ANNALES
POLONICI MATHEMATICI
XLIX (1988)

Functional Banach spaces of holomorphic functions on Reinhardt domains

by JACOB BURBEA (Pittsburgh, Pennsylvania)

Abstract. In this paper, certain functional Banach spaces of holomorphic functions on complete Reinhardt domains in C^n are introduced and studied from a general point of view. The general theory is then applied to functions holomorphic on C^n, on the polydisk and on the ball of C^n, resulting in various sharp norm inequalities.

Introduction. Given any complete Reinhardt domain Ω in C^n and any $0 < p < \infty$, we define $\Omega(p) = \{(z_1, \ldots, z_n) \in C^n: (|z_1|^{p/2}, \ldots, |z_n|^{p/2}) \in \Omega\}$. We also let $\Omega(\infty)$ be the unit polydisk in C^n. Thus, it follows that $\Omega(p)$ is itself a complete Reinhardt domain for any $0 < p \leqslant \infty$ and that $\Omega(2) = \Omega$. With Ω we associate the positive cone $\mathscr{P}(\Omega)$ of functions φ, holomorphic on $\Omega(1)$ and with the power series expansion

$$\varphi(z) = \sum_{\alpha} c_\alpha z^\alpha$$

such that $c_\alpha = c_\alpha(\varphi) \geqslant 0$ for every $\alpha \in Z^n_+$. In particular, $0 \leqslant \varphi(z \cdot \bar{z}) < \infty$ for every $z \in \Omega$. Here, for $z = (z_1, \ldots, z_n)$, $\zeta = (\zeta_1, \ldots, \zeta_n)$ in C^n, $z \cdot \zeta$ is the point $(z_1 \zeta_1, \ldots, z_n \zeta_n)$ in C^n. For any $\varphi \in \mathscr{P}(\Omega)$ we let $\Gamma_\varphi = \{\alpha \in Z^n_+ : c_\alpha(\varphi) > 0\}$, and for any $0 < p < \infty$ we introduce the space l^p_φ which is a weighted l^p-space of sequences $\{a_\alpha\}$ with weights $\{c_\alpha^{1-p}\}$, where α ranges in Γ_φ. Similarly, the space l^∞_φ is a weighted l^∞-space of sequences $\{a_\alpha\}$ with weights $\{c_\alpha^{-1}\}$, $\alpha \in \Gamma_\varphi$.

In this paper we show that for $1 \leqslant p \leqslant \infty$, the space l^p_φ may be identified with a functional Banach space $\mathscr{H}^p_\varphi$ of holomorphic functions on $\Omega(p')$, where $1/p + 1/p' = 1$. When $p = 2$, the space $\mathscr{H}^2_\varphi$ is, of course, a functional Hilbert space of holomorphic functions on $\Omega = \Omega(2)$. The reproducing kernel of this space is given by $k_\varphi(z, \zeta) = \varphi(z \cdot \bar{\zeta})$, where $z \cdot \bar{\zeta} \in \Omega(1)$, and one of the main purposes of this paper is to show that much of the $p = 2$ theory extends to any $1 \leqslant p \leqslant \infty$ by duality principles. In these principles not only the duality between l^p_φ and $l^{p'}_\varphi$ plays a role but also the duality between the domains $\Omega(p)$ and $\Omega(p')$ $(1 < p < \infty)$ is essential, and thus the requirement on Ω to be a complete Reinhardt domain is rather crucial.

The paper gives a systematic account of the general theory of these $\mathscr{H}^p_\varphi$ $(1 \leqslant p \leqslant \infty,\ \varphi \in \mathscr{P}(\Omega))$ spaces and their applications to functions holomorphic on C^n, on the polydisk and on the ball of C^n. Special instances of this general theory may also be found in the earlier works of Lebedev, Lebedev and Milin, and Milin (see [9], pp. 27–32, 197), in the works of Saitoh [10], [11], in the works of the author [2]–[5], and quite recently in the thesis of Kwak [7]. The general approach employed here, however, renders the results in the above mentioned works as particular corollaries of the present general theory.

The paper is organized as follows. Section 1 sets the notation used in this paper and gives some preliminaries. In Section 2 we introduce the above mentioned functional Banach spaces $\mathscr{H}^p_\varphi$ and establish most of their algebraic and topological properties. This includes the relationships that persist amongst the spaces $\mathscr{H}^p_{\varphi_1+\varphi_2}$ and $\mathscr{H}^p_{\varphi_1\varphi_2}$ in terms of the spaces $\mathscr{H}^p_{\varphi_1}$ and $\mathscr{H}^p_{\varphi_2}$, where $\varphi_1, \varphi_2 \in \mathscr{P}(\Omega)$ (see Theorems 2.7 and 2.8 and their corollaries). Section 3 is devoted to a study of decompositions and of certain continuous operators on these spaces (see Theorem 3.4). In Section 4 we study the relationship between the space $\mathscr{H}^p_\varphi$, $\varphi \in \mathscr{P}(\Omega)$, and the space $\mathscr{H}^p_\psi$, where ψ is the composition $F \circ \varphi$, and where F is typically an entire function on C with non-negative Taylor coefficients. The main results of this section are found in Theorem 4.3 and its corollaries. Finally, in Section 5 we give some concrete applications of the general theory to holomorphic functions on C^n, on the polydisk Δ^n and on the unit ball B of C^n. In the latter case we have Theorem 5.5 and its corollaries which deal with exponentiation of functions belonging to certain Sobolev spaces of holomorphic functions on B. A related result is Theorem 5.10 (and its corollary) which is connnected with a recent one-dimensional estimate, due to Chang and Marshall [6], associated with the Dirichlet integral of holomorphic functions on the unit disk of C.

