leqslant m)$ and a certain $t_0, t_0 > 0$, we have

$$\text{(ii)}\qquad M^{i_0}(t_0) > 0\,.$$

In view of (i) we then have $M^{i_0}(t_0) = \tilde{M}^{i_0}(t_0)$ and by lemma in [3] there is a point $X_0 \in S_{t_0}$ such that

$$\text{(iii)}\qquad M^{i_0}(t_0) = \tilde{M}^{i_0}(t_0) = |u^{i_0}(t_0, X_0)| - v^{i_0}(t_0, X_0) > 0\,.$$

In view of the continuity of the function $M^i(t)$, this inequality will be true in a certain neighbourhood V_0 of the point t_0; hence $M^{i_0}(t) = \tilde{M}^{i_0}(t)$ for $t \in V_0$. Since

$$|u^i(t, X)| = v^i(t, X) + \big(|u^i(t, X)| - v^i(t, X)\big)\,,$$

in view of (i) and (iii) we have

$$|u^i(t_0, X_0)| \leqslant v^i(t_0, X_0) + M^i(t_0) \qquad (i = 1, \ldots, m)\,,$$

$$|u^{i_0}(t_0, X_0)| = v^{i_0}(t_0, X_0) + M^{i_0}(t_0)\,.$$

The function $\varphi(X) = |u^{i_0}(t_0, X)| - v^{i_0}(t_0, X)$ attains its maximum, in view of (i) and (iii) at the point X_0; by virtue of (2.2) the point $(t_0, X_0) \in \mathrm{int}\,\Omega$

and, since from 2° it follows that $u^{i_0}(t_0, X_0) \neq 0$, we have $\varphi_X(X_0) = |u^{i_0}(t_0, X_0)|_X - v^{i_0}_X(t_0, X_0) = 0$ and

$$\text{(iv)} \qquad \sum_{j,k=1}^{n} [v^{i_0}_{x_j x_k}(t_0, X_0) - |u^{i_0}(t_0, X_0)|_{x_j, x_k}] \lambda_j \lambda_k \geqslant 0 .$$

Hence, in view of 2° we have

$$0 \leqslant v^{i_0}_X(t_0, X_0) = |u^{i_0}(t_0, X_0)|_X = \big||u^{i_0}(t_0, X_0)|_X\big| = |u^{i_0}_X(t_0, X_0)| .$$

By virtue of lemma in [3], (3.2), (4.2), (W_+), (1.2), (iv) and by the assumption of ellipticity of the functions f^i we have

$$\text{(v)} \qquad \bar{D}_- M^{i_0}(t_0) = \bar{D}_- \tilde{M}^{i_0}(t_0) \leqslant \frac{\partial}{\partial t} |u^{i_0}(t_0, X_0)| - v^{i_0}_t(t_0, X_0)$$

$$\leqslant f^{i_0}\big(t_0, X_0, |U(t_0, X_0)|, |u^{i_0}_X(t_0, X_0)|, |u^{i_0}(t_0, X_0)|_{XX}\big) -$$

$$- f^{i_0}\big(t_0, X_0, V(t_0, X_0), v^{i_0}_X(t_0, X_0), v^{i_0}_{XX}(t_0, X_0)\big)$$

$$= f^{i_0}(t_0, X_0, |U_0|, {}^\circ v^{i_0}_X, |{}^\circ u^{i_0}|_{XX}) - f^{i_0}(t_0, X_0, V_0, {}^\circ v^{i_0}_X, {}^\circ v^{i_0}_{XX}) +$$

$$+ f^{i_0}(t_0, X_0, V_0, {}^\circ v^{i_0}_X, |{}^\circ u^{i_0}|_{XX}) - f^{i_0}(t_0, X_0, V_0, {}^\circ v^{i_0}_X, |{}^\circ u^{i_0}|_{XX})$$

$$\leqslant f^{i_0}\big(t_0, X_0, V_0 + M(t_0), {}^\circ v^{i_0}_X, |{}^\circ u^{i_0}|_{XX}\big) - f^{i_0}(t_0, X_0, V_0, {}^\circ v^{i_0}_X, |{}^\circ u^{i_0}|_{XX}) -$$

