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On Hermite expansions of $1/x$ and $1/|x|$

by S. Lewandowska and J. Mikusiński (Katowice)

Abstract. Distributions $1/x$ and $1/|x|$ are considered as distributional derivatives of $\ln|x|$ and $\operatorname{sgn} x\cdot\ln|x|$, respectively. The Hermite expansion of $1/x$ is given in [1]. In the present paper the Hermite expansion of $1/|x|$ and an asymptotic estimation of coefficients for $1/x$ are given.

1. The distributions $1/x$ and $1/|x|$ are defined as distributional derivatives of $\ln|x|$ and $\operatorname{sgn} x\cdot\ln|x|$ respectively, i.e.,

$$\frac{1}{x} = (\ln|x|)' \quad \text{and} \quad \frac{1}{|x|} = (\operatorname{sgn} x\cdot\ln|x|)'.$$

They are tempered distributions. Consequently, each of them can be expanded into an Hermite series $\sum_{n=0}^{\infty} a_n h_n$, where

$$h_n(x) = (-1)^n(\sqrt{2\pi}n!)^{-1/2}e^{x^2/4}(e^{-x^2/2})^{(n)}$$

and the coefficients a_n are inner products of h_n and $1/x$ or $1/|x|$, respectively, i.e., $a_n = (1/x, h_n)$ or $a_n = (1/|x|, h_n)$.

The espansion $1/x$ into the Hermite series is given in [1] and its coefficients are

$$a_n = 0 \quad \text{for even } n \quad \text{and} \quad a_n = \frac{\sqrt[4]{8\pi}}{\sqrt{n!}}u_n \quad \text{for odd } n$$

with $u_1 = 1$ and $u_{n+1} = 1\cdot 3\cdot\ldots\cdot(n-1)-n\cdot u_{n-1}$.

In this paper we shall prove the following asymptotic equality

$$\lim_{n\to\infty}\sqrt[4]{2n-1}\cdot|a_{2n-1}| = \sqrt{\pi}, \quad \text{where } a_n = \left(\frac{1}{x}, h_n\right). \tag{1}$$

We also give the Hermite espansion for $1/|x|$.

It is interesting that in this expansion the Euler constant $C = 0.57\ldots$ appears. However, an asymptotic estimate for the coefficients of the last espansion seems to be much more difficult and will not be given here.

2. In order to prove (1) we introduce the notation $a_n = \sqrt[4]{\frac{8\pi}{n}}\,\lambda_n$ for odd n. Since $a_1 = \sqrt[4]{8\pi}$, we have $\lambda_1 = 1$.

Moreover, we see that the numbers λ_{n+1} satisfy the following equation

$$\lambda_{n+1} + \sqrt[4]{\frac{n\cdot n}{(n-1)\cdot(n+1)}}\,\lambda_{n-1} = \frac{1}{\sqrt[4]{n+1}}\sqrt{\frac{1\cdot 3\cdot\ldots\cdot(n-1)}{2\cdot 4\cdot\ldots\cdot n}} \tag{2}$$

for $n = 2, 4, \ldots$

They can therefore be expressed as

$$\lambda_{n+1} = b_{n+1}\cdot\mu_{n+1} \quad \text{for } n = 2, 4, \ldots \tag{3}$$

with $\mu_1 = 1$, $b_1 = 1$ and μ_{n+1} such that

$$\mu_{n+1} + \sqrt[4]{\frac{n\cdot n}{(n-1)\cdot(n+1)}}\,\mu_{n-1} = 0. \tag{4}$$

It is easy to verify that the numbers $\mu_1 = 1$ and μ_{n+1} such that

$$\mu_{n+1} = (-1)^{n/2}\sqrt[4]{\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\ldots\cdot\frac{n\cdot n}{(n-1)\cdot(n+1)}}$$

satisfy (4) for even n.

