ANNALES
POLONICI MATHEMATICI
XLII (1983)

On the Tjon–Wu representation of the Boltzmann equation

by TOMASZ DŁOTKO, ANDRZEJ LASOTA (Katowice)

Dedicated to the memory of Jacek Szarski

Abstract. The existence, uniqueness and stability problems for a non-linear version of the Boltzmann equation are discussed. This version was suggested by J. A. Tjon and T. T. Wu [8]. Our main result is a simple proof of the global stability of the stationary solution $u = \exp(-x)$ in the class of all solutions with finite moments.

1. Introduction. Using the Abel transformation it is possible to reduce a class of the Boltzmann equations to the following simple form:

$$\frac{\partial u(t,x)}{\partial t} + u(t,x) = \int_x^\infty \frac{1}{y}\int_0^y u(t,y-z)u(t,z)\,dz\,dy, \tag{1}$$

where $x \in [0,\infty)$, $t \in [0,\infty)$. Equation (1) is considered with the initial condition

$$u(0,x) = u_0(x) \quad \text{for } x \in [0,\infty), \tag{2}$$

where u_0 is a normalized function, namely

$$\int_0^\infty u_0(x)\,dx = \int_0^\infty x u_0(x)\,dx = 1, \qquad u_0(x) \geqslant 0. \tag{3}$$

The function $u(t,x)$ has an important physical interpretation. It is the density function for the distribution of particles in time t with respect to the energy x.

It is easy to see that the function $u(t,x) = \exp(-x)$ is a (stationary) solution of (1). M. F. Barnsley and G. Turchetti successed in arguing that this solution is stable in a class of all initial functions u_0 satisfying the condition

$$\int_0^\infty u_0(x)\exp(\tfrac{1}{2}x)\,dx < \infty.$$

We shall show that this inequality may be replaced by weaker assumption of the existence of the moments

$$\int_0^\infty x^n u_0(x)\,dx < \infty \quad \text{for } n = 0, 1, \ldots$$

In order to make our arguments precise, we shall also show the existence and uniqueness of solution for problem (1)–(2) in the class of initial conditions satisfying (3).

NOTATIONS. Let $I = [0, \infty)$. By B_0 we denote the space

$$\left\{u \in L^1(I);\ \int_0^\infty (1+x)|u(x)|\,dx < \infty\right\},$$

where the norm is given by

$$\|u\|_0 = \int_0^\infty (1+x)|u(x)|\,dx.$$

We set

$$C_0^0(I) = \{u \in C^0(I);\ \lim_{x\to\infty} u(x) = 0\}$$

and we consider the space $B_1 = B_0 \cap C_0^0(I)$, with the norm

$$\|u\|_1 = \|u\|_0 + \|u\|_{C^0(I)}.$$

Condition $u \geqslant 0$ $(u \in B_0)$ denotes $u(x) \geqslant 0$ a.e. The non-linear operator on the right-hand side of (1) is denoted by

$$H(x, u) = \int_x^\infty \frac{1}{y} \int_0^y u(y-z)u(z)\,dz\,dy.$$

Further,

$$M^n(u) = \int_I x^n u(x)\,dx, \qquad M^n(u;\ t) = \int_I x^n u(t, x)\,dx.$$

We shall consider equation (1) as an ordinary differential equation in the space B_0. Thus (1)–(3) may be rewritten in the form

$$(1') \qquad D_t u = -u + H(\cdot, u),$$

where D_t denotes the strong derivative in B_0,

$$(2') \qquad u(0) = u_0,$$

$$(3') \qquad M^0(u_0) = M^1(u_0) = 1, \quad u_0 \geqslant 0.$$

2. Preliminaries. We start with a few elementary lemmas in which we shall exploit the special properties of the right-hand side of equation (1').