1. Notation and Preliminaries. For $z = (z_1, \ldots, z_n) \in C^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in Z^n_+$ and $p > 0$ we use the following notation:

$$|z|^p = (|z_1|^p, \ldots, |z_n|^p) \in R^n_+, \qquad \|z\|_p = (|z_1|^p + \ldots + |z_n|^p)^{1/p} \in R_+,$$

$$\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in C^n, \qquad z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n} \in C,$$

$$|\alpha| = \alpha_1 + \ldots + \alpha_n \in Z_+ \quad \text{and} \quad \alpha! = \alpha_1! \ldots \alpha_n! \in Z_+.$$

Moreover, if also $\zeta = (\zeta_1, \ldots, \zeta_n) \in C^n$, then we let

$$z \cdot \zeta = (z_1 \zeta_1, \ldots, z_n \zeta_n) \in C^n \quad \text{and} \quad \langle z, \zeta \rangle = z_1 \bar{\zeta}_1 + \ldots + z_n \bar{\zeta}_n \in C.$$

We also write $|z|$ for $|z|^1$, $\|z\|$ for $\|z\|_2$ and

$$\|z\|_\infty = \max_{1 \leq j \leq n} |z_j|.$$

We now let

$$\Delta = \{\lambda \in C: |\lambda| < 1\}, \qquad T = \partial\Delta = \{\lambda \in C: |\lambda| = 1\},$$

$$\Delta^n = \{z \in C^n: \|z\|_\infty < 1\}$$

and

$$B = \{z \in C^n\colon \|z\| < 1\}, \qquad S = \partial B = \{z \in C^n\colon \|z\| = 1\}.$$

By $e_k \in Z^n_+$ $(1 \leqslant k \leqslant n)$ we mean the n-tuple that has 1 in the kth entry and 0 everywhere else, and we let

$$I = (1, \ldots, 1) = e_1 + \ldots + e_n.$$

For a complex manifold D, D^* stands for the conjugate manifold of D and $H(D)$ designates the class of all holomorphic functions on D.

An open set Ω in C^n is said to be a *Reinhardt region* if $z \in \Omega$ implies $z \cdot \zeta \in \Omega$ for every $\zeta \in T^n$. One easily verifies that in this case

$$\Omega = \{z \in C^n\colon |z| \in \Omega\},$$

and if we define, for $p > 0$,

$$\Omega(p) = \{z \in C^n\colon |z|^{p/2} \in \Omega\}$$

then $\Omega(p)$ is also a Reinhardt region with $\Omega(2) = \Omega$. An open set Ω in C^n is said to be a complete Reinhardt domain if $z \in \Omega$ implies $z \cdot \zeta \in \Omega$ for every $\zeta \in \bar{\Delta}^n$. In this case Ω is a star-shaped domain and Reinhardt region containing the origin of C^n. Moreover, if Ω is a complete Reinhardt domain in C^n so is $\Omega(p)$, $p > 0$.

We fix a complete Reinhardt domain Ω in C^n and consider $\Omega(p)$ for any $p > 0$. We also define $\Omega(\infty) \equiv \Delta^n$. Typical examples for such Ω will be C^n, Δ^n and B. In these cases we note that $C^n(p) = C^n$ for every $p > 0$, $\Delta^n(p) = \Delta^n$ for every $0 < p \leqslant \infty$ and

$$B(p) = \{z \in C^n\colon \|z\|_p < 1\} \qquad (0 < p \leqslant \infty),$$

with $B(p_1) \subset B(p_2)$ for every $0 < p_1 \leqslant p_2 \leqslant \infty$. We also fix a sequence of non-negative numbers c_α, defined for every $\alpha \in Z^n_+$, so that

$$\sum_\alpha c_\alpha |z|^{2\alpha} < \infty$$

for every $z \in \Omega$. It follows that the power series

$$\sum_\alpha c_\alpha z^\alpha$$

converges *normally* in $\Omega(1)$, i.e., absolutely and uniformly on compacta of $\Omega(1)$, to

$$\varphi(z) = \sum_\alpha c_\alpha z^\alpha \qquad (z \in \Omega(1)). \tag{1.1}$$

In particular, $\varphi \in H(\Omega(1))$ and

$$\varphi(z \cdot \bar{z}) = \varphi(|z|^2) < \infty \qquad (z \in \Omega). \tag{1.2}$$

We define

$$\Lambda_\varphi = \{\alpha \in Z^n_+ : c_\alpha = 0\},$$

and thus $\partial^\alpha \varphi(0) = 0$ for every α in Λ_φ. Here for $\alpha = (\alpha_1, \ldots, \alpha_n)$ in Z^n_+, $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, where $\partial_j = \partial/\partial z_j$, $1 \leqslant j \leqslant n$.

For $1 \leqslant p \leqslant \infty$ we let $p' = p/(p-1)$, and hence $1 \leqslant p' \leqslant \infty$ with $p'' = p$. An application of Hölder's inequality with $1 < p < \infty$ shows that for any $z, \zeta \in C^n$,

$$\varphi(|z \cdot \bar{\zeta}|) \leqslant \{\varphi(|z|^p)\}^{1/p} \{\varphi(|\zeta|^{p'})\}^{1/p'} \quad (1 < p < \infty). \tag{1.3}$$

We now define

$$k_\varphi(z, \zeta) = \varphi(z \cdot \bar{\zeta}). \tag{1.4}$$

This function is sesqui-holomorphic, i.e., holomorphic in the first variable and anti-holomorphic in the second, on the domain $\{(z, \zeta) \in C^{2n} : |z \cdot \zeta| \in \Omega(1)\}$. Moreover, for any $1 \leqslant p \leqslant \infty$, $k_\varphi \in H(\Omega(p) \times \Omega(p')^*)$. This follows from (1.3) when $1 < p < \infty$, while for $p = 1$ or $p = \infty$ this follows from the fact that $\Omega(\infty) = \Delta^n$. Note also that k_φ is hermitian, i.e., $\overline{k_\varphi(z, \zeta)} = k_\varphi(\zeta, z)$ for $z, \zeta \in C^n$ with $|z \cdot \zeta| \in \Omega(1)$.