$$- [f^{i_0}(t_0, X_0, V_0, {}^\circ v^{i_0}_X, {}^\circ v^{i_0}_{XX}) - f^{i_0}(t_0, X_0, V_0, {}^\circ v^{i_0}_X, |{}^\circ u^{i_0}|_{XX})]$$

$$\leqslant \sigma^{i_0}\big(t_0, M^1(t_0), \ldots, M^m(t_0)\big) .$$

The implication (ii) $\Rightarrow$ (v) holding true, by lemma 1.1 we have $M^i(t) \leqslant 0$ for $0 \leqslant t < T$, which completes the proof.

THEOREM 2.2. *Let the system of partial differential equations of the second order be given in the following form* (1.1), *i.e.*

$$z^i_t(t, X) = f^i(t, X, Z, z^i_X, z^i_{XX}) \qquad (i = 1, \ldots, m) ,$$

where

1° *the functions* $f^i(t, X, Z, P, Q)$ *are defined for* $(t, X) \in \Omega$ *and arbitrary* Z, P *and* Q, *satisfy condition* (W_+) *with regard to* Z *and inequality* (1.2), *where functions* σ *satisfy the assumptions of lemma* 1.1 *and*

$$\text{(5.2)} \qquad f^i(t, X, Z, P, Q) = -f^i(t, X, -Z, -P, -Q) ,$$

2° *the sequence of the functions* $v^i(t, X)$ $(i = 1, \ldots, m)$ *is a regular parabolic solution of system* (1.1) *in* Ω, *the functions* $\omega^i(t, X)$ $(i = 1, \ldots, m)$ *are continuous for* $(t, X) \in \bar{\Omega}$, $\omega^i(t, X) \geqslant 0$ *for* $(t, X) \in \Omega$ *and*

$$\text{(6.2)} \qquad |v^i(t, X)| \leqslant \omega^i(t, X) \quad \textit{for} \quad (t, X) \in S_0 + \Sigma ,$$

3° *if for a certain point* $(t_0, X_0) \in \Omega$ *and a certain index* i_0 $(1 \leqslant i_0 \leqslant m)$

$$|v^{i_0}(t_0, X_0)| > \omega^{i_0}(t_0, X_0)$$

is true, then function ω^{i_0} has in the neighbourhood of this points continuous derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x_j}, \frac{\partial^2}{\partial x_j \partial x_k}$ and satisfies the differential inequality

$$\omega_t^{i_0}(t_0, X_0) \geqslant f^{i_0}\big(t_0, X_0, \omega(t_0, X_0), \omega_X^{i_0}(t_0, X_0), \omega_{XX}^{i_0}(t_0, X_0)\big).$$

Under the assumptions referred to the following inequality is true:

$$|v^i(t, X)| \leqslant \omega^i(t, X) \quad \text{for} \quad (t, X) \in \Omega \quad (i = 1, \ldots, m).$$

Proof. The proof is conducted in two stages. First, by an argument similar to that in the proof of theorem 1.2 we prove the inequality $v^i(t, X) \leqslant \omega^i(t, X)$ accepting now

$$\tilde{M}^i(t) = \max_{X \in S_t} \big(v^i(t, X) - \omega^i(t, X)\big) \quad \text{and} \quad \varphi(X) = v^{i_0}(t_0, X) - \omega^{i_0}(t_0, X).$$

To prove the second part of the inequality from our assertion, i.e. $-\omega^i(t, X) \leqslant v^i(t, X)$, we make use of assumption (5.2) additionally.

Remark 2. To prove the theorem it is sufficient to assume that 2° and (5.2) are satisfied only for the index i_0 at the exceptional point (t_0, X_0). It is worth noticing that the functions $f^i(t, X, Z, P, Q)$ linear with regard to Z, P, Q satisfy assumption (5.2).