LEMMA 1. *If* $u \in B_0$ *and*

$$M^n(u) = \int_I x^n u(x)\,dx < \infty,$$

then

$$\int_I x^n H(x, u)\,dx = \frac{1}{n+1}\int_0^\infty u(z)\int_0^\infty u(r)(z+r)^n\,dr\,dz \tag{4}$$
$$= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k} M^k(u)M^{n-k}(u).$$

Proof. Under our assumption all moments $M^k(u)$ with $k \leqslant n$ are finite. Since we are going to prove that the integral on the left-hand side of (4) exists, we start from rewritting the right-hand side. By the definition of $M^n(u)$ we have

$$R_n := \frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k} M^k(u)M^{n-k}(u)$$
$$= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}\int_0^\infty z^k u(z)\,dz\cdot\int_0^\infty r^{n-k}u(r)\,dr$$
$$= \frac{1}{n+1}\int_0^\infty u(z)\int_0^\infty (z+r)^n u(r)\,dr\,dz.$$

Further, by classical applications of the Fubini theorem,

$$R_n = \frac{1}{n+1}\int_0^\infty u(z)\int_0^\infty u(y-z)y^n\,dy\,dz = \int_0^\infty\int_0^y u(y-z)u(z)\frac{y^n}{n+1}\,dz\,dy$$
$$= \int_0^\infty\int_0^y u(y-z)u(z)\int_0^y \frac{x^n}{y}\,dx\,dz\,dy$$
$$= \int_0^\infty\int_0^y \frac{x^n}{y}\int_0^y u(y-z)u(z)\,dz\,dx\,dy = \int_0^\infty x^n H(x, u)\,dx.$$

In particular, for $n = 0$, $n = 1$, we get

$$\int_0^\infty H(x, u)\,dx = \Big(\int_0^\infty u(x)\,dx\Big)^2, \tag{5}$$

$$\int_0^\infty xH(x, u)\,dx = \Big|\int_0^\infty u(x)\,dx\cdot\int_0^\infty xu(x)\,dx. \tag{6}$$

The integral H can be considered as a non-linear operator from the space B_0 into itself.

We have the following

LEMMA 2. *If u is a solution of* (1')–(3') *and*

$$u \in C^1([0, \infty): B_0),$$

then

$$M^0(u;\ t) \equiv M^1(u;\ t) \equiv 1 \quad \textit{for all } t \geqslant 0.$$

Proof. Integrating (1) and using (5) we obtain

$$\frac{d}{dt} M^0(u;\ t) = -M^0(u;\ t) + \big(M^0(u;\ t)\big)^2.$$

Moreover, from (3') we have $M^0(u_0) = 1$. This implies $M^0(u;\ t) \equiv 1$ for all $t \geqslant 0$. Analogously, from (6) it follows

$$\frac{d}{dt} M^1(u;\ t) = -M^1(u;\ t) + M^0(u;\ t)\cdot M^1(u;\ t) = 0$$

and consequently $M^1(u;\ t) \equiv M^1(u_0) \equiv 1$.

We shall also use the following elementary observation:

LEMMA 3. *If the continuous function $f\colon I \to R$ has the limit*

$$\lim_{t\to\infty} f(t) = f_0,$$

then the solution w of the problem

$$w'(t) = -\lambda w(t) + f(t), \qquad \lambda > 0,$$

converges to f_0/λ, where t tends to infinity. Moreover, for all $t \geqslant 0$,

$$\sup w(t) \leqslant \max\{w(0), \sup\big(f(t)/\lambda\big)\}. \tag{7}$$

3. Existence and uniqueness of solutions.

THEOREM 1. *For any initial condition $u_0 \in B_0$ satisfying* (3') *there exists unique solution u of problem* (1')–(2'), *and*

$$u \in C^1([0, \infty): B_0), \qquad u(t) \geqslant 0. \tag{8}$$

Proof. It is easy to show that the non-linear term $H(\cdot, v)$ satisfies in B_0 the Lipschitz condition with respect to v. In fact, for $u, v \in B_0$ we have

$$\int_0^\infty (1+x)|H(x, u) - H(x, v)|\,dx$$

$$\leqslant \int_0^\infty (1+x) \int_x^\infty \frac{1}{y} \int_0^y \big[|u(y-z)||u(z)-v(z)| + |v(z)||u(y-z)-v(y-z)|\big]\,dz\,dy\,dx$$