By dv we denote the Lebesgue measure in C^n, and if D is a domain with C^2 boundary (or a product of such domains), then $d\sigma_D$ represents the normalized surface measure on the distinguished boundary $\partial_0 D$ of D. For q and m with $q(q+m) \neq 0$, we let

$$(q)_m = \Gamma(q+m)/\Gamma(q).$$

Typical examples for the above function φ are as follows:

EXAMPLE 1.1. ($\Omega = C^n$). For any $r, q > 0$ we let $\varphi = \varphi_{r,q}$ be given by

$$\varphi_{r,q}(z) = (n)_{r-1} \sum_\alpha \frac{q^{|\alpha|}}{\alpha!\,(n+|\alpha|)_{r-1}} z^\alpha.$$

Then $\varphi_{r,q} \in H(C^n)$, and for any $z, \zeta \in C^n$

$$k_{r,q}(z, \zeta) \equiv \varphi_{r,q}(z \cdot \bar{\zeta}) = F(n; r+n-1 : q\langle z, \zeta\rangle), \tag{1.5}$$

where $F(a; b: \lambda)$ is the well-known confluent hypergeometric function

$$F(a; b: \lambda) = \sum_{m=0}^{\infty} \frac{(a)_m}{m!\,(b)_m} \lambda^m.$$

In particular, $k_{1,q}(z, \zeta) = e^{q\langle z, \zeta\rangle}$. Note also that $\Lambda_{\varphi_{r,q}} = \emptyset$.

EXAMPLE 1.2. ($\Omega = \Delta^n$). For $\boldsymbol{q} = (q_1, \ldots, q_n) \in \boldsymbol{R}^n_+ \setminus \{0\}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{Z}^n_+$ we let

$$(\boldsymbol{q})_\alpha = (q_1)_{\alpha_1} \ldots (q_n)_{\alpha_n}.$$

We now define $\varphi = \varphi_{\boldsymbol{q}}$ by

$$\varphi_{\boldsymbol{q}}(z) = \sum_\alpha \frac{(\boldsymbol{q})_\alpha}{\alpha!} z^\alpha \qquad (\boldsymbol{q} \in \boldsymbol{R}^n_+ \setminus \{0\},\ z \in \Delta^n).$$

It follows that

$$\varphi_{\boldsymbol{q}}(z) = \prod_{j=1}^n (1 - z_j)^{-q_j} \qquad (z = (z_1, \ldots, z_n) \in \Delta^n),$$

with $\varphi_{\boldsymbol{q}} \in H(\Delta^n)$ and $\Lambda_{\varphi_{\boldsymbol{q}}} = \emptyset$.

EXAMPLE 1.3. ($\Omega = B$). For $q \geqslant 0$ we define $\varphi = \varphi_q$ by

$$\varphi_q(z) = \sum_\alpha \frac{(q)_{|\alpha|}}{\alpha!} z^\alpha \qquad (q > 0),$$

$$\varphi_0(z) = \sum_{\alpha > 0} \frac{\Gamma(|\alpha|)}{\alpha!} z^\alpha \qquad (q = 0).$$

It follows that $\varphi_q \in H(B(1))$ with $\Lambda_{\varphi_0} = \{0\}$ and $\Lambda_{\varphi_q} = \emptyset$ for every $q > 0$. Moreover, the function

$$k_q(z, \zeta) \equiv \varphi_q(z \cdot \bar{\zeta})$$

is of the form

$$k_q(z, \zeta) = \begin{cases} (1 - \langle z, \zeta \rangle)^{-q} & q > 0, \\ -\log(1 - \langle z, \zeta \rangle), & q = 0, \end{cases}$$

and is sesqui-holomorphic in the domain $\{(z, \zeta) \in \boldsymbol{C}^{2n}\colon |\langle z, \zeta \rangle| < 1\}$. Note also that $\varphi_0(0) = 0$, $\varphi_q = \exp(q\varphi_0)$ for $q > 0$ and that $\varphi_0 = \lim_{q \to 0^+} (\varphi_q - 1)/q$.

2. Functional Banach spaces over $\Omega(p')$. If D is a complete Reinhardt domain in $\boldsymbol{C}^n$ and if $f \in H(D)$, then there exists a unique power series such that

$$f(z) = \sum_\alpha a_\alpha z^\alpha \qquad (z \in D)$$

with normal convergence in D, and with

$$a_\alpha = \{\partial^\alpha f(0)\}/\alpha! \qquad (\alpha \in \boldsymbol{Z}^n_+).$$

For any subset Λ of $\boldsymbol{Z}^n_+$, we define

$$H(D\colon \Lambda) = \{f \in H(D)\colon\ \partial^\alpha f(0) = 0,\ \alpha \in \Lambda\}$$

which is a subspace of $H(D)$, and we note that $H(D\colon \emptyset) = H(D)$.

We now return to our fixed complete Reinhardt domain Ω and function φ. Let

$$\Gamma_\varphi = Z_+^n \setminus \Lambda_\varphi = \{\alpha \in Z_+^n:\ c_\alpha > 0\},$$

and let P be a sufficiently small polydisk about the origin. For $f \in H(P: \Lambda_\varphi)$ and $0 < p \leqslant \infty$, we define

$$\|f\|_{p,\varphi} = \{\sum_\alpha c_\alpha^{1-p}|a_\alpha|^p\}^{1/p} \quad (0 < p < \infty) \tag{2.1}$$

and

$$\|f\|_{\infty,\varphi} = \sup_\alpha \{c_\alpha^{-1}|a_\alpha|\}, \tag{2.2}$$

where $a_\alpha = \{\partial^\alpha f(0)\}/\alpha!$ and α ranges over Γ_φ. Similarly, if $f,\ g \in H(P: \Lambda_\varphi)$ with $a_\alpha = \{\partial^\alpha f(0)\}/\alpha!$ and $b_\alpha = \{\partial^\alpha g(0)\}/\alpha!$ such that

$$\sum_\alpha c_\alpha^{-1}|a_\alpha b_\alpha| < \infty \quad (\alpha \in \Gamma_\varphi),$$

then we define

$$\langle f, g\rangle_\varphi = \sum_\alpha c_\alpha^{-1} a_\alpha \bar{b}_\alpha \quad (\alpha \in \Gamma_\varphi). \tag{2.3}$$