§ 3. THEOREM 1.3. *Suppose that*

1° *the functions $f^i(t, X, Z, P, Q)$ $(i = 1, \ldots, m)$ are defined for $(t, X) \in \Omega$, $Z \geqslant 0$, $P \geqslant 0$ and arbitrary Q, and satisfy the condition (W_+) with regard to Z,*

2° *the functions $u^i(t, X)$, $v^i(t, X)$ $(i = 1, \ldots, m)$ are continuous for $(t, X) \in \bar{\Omega}$, $v^i(t, X) > 0$, $v_X^i(t, X) \geqslant 0$ for $(t, X) \in \Omega$ and the functions f^i are elliptic with regard to the sequence $v^i(t, X)$ $(i = 1, \ldots, m)$,*

$$(1.3) \qquad |u^i(t, X)| < v^i(t, X) \quad \text{for} \quad (t, X) \in S_0 + \Sigma,$$

3° *if for a certain point $(t_0, X_0) \in \Omega$ and a certain index i_0 $(1 \leqslant i_0 \leqslant m)$*

$$|u^{i_0}(t_0, X_0)| = v^{i_0}(t_0, X_0)$$

is true, then the functions u^{i_0}, v^{i_0} have in the neighbourhood of this point continuous derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x_j}, \frac{\partial^2}{\partial x_j \partial x_k}$ and satisfy the differential inequalities

$$(2.3) \quad |u_t^{i_0}(t_0, X_0)| < f^{i_0}\big(t_0, X_0, |U(t_0, X_0)|, |u_X^{i_0}(t_0, X_0)|, |u^{i_0}(t_0, X_0)|_{XX}\big),$$

$$(3.3) \quad v_t^{i_0}(t_0, X_0) \geqslant f^{i_0}\big(t_0, X_0, V(t_0, X_0), v_X^{i_0}(t_0, X_0), v_{XX}^{i_0}(t_0, X_0)\big).$$

Under our assumptions the inequalities

$$|u^i(t, X)| < v^i(t, X)$$

hold for $(t, X) \in \Omega$ $(i = 1, \ldots, m)$.

The proof is similar to those in paper [1].

Remark 3. We may formulate theorems analogous to theorem 2.2 but containing a strong inequality.

In theorems 1.2 and 2.2 we may assume another more general boundary inequality, corresponding to a certain extent to the first and the third classical problems of Fourier.

Let the functions

$$(4.3)\qquad a_i(t,X),\quad \beta_i(t,X)\quad (i=1,\ldots,m),\quad a_i(t,X)\geqslant 0,\quad \beta_i(t,X)>0$$

be defined in Σ and suppose that for every point $(t,X)\in\Sigma_{a_i}$ the directions $l_i=l_i(t,X)$ orthogonal to the axis t and penetrating into the closed region Ω are given. We will assume that

$$(5.3)\qquad \frac{\partial[u^i\pm v^i]}{\partial l_i}\overset{\mathrm{df}}{=}0\quad \text{for}\quad (t,X)\in\Sigma-\Sigma_{a_i}.$$

THEOREM 2.3. *Under the assumptions of theorem* 1.2, *with the exception of inequality* (2.2), *which is substituted by the following more general inequality*

$$(6.3)\qquad a_i(t,X)\frac{\partial[|u^i|-v^i]}{\partial l_i}-\beta_i(t,X)[|u^i|-v^i]\geqslant 0\quad \textit{for}\quad (t,X)\in\Sigma$$

and

$$|u^i(0,X)|\leqslant v^i(0,X)\quad \textit{for}\quad X\in S_0,$$

where the functions a_i, β_i and the directions l_i satisfy (4.3) *and* (5.3), *the inequality of theorem* 1.2 *is true, i.e.*

$$|u^i(t,X)|\leqslant v^i(t,X)\quad \textit{for}\quad (t,X)\in\Omega\quad (i=1,\ldots,m).$$

Remark 4. In the particular case when $a_i\equiv 0$, $\beta_i\equiv 1$ the boundary inequality (6.3) is reduced to inequality (2.2).