From (4), (3) and (2) we obtain by a simple calculation $b_1 = 1$ and

$$b_{n+1} = b_{n-1} + \frac{1}{\sqrt[4]{n+1}}\sqrt{\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot n}}\,\mu_{n+1}^{-1}$$

for $n = 2, 4, \ldots$

Hence it follows that

$$b_{n+1} = 1 - \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} - \ldots + (-1)^{n/2}\frac{1\cdot 3\cdot\ldots\cdot(n-1)}{2\cdot 4\cdot\ldots\cdot n} \quad \text{for } n = 2, 4, \ldots$$

Now we are going to find $\lim_{n\to\infty}|\mu_{n+1}|$ and $\lim_{n\to\infty} b_{n+1}$, with even n.

By simple calculations, we obtain

$$|\mu_{n+1}| = \sqrt[4]{2\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{(n-1)^2}\right)}\cdot\sqrt[4]{\frac{n}{n+1}}.$$

Hence we infer that

$$\lim_{n\to\infty}|\mu_{n+1}| = \sqrt[4]{2\lim_{n\to\infty}\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{(n-1)^2}\right)}. \tag{5}$$

From the equality

$$\sin \pi x = \pi \cdot x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\dots$$

it follows that

$$\frac{\sin \pi x}{2(1-x^2)} = \sin \pi \frac{x}{2} \cdot \left(1-\frac{x^2}{3^2}\right)\left(1-\frac{x^2}{5^2}\right)\dots$$

Hence, if x tends to 1, we have

$$\frac{\pi}{4} = \left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\dots$$

By the last equality and (5) we infer that

$$(6) \qquad \lim_{n\to\infty} |\mu_{n+1}| = \sqrt[4]{\frac{\pi}{2}}.$$

It remains to find $\lim\limits_{n\to\infty} b_{n+1}$.

We have

$$(1+x)^{-1/2} = \binom{-1/2}{0} + \binom{-1/2}{1}x + \binom{-1/2}{2}x^2 + \dots$$

Hence if x tends to 1, we get

$$\frac{1}{\sqrt{2}} = \binom{-1/2}{0} + \binom{-1/2}{1} + \binom{-1/2}{2} + \dots = 1 - \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} - \dots = \lim_{n\to\infty} b_{n+1}.$$

Finally, by (6) and (3) we have

$$\lim_{n\to\infty} |\lambda_{n+1}| = \sqrt[4]{\frac{\pi}{2}} \cdot \frac{1}{\sqrt{2}} = \sqrt[4]{\frac{\pi}{8}}.$$

Hence (1) follows, which was to be proved.

3. In this section we shall expand $1/|x|$ into the Hermite series. Since $1/|x|$ is even, $c_n = 0$ for odd n and

$$(7) \qquad c_n = \big((\operatorname{sgn} x \cdot \ln|x|)', h_n(x)\big) = -\big(\operatorname{sgn} x \cdot \ln|x|, h_n'(x)\big)$$

$$= -2\int_0^\infty \ln x \cdot h_n'(x)\,dx \quad \text{for even } n.$$

It is known that (see [1])

$$\sqrt{n+1}\cdot h_{n+1}(x) = x h_n(x) - \sqrt{n}\, h_{n-1}(x).$$

Hence

$$\int_0^\infty \frac{1}{x}\left(\sqrt{n+1}\,h_{n+1}(x)+\sqrt{n}\,h_{n-1}(x)\right) = \int_0^\infty h_n(x)\,dx.$$

Integrating by parts, we find

$$-\int_0^\infty \ln x\left(\sqrt{n+1}\,h'_{n+1}(x)+\sqrt{n}\,h'_{n-1}(x)\right)dx = \int_0^\infty h_n(x)\,dx.$$

Consequently, by (7), we can write

$$\sqrt{n+1}\,c_{n+1}+\sqrt{n}\,c_{n-1} = 2\int_0^\infty h_n(x)\,dx. \tag{8}$$

Using the known formula (see [1])

$$\sqrt{n+1}\,h_{n+1} = -2h'_n+\sqrt{n}\,h_{n-1},$$

we obtain

$$\sqrt{n+1}\int_0^\infty h_{n+1}(x)\,dx-\sqrt{n}\int_0^\infty h_{n-1}(x)\,dx = 2h_n(0). \tag{9}$$