$$= \int_0^\infty (1+\tfrac{1}{2}y) \int_0^y [|u(y-z)|\,|u(z)-v(z)| + |v(z)|\,|u(y-z)-v(y-z)|]\,dz\,dy$$

$$\leqslant \int_0^\infty |u(z)-v(z)| \int_z^\infty (1+y-z)\,|u(y-z)|\,dy\,dz +$$

$$+ \int_0^\infty |v(z)| \int_z^\infty [(1+y-z)+z]\,|u(y-z)-v(y-z)|\,dz\,dy$$

$$\leqslant 2\,\|u-v\|_0 [\|u\|_0 + \|v\|_0].$$

Finally

$$\|H(x, u) - H(x, v)\|_0 \leqslant 2\,\|u-v\|_0 [\|u\|_0 + \|v\|_0]. \tag{9}$$

Now we may apply to (1′)–(3′) the classical method of successive approximations, setting

$$u_0(t) = u_0, \qquad D_t u_{i+1} + u_{i+1} = H(\cdot, u_i),$$

or

$$u_{i+1}(t) = u_0 \exp(-t) + \int_0^t \exp(\tau - t) H(\cdot, u_i)(\tau)\,d\tau. \tag{10}$$

From the last formula it follows that $u_i(t) \geqslant 0$ for all i. Moreover, according to (5) and (6),

$$M^0(u_{i+1};\ t) = M^0(u_0)\exp(-t) + \int_0^t \exp(\tau - t)\big(M^0(u_i;\ \tau)\big)^2\,d\tau, \tag{11}$$

$$M^1(u_{i+1};\ t) = M^1(u_0)\exp(-t) + \int_0^t \exp(\tau - t) M^0(u_i;\ \tau) M^1(u_i;\ \tau)\,d\tau. \tag{12}$$

Since $M^0(u_0) = M^1(u_0) = 1$, this implies by induction argument

$$M^0(u_i;\ t) = M^1(u_i;\ t) = 1 \quad \text{for all } t \geqslant 0,\ i = 0, 1, 2, \ldots$$

Consequently

$$\|u_i\|_0 = M^0(u_i) + M^1(u_i) = 2. \tag{13}$$

This fact and the Lipschitz condition (9) assure the convergence (uniform on compact intervals) of successive approximations on the whole half line $[0, \infty)$. The limiting function

$$u(t) = \lim_{i \to \infty} u_i(t)$$

is of course a solution of (1′)–(3′) and satisfies (8).

In the proof of the convergence the non-negativity of u_i plays an important role. It assures the uniformness of the Lipschitz condition. So the uniqueness should be proved separately.

For any u, v which are the solutions of (1')–(3') we have

$$\begin{aligned}\frac{d}{dt}\|u-v\|_0 &\leqslant \left\|\frac{d}{dt}(u-v)\right\|_0 \\ &\leqslant \|u-v\|_0+\|H(\cdot,u)-H(\cdot,v)\|_0 \\ &\leqslant \|u-v\|_0(1+\|u\|_0+\|v\|_0) = 3\|u-v\|_0.\end{aligned}$$

Since $\|u(0)-v(0)\|_0 = 0$ this by standard argument implies $u = v$.

It is easy also to prove the existence of solutions of equation (1')–(3') in the space B_1.

Remark 1. For any $u_0 \in B_1$ the unique solution u of (1')–(3') given by Theorem 1 satisfies

$$u \in C^1([0,T): B_1),$$

where T is a sufficiently small number.