For $1 \leqslant p \leqslant \infty$ and $p' = p/(p-1)$ we let

$$\mathscr{H}_\varphi^p \equiv \mathscr{H}_\varphi^p[\Omega(p')] = \{f \in H(\Omega(p'): \Lambda_\varphi):\ \|f\|_{p,\varphi} < \infty\},$$

and thus $\mathscr{H}_\varphi^p$ is a space of holomorphic functions f on $\Omega(p')$ with $\partial^\alpha f(0) = 0$ for $\alpha \in \Lambda_\varphi$. Denote by $l_\varphi^p = l_\varphi^p(\Gamma_\varphi)$, $0 < p \leqslant \infty$, the space of complex sequences $\boldsymbol{a} = \{a_\alpha\}$, $\alpha \in \Gamma_\varphi$, with $\|\boldsymbol{a}\|_{p,\varphi} < \infty$, where $\|\boldsymbol{a}\|_{p,\varphi}$ is defined by the right-hand side of (2.1)–(2.2). Thus, l_φ^p is a complete topological vector space which is a separable for $0 < p < \infty$. For $0 < p < 1$, it is a metric space with the translation-invariant metric $\varrho(\boldsymbol{a}, \boldsymbol{b}) = \|\boldsymbol{a}-\boldsymbol{b}\|_{p,\varphi}^p$, and for $1 \leqslant p \leqslant \infty$ it is a Banach space with the norm $\|\cdot\|_{p,\varphi}$. When $1 \leqslant p \leqslant \infty$, we define a pairing $\langle \boldsymbol{a}, \boldsymbol{b}\rangle_\varphi$ between $\boldsymbol{a} = \{a_\alpha\} \in l_\varphi^p$ and $\boldsymbol{b} = \{b_\alpha\} \in l_\varphi^{p'}$ by identifying it with the right-hand side of (2.3). It follows that the dual $(l_\varphi^p)^*$ of l_φ^p, $1 \leqslant p < \infty$, is $l_\varphi^{p'}$, with the duality given by the pairing $\langle\ ,\ \rangle_\varphi$, and that l_φ^p, $1 < p < \infty$, is a reflexive Banach space. In particular, l_φ^2 is a separable Hilbert space with inner-product $\langle\ ,\ \rangle_\varphi$ and norm $\|\cdot\|_{2,\varphi}$.

The following theorem shows that $\mathscr{H}_\varphi^p$ may be identified with l_φ^p for $1 \leqslant p \leqslant \infty$.

THEOREM 2.1. *Let $1 \leqslant p \leqslant \infty$. For $f \in \mathscr{H}_\varphi^p$ with $a_\alpha = \{\partial^\alpha f(0)\}/\alpha!$, $\alpha \in \Gamma_\varphi$, let $Tf = \{a_\alpha\}$. Then T is a one-to-one linear transformation of $\mathscr{H}_\varphi^p$ onto l_φ^p with $\|Tf\|_{p,\varphi} = \|f\|_{p,\varphi}$ for every $f \in \mathscr{H}_\varphi^p$.*

Proof. That T is a one-to-one linear transformation of $\mathscr{H}_\varphi^p$ into l_φ^p is clear. It is also clear that $\|Tf\|_{p,\varphi} = \|f\|_{p,\varphi}$ for $f \in \mathscr{H}_\varphi^p$. It remains to be shown that T is also surjective. For this purpose, we let $\boldsymbol{a} = \{a_\alpha\} \in l^p_\varphi$ and $z \in \Omega(p')$. By Hölder's inequality for $1 < p \leqslant \infty$,

$$\sum_\alpha |a_\alpha z^\alpha| \leqslant \|\boldsymbol{a}\|_{p,\varphi} \{\varphi(|z|^{p'})\}^{1/p'} < \infty \qquad (\alpha \in \Gamma_\varphi).$$

Similarly, since $\Omega(\infty) = \Delta^n$, we have for $p = 1$

$$\sum_\alpha |a_\alpha z^\alpha| \leqslant \|\boldsymbol{a}\|_1 < \infty \qquad (\alpha \in \Gamma_\varphi).$$

It follows that the function

$$f(z) = \sum_\alpha a_\alpha z^\alpha \qquad (\alpha \in \Gamma_\varphi)$$

is in $H(\Omega(p'))$ with

$$\partial^\alpha f(0) = 0, \qquad \alpha \in \Lambda_\varphi.$$

Since also $\|f\|_{p,\varphi} = \|\boldsymbol{a}\|_{p,\varphi} < \infty$, we deduce that $f \in \mathscr{H}^p_\varphi$ and that $\boldsymbol{a} = Tf$. This concludes the proof.

Corollary 2.2. *Let $1 \leqslant p \leqslant \infty$. Then $\mathscr{H}^p_\varphi$ is a Banach space of holomorphic functions f on $\Omega(p')$ with $\partial^\alpha f(0) = 0$ for $\alpha \in \Lambda_\varphi$ and with norm $\|f\|_{p,\varphi}$. It is separable for $1 \leqslant p < \infty$ and reflexive for $1 < p < \infty$, and is a Hilbert space for $p = 2$ with inner-product $\langle\ ,\ \rangle_\varphi$ and norm $\|\cdot\|_{2,\varphi}$. Moreover, the dual $(\mathscr{H}^p_\varphi)^*$ of $\mathscr{H}^p_\varphi$, $1 \leqslant p < \infty$, is $\mathscr{H}^{p'}_\varphi$ with the duality given by the pairing $\langle\ ,\ \rangle_\varphi$.*

The next theorem shows that the function k_φ defined in (1.4) plays a central role in the space $\mathscr{H}^p_\varphi$.