By (8), we can write

$$\sqrt{n+1}\cdot\sqrt{n+2}\cdot c_{n+2}+(n+1)\cdot c_n = 2\sqrt{n+1}\int_0^\infty h_{n+1}(x)\,dx$$

and

$$n\cdot c_n+\sqrt{n}\cdot\sqrt{n-1}\cdot c_{n-2} = 2\sqrt{n}\int_0^\infty h_{n-1}(x)\,dx.$$

Hence, by (9), we have

$$\sqrt{(n+1)(n+2)}\,c_{n+2}+c_n-\sqrt{n(n-1)}\,c_{n-2} = 4h_n(0) \quad \text{for even } n. \tag{10}$$

It is easy to verify that

$$c_0 = \frac{1}{\sqrt[4]{2\pi}}\int_0^\infty x\ln x e^{-x^2/4}\,dx$$

and

$$c_2 = -\frac{1}{\sqrt[4]{8\pi}}\int_0^\infty (5x-x^3)\ln x\cdot e^{-x^2/4}\,dx.$$

By a simple calculation, we obtain

$$c_2 = \frac{1}{\sqrt{2}}\left(\frac{4}{\sqrt[4]{2\pi}} - c_0\right).$$

Hence, since

$$\int_0^\infty x \ln x \cdot e^{-x^2/4} dx = \ln 4 - C,$$

where C is the Euler constant we can finally write

$$c_0 = \frac{1}{\sqrt[4]{2\pi}} (\ln 4 - C)$$

and

$$c_2 = \frac{1}{\sqrt[4]{8\pi}} (4 - \ln 4 + C).$$

Moreover, we have (see [1])

$$h_n(0) = \frac{(-1)^{n/2}}{\sqrt[4]{2\pi}} \sqrt{\frac{1}{2} \cdot \ldots \cdot \frac{3}{4} \cdot \ldots \cdot \frac{n-1}{n}} \qquad \text{for even } n.$$

Thus the sequence $\{c_n\}$ is defined.

Substituting $v_n = \sqrt[4]{2\pi} \cdot \sqrt{n!} \cdot c_n$ into (10) we obtain the following equation

(11) $$v_{n+2} + v_n - n(n-1) v_{n-2} = 4(-1)^{n/2} 1 \cdot 3 \cdot \ldots \cdot (n-1).$$

The general solution of (11) has the following form:

$$v_n = A\alpha_n + B\beta_n, \quad \text{where } \alpha_0 = 1,\ \beta_2 = -1$$

and $\alpha_{n+2} + \alpha_n - n(n-1)\alpha_{n-2} = 0$, and $\beta_0 = 0$, $\beta_2 = 1$, and

$$\beta_{n+2} + \beta_n - n(n-1)\beta_{n-2} = (-1)^{n/2} 1 \cdot 3 \cdot \ldots \cdot (n-1).$$

Therefore we have

$$c_n = \frac{1}{\sqrt[4]{2\pi}\sqrt{n!}} (A\alpha_n + B\beta_n).$$

By simple calculations we find that $A = \ln 4 - C$ and $B = 4$.
Finally, we can write

(12) $$c_n = \begin{cases} \dfrac{1}{\sqrt[4]{2\pi}\sqrt{n!}} [(\ln 4 - C)\alpha_n + 4\beta_n] & \text{for even } n, \\ 0 & \text{for odd } n. \end{cases}$$

In particular, the initial three coefficients of the expansion are

$$\frac{1}{|x|} = \frac{1}{\sqrt[4]{2\pi}}\left((\ln 4 - C)h_0 + [4 - (\ln 4 - C)]\frac{1}{\sqrt{2!}}h_2 + \right. \tag{13}$$

$$\left. + [3(\ln 4 - C) - 8]\frac{1}{\sqrt{4!}}h_4 + \ldots\right).$$

References

[1] P. Antosik, J. Mikusiński and R. Sikorski, *Theory of distributions. The sequential approach*, Amsterdam–Warszawa 1973.

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