In fact, for $u, v \in B_1$ and $\|u\|_{C^0(I)}, \|v\|_{C^0(I)} \leqslant m$,

$$\begin{aligned}&\left|\int_x^\infty \frac{1}{y}\int_0^y [u(y-z)u(z)-v(y-z)v(z)]dzdy\right| \\ &\leqslant 2\int_0^{1/m}\frac{1}{y}\int_0^y m|u(z)-v(z)|dzdy+\int_{1/m}^\infty \frac{1}{y}\int_0^y |u(y-z)||u(z)-v(z)|dzdy+ \\ &\qquad +\int_{1/m}^\infty \frac{1}{y}\int_0^y |v(z)||u(y-z)-v(y-z)|dzdy \\ &\leqslant 2\sup_I |u(x)-v(x)|+m\int_0^\infty |u(z)-v(z)|\int_z^\infty |u(y-z)|dzdy+ \\ &\qquad +m\int_0^\infty |v(z)|\int_z^\infty |u(y-z)-v(y-z)|dzdy \\ &= 2\|u-v\|_{C^0(I)}+m\|u-v\|_0[\|u\|_0+\|v\|_0].\end{aligned}$$

Consequently

$$\|H(\cdot,u)-H(\cdot,v)\|_{C^0(I)} \leqslant 2\|u-v\|_{C^0(I)}+m\|u-v\|_0[\|u\|_0+\|v\|_0].$$

From this and inequality (9) follows

$$\begin{aligned}&\|H(\cdot,u)-H(\cdot,v)\|_1 \\ &\qquad \leqslant \|u-v\|_0 \max\{\|u\|_{C^0(I)}, \|v\|_{C^0(I)}\}(\|u\|_0+\|v\|_0)+2\|u-v\|_{C^0(I)}\end{aligned}$$

or

$$\|H(\cdot, u)-H(\cdot, v)\|_1 \leqslant \big(2+(\|u\|_1+\|v\|_1)^2\big)\|u-v\|_1.$$

The last inequality implies that the sequence $u_i(t)$ of successive approximations given by formula (10) is convergent in B_1 norm uniformly for $t \in [0, T)$ with sufficiently small T. We do not know if $u_i(t)$ converges to $u(t)$ in B_1 norm on the whole half line.

4. Moments of solutions. We shall study the behaviour of moments $M^n(u; t)$. Our starting point is the following

LEMMA 4. *Suppose that the initial condition $u_0 \in B_0$ in problem* (1)–(3) *admits all moments up to the order L, i.e.*

$$M^l(u_0) < \infty \quad \textit{for } l = 0, 1, \ldots, L.$$

Then there exists $T_0 = t_0(L)$ such that all moments $M^l(u; t)$ of the solution u exist for $t \in [0, T_0)$, $l = 0, 1, \ldots, L$, and are continuously differentiable functions of t.

Proof. Using (10), it is easy to obtain a formal system of equations which must be satisfied by moments, if they exist. Namely, setting $M_i^l(t) = M^l(u_i; t)$, we have

$$\begin{aligned} M_{i+1}^l(t) &= M^l(0)\exp(-t)+ \\ &\quad + \int_0^t \exp(\tau-t)\int_0^\infty x^l \int_x^\infty \frac{1}{y}\int_0^y u_i(\tau, y-z)u_i(\tau, z)\,dz\,dy\,dx\,d\tau \\ &= M^l(0)\exp(-t)+\int_0^t \exp(\tau-t)\sum_{k=0}^{l}\binom{l}{k} M_i^k(\tau)M_i^{l-k}(\tau)\,d\tau, \qquad (14)\\ M^l(0) &= \int_0^\infty x^l u_0(x)\,dx. \end{aligned}$$

Write $A = \max\{m^l;\ l = 0, 1, \ldots, L\}$, where

$$m^0 = 1, \quad m^1 = 1, \quad m^l = \max\left\{M^l(0), \frac{1}{l-1}\, 2^l (m^{l-1})^2\right\} \tag{15}$$

and denote by $\mu(t)$ the solution of the problem

$$\mu(t)+c\mu^2(t) = \mu'(t) \quad \big(c = \big(1/(L+1)\big)2^L\big), \quad \mu(0) = A.$$

The function $\mu(t)$ is positive and defined in an interval $[0, T_0)$. By an induction argument it is easy to show that

$$M_i^l(t) \leqslant \mu(t) \quad \text{for } l \leqslant L,\ i = 0, 1, \ldots,\ t \in [0, T_0). \tag{16}$$

In fact, $M_0^l(t) = M^l(u_0) \leqslant A$ by the definition of A, and $A \leqslant \mu(t)$ for all t. Now assume that inequality (16) holds for $l = 0, 1, \ldots, L$ and $i \leqslant j$.