Theorem 2.3. *Let $1 \leqslant p \leqslant \infty$. Then:*

(i) *For any fixed $\zeta \in \Omega(p')$, the function $k_\varphi(\cdot, \zeta)$ is in $\mathscr{H}^{p'}_\varphi$ with*

$$\|k_\varphi(\cdot, \zeta)\|_{p',\varphi} = \begin{cases} \{\varphi(|\zeta|^{p'})\}^{1/p'}, & 1 \leqslant p' < \infty, \\ \sup\limits_{\alpha \in \Gamma_\varphi} |\zeta^\alpha|, & p' = \infty, \end{cases}$$

and thus $\|k_\varphi(\cdot, \zeta)\|_{\infty,\varphi} \leqslant 1$.

(ii) *For any fixed $\zeta \in \Omega(p')$ and any $f \in \mathscr{H}^p_\varphi$,*

$$f(\zeta) = \langle f, k_\varphi(\cdot, \zeta)\rangle_\varphi$$

and so

$$|f(\zeta)| \leqslant \|f\|_{p,\varphi} \|k_\varphi(\cdot, \zeta)\|_{p',\varphi}.$$

(iii) *For any $(z, \zeta) \in \Omega(p) \times \Omega(p')$,*

$$\varphi(\zeta \cdot \bar{z}) = k_\varphi(\zeta, z) = \langle k_\varphi(\cdot, z), k_\varphi(\cdot, \zeta)\rangle_\varphi.$$

(iv) *For any $z_1, \ldots, z_n \in \Omega(p) \cap \Omega(p')$ and any $a_1, \ldots, a_N \in \boldsymbol{C}$,*

$$\sum_{i,j=1}^N a_i \bar{a}_j k_\varphi(z_i, z_j) \geqslant 0.$$

Proof. To prove (i), we assume first that $1 < p \leqslant \infty$ and that ζ is in $\Omega(p')$. In this case it is clear that $k_\varphi(\cdot, \zeta) \in H(\Omega(p): \Lambda_\varphi)$ and that

$$\|k_\varphi(\cdot\,\zeta)\|_{p',\varphi}^{p'} = \sum_\alpha c_\alpha^{1-p'} |c_\alpha \bar{\zeta}^\alpha|^{p'} = \varphi(|\zeta|^{p'}) \qquad (\alpha \in \Gamma_\varphi).$$

It follows that $k_\varphi(\cdot, \zeta) \in \mathscr{H}_\varphi^{p'}$ with norm as given in (i). When $p = 1$ and $\zeta \in \Omega(\infty) = \Delta^n$, we have $k_\varphi(\cdot, \zeta) \in H(\Omega(1): \Lambda_\varphi)$ with

$$\|k_\varphi(\cdot, \zeta)\|_{\infty,\varphi} = \sup_\alpha \{c_\alpha^{-1} |c_\alpha \bar{\zeta}^\alpha|\} = \sup_\alpha |\zeta^\alpha| \leqslant 1 \qquad (\alpha \in \Gamma_\varphi).$$

This norm is exactly 1 if $0 \in \Gamma_\varphi$, and (i) follows. The identity in (ii) follows from (i) and the definition of $\mathscr{H}_\varphi^p$. The inequality follows from Hölder's inequality, and (ii) is proved. To prove (iii) we use (i) to deduce that $k_\varphi(\cdot, z) \in \mathscr{H}_\varphi^p$ for $z \in \Omega$ and then apply (ii). Finally, to prove (iv) we note that the power series for $k_\varphi(z_i, z_j) = \varphi(z_i \cdot \bar{z}_j)$ is absolutely convergent when $z_i, z_j \in \Omega(p) \cap \Omega(p')$. It follows that

$$\sum_{i,j=1}^N a_i \bar{a}_j k_\varphi(z_i, z_j) = \sum_{i,j=1}^N a_i \bar{a}_j \varphi(z_i \cdot \bar{z}_j) = \sum_\alpha c_\alpha \Big|\sum_{i=1}^N a_i z_i^\alpha\Big|^2 \geqslant 0 \qquad (\alpha \in \Gamma_\varphi),$$

and the proof is complete.

Theorem 2.3 shows, in particular, that $\mathscr{H}_\varphi^p$ ($1 \leqslant p \leqslant \infty$) is a *functional Banach space* over $\Omega(p')$, i.e., that for any $\zeta \in \Omega(p')$, the evaluation $f \mapsto f(\zeta)$ from $\mathscr{H}_\varphi^p$ into C is continuous. The norm of this evaluation is $\|k_\varphi(\cdot, \zeta)\|_{p',\varphi}$, and in view of (ii), the function k_φ may be called the *reproducing kernel* for in $\mathscr{H}_\varphi^p$ and $\mathscr{H}_\varphi^{p'}$. Moreover, property (iv) of the theorem shows that this kernel is *positive-definite* on $\Omega(p) \cap \Omega(p')$. The case $p = 2$ is of particular interest for this case $\mathscr{H}_\varphi^2$ is a functional Hilbert space over Ω with the reproducing kernel k_φ. We refer to [2]–[5], [10], [11] and the references therein for additional details.

We now proceed to establish some further properties of the spaces $\mathscr{H}_\varphi^p$.