Proof. The proof is quite similar to that of theorem 1.2 with the exception that now, in order to show that the point $(t_0,X_0)\in\mathrm{int}\,\Omega$, we make use not of inequality (2.2) but of inequality (6.3) (which is true for the index i_0 and the point (t_0,X_0)). Suppose that the point $(t_0,X_0)\in\Sigma$. In view of (iii) we have

$$|u^{i_0}(t_0,X_0)|-v^{i_0}(t_0,X_0)>0$$

and introducing the parametric equations of the straight half-line l_{i_0} we find that the maximum is attained at the left end of the interval, whence

$$\frac{\partial[|u^{i_0}(t_0,X_0)|-v^{i_0}(t_0,X_0)]}{\partial l_{i_0}}\leqslant 0.$$

Making use of (4.3) we obtain

$$\alpha_{i_0}(t_0, X_0)\frac{\partial[|u^{i_0}(t_0, X_0)|-v^{i_0}(t_0, X_0)]}{\partial l_{i_0}}-\beta_{i_0}(t_0, X_0)[|u^{i_0}(t_0, X_0)|-v^{i_0}(t_0, X_0)]<0,$$

which is inconsistent with (6.3), i.e. the point $(t_0, X_0) \in \operatorname{int}\Omega$.

THEOREM 3.3. *Under the assumptions of theorem 2.2 with the exception of inequality* (6.2), *which is replaced by the more general boundary inequality*

$$(7.3) \qquad \alpha_i(t, X)\frac{\partial[(-1)^{\mu}v^i+\omega^i]}{\partial l_i}-\beta_i(t, X)[(-1)^{\mu}v^i+\omega^i]\leqslant 0 \quad (\mu=0,1)$$

$$\textit{for} \quad (t, X)\in\Sigma$$

and

$$|v^i(0, X)|\leqslant\omega^i(0, X) \quad \textit{for} \quad X\in S_0,$$

where the functions α_i, β_i and the direction l_i satisfy (4.3) *and* (5.3), *the inequality of theorem* 2.2 *is true, i.e.*

$$|v^i(t, X)|\leqslant\omega^i(t, X) \quad \textit{for} \quad (t, X)\in\Omega \quad (i=1,\dots,m).$$

The proof is quite similar to the preceding ones.

Remark 5. We may formulate an analogous theorem but containing strong inequalities.

EXAMPLE. Let us consider the heat equation

$$\frac{\partial u(t, x)}{\partial t}=\frac{\partial^2 u(t, x)}{\partial x^2} \quad \text{in the region} \quad \Omega: \begin{cases}0\leqslant t<+\infty,\\ 0\leqslant x\leqslant 2\pi,\end{cases}$$

which is in form (1.1) and satisfies assumption (5.2). The function

$$f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)=\frac{\partial^2 u}{\partial x^2}$$

satisfies the assumptions of theorem 2.2 automatically. In view of the fact that now the parabolicity in J. Szarski's sense holds true with regard to every function of the class C^1, the function $v(t, x)=\frac{1}{4}e^{-t}\sin\frac{1}{2}x$ is a parabolic solution of the heat equation in Ω. Assuming $\omega(t, x)=\frac{2}{t+4}\sin\frac{x}{2}$ we have $\omega(t, x)\geqslant 0$ and

$$|v(0, x)|=\frac{1}{4}|\sin x|\leqslant\omega(0, x)=\frac{1}{2}\sin\frac{x}{2} \quad \text{for} \quad 0\leqslant x\leqslant 2\pi,$$

$$|v(t, 0)|=|v(t, 2\pi)|=0\leqslant\omega(t, 0)=\omega(t, 2\pi)=0 \quad \text{for} \quad 0\leqslant t<+\infty,$$

$$-\frac{2}{(t+4)^2}\sin\frac{x}{2}\geqslant-\frac{1}{2(t+4)}\sin\frac{x}{2},$$

whence $\omega_t \geqslant \omega_{xx}$ for every point $(t, x) \in \operatorname{int} \Omega$; therefore, by virtue of theorem 2.2 we have

$$\left|\frac{1}{4} e^{-t} \sin x\right| \leqslant \frac{2}{t+4} \sin\frac{x}{2} \quad \text{for} \quad (t, X) \in \Omega .$$

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