Then

$$M_{j+1}^{l}(t) = M^{l}(0)\exp(-t)+\int\limits_{0}^{t}\frac{\exp(\tau-t)}{l+1}\sum_{k=0}^{l}\binom{l}{k}M_j^k(\tau)M_j^{l-k}(\tau)\,d\tau$$
$$\leqslant A\exp(-t)+\int\limits_{0}^{t}\frac{\exp(\tau-t)}{L+1}2^L\mu^2(\tau)\,d\tau,$$

which proves (16).

Therefore, formulae (14) are meaningfull. The function $M_i^l(t)$ represents the sequence of successive approximations for the system

$$M^0(t) = 1, \quad M^1(t) = 1,$$

$$\frac{d}{dt}M^l(t) = -M^l(t)+\frac{1}{l+1}\sum_{k=0}^{l}\binom{l}{k}M^k(t)M^{l-k}(t)$$

$$= -\frac{l-1}{l+1}M^l(t)+\frac{1}{l+1}\sum_{k=0}^{l}\binom{l}{k}M^k(t)M^{l-k}(t), \tag{17}$$

$$M^l(0) = \int\limits_{0}^{\infty}x^l u_0(x)\,dx$$

of ordinary differential equations. Then $\{M_i^l(t)\}_{i=1}^{\infty}$ $(l\leqslant L)$ converge on $[0, T_0)$ (uniformly on compact subintervals) to the solution $M^l(t)$ of (17).

On the other hand, for each $r > 0$

$$\int\limits_{0}^{r}x^l u_i(t, x)\,dx \leqslant M_i^l(t)$$

and consequently, since $u_i(t,\cdot)\to u(t,\cdot)$ in B_0,

$$\int\limits_{0}^{r}x^l u(t, x)\,dx \leqslant M^l(t).$$

Again passing to the limit as $r\to\infty$, we obtain

$$M^l(u;\ t) = \int\limits_{0}^{\infty}x^l u(t, x)\,dx \leqslant M^l(t).$$

Now, knowing that $M^l(u;\ t)$ exists, we may multiply equation

$$u(t, x) = u_0(x)\exp(-t)+\int\limits_{0}^{t}\exp(\tau-t)H(\cdot, u)(\tau)\,d\tau$$

by x^l ($l \leqslant L$) and integrate on the half line $[0, \infty)$. This shows that $M^l(u;\, t)$ are actually the solutions of (17) and therefore $M^l(u;\, t) = M^l(t)$ ($t \in [0, T_0)$, $l \leqslant L$). The proof is finished.

Now we are going to show that the moments $M^n(u;\, t)$ exist for all $t \geqslant 0$ and that they have finite limits as $t \to \infty$.

THEOREM 2. *If the initial condition $u_0 \in B_0$ has all moments $M^l(u_0)$ for $l \leqslant L$ finite, then also the solution $u(t, \cdot)$ has all moments $M^l(u;\, t)$ finite up to the order L for every $t \geqslant 0$. They satisfy equations* (17).

Proof. As was shown in Lemma 4, the moments $M^l(u;\, t)$ of the solution u of (1′)–(3′) satisfy (17) for $t \in [0, T_0)$. It is easy to show by an induction argument (using Lemma 3) that the numbers m^l, given by (15) are the upper bounds for $M^k(t)$, namely

$$(18) \qquad 0 \leqslant \sup_{0 \leqslant k \leqslant l} M^k(t) \leqslant m^l \quad \text{for all } t \geqslant 0.$$

In fact, $M^0(u_0) = 1 = m^0$. Setting

$$f(t) = \frac{1}{l+1} \sum_{k=0}^{l} \binom{l}{k} M^k(t)\; M^{l-k}(t)$$

and applying Lemma 3 to equations (17), we obtain

$$M^l(t) \leqslant \max\left\{M^l(0), \frac{1}{l-1} 2^l \big(\sup_{\substack{t \geqslant 0 \\ k \leqslant l-1}} M^k(t)\big)^2\right\}.$$

Assuming by induction that $M^k(t) \leqslant m^k$, $k \leqslant l-1$, we have immediately (18).