THEOREM 2.4. *Let $c > 0$ and $1 \leqslant p < q \leqslant \infty$. Then the following statements are equivalent:*

(i) *$c_\alpha \geqslant c$ for every $\alpha \in \Gamma_\varphi$;*

(ii) *$\mathscr{H}_\varphi^p \subset \mathscr{H}_\varphi^q$ and $\|f\|_{q,\varphi} \leqslant c^{1/q-1/p} \|f\|_{p,\varphi}$ for every $f \in \mathscr{H}_\varphi^p$.*

Proof. To prove the implication (i) $\Rightarrow$ (ii), we first assume that $f \in H(P: \Lambda_\varphi)$, where P is a sufficiently small polydisk about the origin of C^n. Then

$$f(z) = \sum a_\alpha z^\alpha \qquad (\alpha \in \Gamma_\varphi)$$

with normal convergence in P. We first assume that $q < \infty$. Then, since $0 < p/q < 1$,

$$\begin{aligned}\|f\|_{p,\varphi}^p = \Big\{\sum_\alpha c_\alpha^{1-q} |a_\alpha|^q\Big\}^{p/q} &\leqslant \sum_\alpha c_\alpha^{p/q-p} |a_\alpha|^p \\ &= \sum_\alpha c_\alpha^{p/q-1} c_\alpha^{1-p} |a_\alpha|^p \leqslant c^{p/q-1} \|f\|_{p,\varphi}^p \qquad (\alpha \in \Gamma_\varphi).\end{aligned}$$

and thus $\|f\|_{q,\varphi} \leqslant c^{1/q-1/p}\|f\|_{p,\varphi}$ for $q < \infty$. When $q = \infty$, we have

$$\|f\|_{p,\varphi} = \{\sum_{\alpha} c_\alpha^{1-p}|a_\alpha|^p\}^{1/p} \geqslant c_\alpha^{1/p} c_\alpha^{-1}|a_\alpha| \geqslant c^{1/p} c_\alpha^{-1}|a_\alpha|$$

for every $\alpha \in \Gamma_\varphi$. Hence $\|f\|_{p,\varphi} \geqslant c^{1/p}\|f\|_{\infty,\varphi}$. It follows that for $f \in H(P\colon \Lambda_\varphi)$

$$\|f\|_{q,\varphi} \leqslant c^{1/q-1/p}\|f\|_{p,\varphi} \quad (0 < p \leqslant q \leqslant \infty). \tag{2.4}$$

We now assume that $f \in \mathscr{H}_\varphi^p$ with $1 \leqslant p < q \leqslant \infty$ to prove (ii). Then $\|f\|_{p,\varphi} < \infty$ and $f \in H(\Omega(p')\colon \Lambda_\varphi)$. It follows from Theorem 2.1 and (2.4) that $f \in \mathscr{H}_\varphi^q$ with $\|f\|_{q,\varphi} \leqslant c^{1/q-1/p}\|f\|_{p,\varphi}$ and (ii) is proved.

To prove the implication (ii)$\Rightarrow$(i), we observe that for any $\alpha \in \Gamma_\varphi$, the monomial $f_\alpha(z) = z^\alpha$ is in $\mathscr{H}_\varphi^p \cap \mathscr{H}_\varphi^q$ with

$$\|f_\alpha\|_{p,\varphi} = c_\alpha^{1/p-1}, \quad \|f_\alpha\|_{q,\varphi} = c_\alpha^{1/q-1}.$$

Thus, it follows from (ii) that $c_\alpha \geqslant c$ for any $\alpha \in \Gamma_\varphi$. This concludes the proof.

By $\mathscr{P}(\Omega)$ we denote the class of functions φ on Ω as described in (1.1)–(1.2). Thus, associated with $\varphi \in \mathscr{P}(\Omega)$ we have $c_\alpha = c_\alpha(\varphi) \geqslant 0$, $\alpha \in \boldsymbol{Z}_+^n$, $\Lambda_\varphi = \{\alpha \in \boldsymbol{Z}_+^n\colon c_\alpha(\varphi) = 0\}$, $\Gamma_\varphi = \boldsymbol{Z}_+^n \setminus \Lambda_\varphi$, k_φ and the spaces $\mathscr{H}_\varphi^p$ for $1 \leqslant p \leqslant \infty$.

The proof of the next proposition is straightforward.

PROPOSITION 2.5. *Let $\varphi \in \mathscr{P}(\Omega)$ and $a > 0$. Then $a\varphi \in \mathscr{P}(\Omega)$ with $\Lambda_{a\varphi} = \Lambda_\varphi$ and $\mathscr{H}_\varphi^p = \mathscr{H}_{a\varphi}^p$ for every $1 \leqslant p \leqslant \infty$. Moreover, for $f \in \mathscr{H}_\varphi^p$ with $1 \leqslant p \leqslant \infty$,*

$$\|f\|_{p,a\varphi} = a^{-1/p'}\|f\|_{p,\varphi}.$$

THEOREM 2.6. *Let $\varphi_j \in \mathscr{P}(\Omega)$ with $c_\alpha(j) = c_\alpha(\varphi_j)$, $\alpha \in \boldsymbol{Z}_+^n$, $k_j = k_{\varphi_j}$, $\Lambda_j = \Lambda_{\varphi_j}$ and $\Gamma_j = \Gamma_{\varphi_j}$, $j = 1, 2$. Let $a > 0$. Then the following conditions are equivalent:*

(i) *The kernel $K \equiv ak_1 - k_2$ is positive-definite on Ω;*

(ii) *$c_\alpha(2) \leqslant ac_\alpha(1)$ for every $\alpha \in \boldsymbol{Z}_+^n$;*

(iii) *$\mathscr{H}_{\varphi_2}^p \subset \mathscr{H}_{\varphi_1}^p$ with $\|f\|_{p,\varphi_1} \leqslant a^{1/p'}\|f\|_{p,\varphi_2}$ for any $f \in \mathscr{H}_{\varphi_2}^p$ for every $1 \leqslant p \leqslant \infty$;*

(iv) *$\mathscr{H}_{\varphi_2}^p \subset \mathscr{H}_{\varphi_1}^p$ with $\|f\|_{p,\varphi_1} \leqslant a^{1/p'}\|f\|_{p,\varphi_2}$ for any $f \in \mathscr{H}_{\varphi_2}^p$ for some $1 \leqslant p \leqslant \infty$.*

Proof. To prove the implication (i)$\Rightarrow$(ii), we let $c_\alpha = ac_\alpha(1) - c_\alpha(2)$ for any $\alpha \in \boldsymbol{Z}_+^n$ and note that

$$K(z, \zeta) = \sum_\alpha c_\alpha z^\alpha \bar{\zeta}^\alpha \quad (z, \zeta \in \Omega),$$

with normal convergence in $\Omega \times \Omega$. Fix an arbitrary $\beta \in \boldsymbol{Z}_+^n$ with $m = |\beta|$. We shall prove that $c_\beta \geqslant 0$ from which (ii) will follow. To this end we observe that the $N \equiv \binom{n+m}{n}$ monomials $f_\alpha(z) = z^\alpha$, $\alpha \in Z^n_+$, $|\alpha| \leqslant m$, are linearly independent on Ω. Thus there are N points $\zeta_1, \ldots, \zeta_N$ in Ω so that

$$\det[f_\alpha(\zeta_j)] \neq 0 \qquad (|\alpha| \leqslant m,\ 1 \leqslant j \leqslant N).$$

It follows that there exist scalars $a_1, \ldots, a_N \in C$ such that

$$\sum_{j=1}^{N} a_j f_\alpha(\zeta_j) = \delta_{\alpha\beta} \qquad (|\alpha| \leqslant m),$$

where $\delta_{\alpha\beta}$ is the "delta of Kronecker" of α and β. Let $0 \leqslant \varepsilon \leqslant 1$ and define $z_j = \varepsilon\zeta_j$ $(j = 1, \ldots, N)$. Then $z_1, \ldots, z_N \in \Omega$ and by assumption

$$\begin{aligned} 0 \leqslant \sum_{i,j=1}^{N} a_i \bar{a}_j K(z_i, z_j) &= \sum_\alpha c_\alpha \Big|\sum_{j=1}^{N} a_i z_i^\alpha\Big|^2 \\ &= \sum_\alpha c_\alpha \varepsilon^{2|\alpha|} \Big|\sum_{j=1}^{N} a_j f_\alpha(\zeta_j)\Big|^2 \\ &= c_\beta \varepsilon^{2m} + \sum_{|\alpha| \geqslant m+1} c_\alpha \varepsilon^{2|\alpha|} \Big|\sum_{j=1}^{N} a_j f_\alpha(\zeta_j)\Big|^2. \end{aligned}$$

Upon dividing by ε^{2m} and then letting $\varepsilon \to 0$, we obtain that $c_\beta \geqslant 0$ and (ii) follows.

We now prove (ii)$\Rightarrow$(iii). To this end we observe that condition (ii) implies in particular that $\Lambda_1 \subset \Lambda_2$, and hence $\Gamma_2 \subset \Gamma_1$. Let $f \in \mathscr{H}^p_{\varphi_2}$ with $a_\alpha = \{\partial^\alpha f(0)\}/\alpha!$, $\alpha \in Z^n_+$. Then $f \in H(\Omega(p')\colon \Lambda_2)$ and $a_\alpha = 0$ for $\alpha \in \Lambda_2$. It follows that $f \in H(\Omega(p')\colon \Lambda_1)$. Moreover, for $1 \leqslant p < \infty$,

$$\begin{aligned} \|f\|^p_{p,\varphi_1} &= \sum_{\alpha\in\Gamma_1} [c_\alpha(1)]^{1-p}|a_\alpha|^p = \sum_{\alpha\in\Gamma_2} [c_\alpha(1)]^{1-p}|a_\alpha|^p \\ &= \sum_{\alpha\in\Gamma_2} \left[\frac{c_\alpha(2)}{c_\alpha(1)}\right]^{p-1} [c_\alpha(2)]^{1-p}|a_\alpha|^p \leqslant a^{p-1}\|f\|^p_{p,\varphi_2}. \end{aligned}$$

Similarly, for $p = \infty$, $\|f\|_{\infty,\varphi_1} \leqslant a\|f\|_{\infty,\varphi_2}$, and thus (iii) follows.

The implication (iii)$\Rightarrow$(iv) is trivial. To prove (iv)$\Rightarrow$(i), we observe that for any $\alpha \in \Gamma_2$, the monomial $f_\alpha(z) = z^\alpha$ is in $\mathscr{H}^p_{\varphi_2}$. It follows from (iv) that $f_\alpha \in \mathscr{H}^p_{\varphi_1}$ and that $c_\alpha(2) \leqslant ac_\alpha(1)$ for every $\alpha \in \Gamma_2$. Since for $\alpha \in Z^n_+ \setminus \Gamma_2 = \Lambda_2$, $c_\alpha(2) = 0$ we deduce that $c_\alpha \equiv ac_\alpha(1) - c_\alpha(2) \geqslant 0$ for every $\alpha \in Z^n_+$. It follows that

$$K(z, \zeta) = \sum_\alpha c_\alpha z^\alpha \bar{\zeta}^\alpha \qquad (z, \zeta \in \Omega)$$

with $c_\alpha \geqslant 0$ for every $\alpha \in Z^n_+$. Hence by Theorem 2.3 (iv) (with $p = 2$ and $k_\varphi = K$) we have that K is positive-definite on Ω, and (i) follows. This concludes the proof.

THEOREM 2.7. *Let* $\varphi_j \in \mathscr{P}(\Omega)$ *with* $c_\alpha(j) = c_\alpha(\varphi_j)$, $\alpha \in \mathbf{Z}_+^n$, $\Lambda_j = \Lambda_{\varphi_j}$ *and* $\Gamma_j = \Gamma_{\varphi_j}$, $j = 1, 2$. *Then* $\psi = \varphi_1 + \varphi_2$ *is in* $\mathscr{P}(\Omega)$ *with* $\Lambda_\varphi = \Lambda_1 \cap \Lambda_2$ *and* $\Gamma_\varphi = \Gamma_1 \cup \Gamma_2$. *Moreover, for* $1 \leqslant p \leqslant \infty$ *let* $f \in \mathscr{H}^p_{\varphi_1}$ *and* $g \in \mathscr{H}^p_{\varphi_2}$. *Then* $f+g \in \mathscr{H}^p_\psi$ *with*

$$\|f+g\|_{p,\psi} \leqslant (\|f\|^p_{p,\varphi_1} + \|g\|^p_{p,\varphi_2})^{1/p} \quad (1 \leqslant p < \infty)$$

and

$$\|f+g\|_{\infty,\psi} \leqslant \max(\|f\|_{\infty,\varphi_1}, \|g\|_{\infty,\varphi_2}) \quad (p = \infty).$$