Now we may repeat the proof of Lemma 4 step by step in the intervals $[\frac{3}{4}T_0, \frac{7}{4}T_0)$, $[\frac{6}{4}T_0, \frac{10}{4}T_0)$, ... This gives the global (for all $t \geqslant 0$) existence of all moments up to the order L, and completes the proof.

5. Asymptotic behaviour of solutions of (1′)–(3′). System (17) has the stationary solution, independent on t,

$$M^n \equiv n!.$$

The solution is globally asymptotic stable, namely for any initial condition with $M^0(0) = M^1(0) = 1$, $M^i(0) \geqslant 0$, $i = 2, 3, \ldots$,

$$(19) \qquad M^n(t) \to n! \quad \text{for } t \to \infty.$$

This fact follows immediately from Lemma 3.

Using this, we may prove the following

THEOREM 3. *If the initial condition $u_0 \in B_0$ has finite all moments, then*

$$(20) \qquad \lim_{t \to \infty} \int_0^y u(t, x)\, dx = \int_0^y \exp(-x)\, dx \quad \textit{for each } y \geqslant 0.$$

Proof. It is well known that the function $\exp(-x)$ is uniquely determined by all its moments (e.g. see [6]). Its moments are equal to $n!$. From (19) by classical "moments convergence theorem" (see [4]) we obtain (20). This completes the proof.

From Theorem 3 it follows that $u(t, x) = \exp(-x)$ is the unique stationary solution of (1)–(3) with all finite moments such that $M^0(u;\ 0) = M^1(u;\ 0) = 1$.

Final remarks. We have proved in Theorem 1 that the unique solution of (1')–(3') with non-negative initial value u_0 is non-negative for all t. However, the proof was restricted to solutions satisfying the additional condition

$$M^0(u_0) = M^1(u_0) = 1.$$

By the method of differential inequalities it is easy to prove that any solution of (1')–(2') satisfying $u_0 \geqslant 0$, $u_0 \in B_0$ is non-negative in its domain of existence. In fact, if u is a solution of (1')–(2'), then $v(t) = u(t)\exp(t)$ is a solution of

$$D_t v = \exp(-t)H(\cdot, v), \quad v(0) = u_0 \geqslant 0. \tag{21}$$

The right-hand side of (21) is a Lipschitzian monotonic operator in B_0. Thus since $\bar{v} \equiv 0$ is a solution of (21) and $v(0) \geqslant \bar{v}(0)$, we have (see [7], [5])

$$v(t) \geqslant \bar{v}(t) \equiv 0$$

whenever v exists.

References

[1] M. F. Barnsley and G. Turchetti, *New results on the nonlinear Boltzmann equation,* Bifurcation Phenomena in Mathematical Physics and Related Topics, p. 351–370.

[2] T. Carleman, *Problèmes mathématique dans la théorie cinétique des gaz,* Uppsala 1957.

[3] A. Lasota, A. Strauss and W. Walter, *Infinite systems of differential inequalities defined recursively,* J. Differ. Equ. 9 (1971), p. 93–107.

[4] M. Loeve, *Probability theory,* Van Nostrand, 1960.

[5] Ja. D. Mamedov, S. Aširov, S. Atgaev, *Inequalities theory,* Ašhabad 1980 (in Russian).

[6] J. A. Shohat and J. D. Tamarkin, *The problem of moments,* New York 1943.

[7] J. Szarski, *Differential inequalities,* P.W.N., Warszawa 1967.

[8] J. A. Tjon, T. T. Wu, *Numerical aspects of the approach to a Maxwellian distribution,* Phys. Rev. A, 19 (1978), p. 883–888.

Reçu par la Rédaction le 8.03.1981