Equality holds if there exists a sequence $\{\varkappa_\alpha\}$ *such that* $\partial^\alpha f(0) = \alpha!\varkappa_\alpha c_\alpha(1)$ *and* $\partial^\alpha g(0) = \alpha!\varkappa_\alpha c_\alpha(2)$ *for every* $\alpha \in \mathbf{Z}_+^n$, *and such that* $\sup_\alpha |\varkappa_\alpha| < \infty$ *when* $p = \infty$ *and* $\sum_\alpha c_\alpha(j)|\varkappa_\alpha|^p < \infty$ $(j = 1, 2)$ *when* $1 \leqslant p < \infty$. *This condition is also necessary when* $1 \leqslant p < \infty$ *but not when* $p = \infty$.

Proof. The first assertion is completely trivial, so we turn to the second assertion. We let $a_\alpha = \partial^\alpha f(0)/\alpha!$ and $b_\alpha = \partial^\alpha g(0)/\alpha!$, $\alpha \in \mathbf{Z}_+^n$, and observe that $a_\alpha = 0$ for $\alpha \in \Lambda_1$ and $b_\alpha = 0$ for $\alpha \in \Lambda_2$. The last observation allows one to let α range over all $\mathbf{Z}_+^n$ when computing the norms of f and g as weighted l^p-norms. In fact these norms are unchanged if we employ a (temporary) convention of letting $c_\alpha(j) \equiv 1$ for $\alpha \in \Lambda_j$ $(j = 1, 2)$. We shall use this convention freely in order to avoid lengthy discussions. Now, to prove the above inequalities we assume first that $p = \infty$. In this case we have, for any $\alpha \in \mathbf{Z}_+^n$

$$\max(\|f\|_{\infty,\varphi_1}, \|g\|_{\infty,\varphi_2}) \geqslant \max\left(\frac{|a_\alpha|}{c_\alpha(1)}, \frac{|b_\alpha|}{c_\alpha(2)}\right) \geqslant \frac{|a_\alpha|+|b_\alpha|}{c_\alpha(1)+c_\alpha(2)} \geqslant \frac{|a_\alpha+b_\alpha|}{c_\alpha(1)+c_\alpha(2)},$$

and the inequality for $p = \infty$ follows. Equality holds if $a_\alpha = \varkappa_\alpha c_\alpha(1)$ and $b_\alpha = \varkappa_\alpha c_\alpha(2)$ for every $\alpha \in \mathbf{Z}_+^n$ for some sequence $\{\varkappa_\alpha\}$ with $\sup_\alpha |\varkappa_\alpha| < \infty$. That this condition is not necessary can be easily seen by letting $c_\alpha(j) = 1$ $(j = 1, 2)$ for every α in $\mathbf{Z}_+^n$, $a_\alpha = b_\alpha = 1$ for every α in $\mathbf{Z}_+^n \setminus \{0\}$, $a_0 = 1$ and $b_0 = 1/2$.

We now assume that $1 \leqslant p < \infty$. By the convexity of the function $x \mapsto x^p$, $x > 0$, we deduce that for any $\alpha \in \mathbf{Z}_+^n$

$$\begin{aligned}
[c_\alpha(1)+c_\alpha(2)]\left|\frac{a_\alpha+b_\alpha}{c_\alpha(1)+c_\alpha(2)}\right|^p &\leqslant [c_\alpha(1)+c_\alpha(2)]\left[\frac{|a_\alpha|+|b_\alpha|}{c_\alpha(1)+c_\alpha(2)}\right]^p \\
&= [c_\alpha(1)+c_\alpha(2)]\left[\frac{c_\alpha(1)}{c_\alpha(1)+c_\alpha(2)}\left(\frac{|a_\alpha|}{c_\alpha(1)}\right)+\frac{c_\alpha(2)}{c_\alpha(1)+c_\alpha(2)}\left(\frac{|b_\alpha|}{c_\alpha(2)}\right)\right]^p \\
&\leqslant [c_\alpha(1)+c_\alpha(2)]\left[\frac{c_\alpha(1)}{c_\alpha(1)+c_\alpha(2)}\left(\frac{|a_\alpha|}{c_\alpha(1)}\right)^p+\frac{c_\alpha(2)}{c_\alpha(1)+c_\alpha(2)}\left(\frac{|b_\alpha|}{c_\alpha(2)}\right)^p\right] \\
&= [c_\alpha(1)]^{1-p}|a_\alpha|^p + [c_\alpha(2)]^{1-p}|b_\alpha|^p.
\end{aligned}$$

This gives the desired inequality for $1 \leqslant p < \infty$. The above argument also shows that equality holds if and only if $a_\alpha \bar{b}_\alpha \geqslant 0$ and $|a_\alpha|/c_\alpha(1) = |b_\alpha|/c_\alpha(2)$ for every $\alpha \in Z^n_+$. This is equivalent to the existence of a sequence $\{\varkappa_\alpha\}$ of complex numbers such that $a_\alpha = \varkappa_\alpha c_\alpha(1)$ and $b_\alpha = \varkappa_\alpha c_\alpha(2)$ for every $\alpha \in Z^n_+$. Moreover, the fact that $\|f\|_{p,\varphi_1} < \infty$ and $\|g\|_{p,\varphi_2} < \infty$ is now equivalent to $\sum_\alpha c_\alpha(j)|\varkappa_\alpha|^p < \infty$ $(j = 1, 2)$, and the proof is complete.

THEOREM 2.8. *Let φ and ψ be in $\mathscr{P}(\Omega)$. Then $\varphi\psi \in \mathscr{P}(\Omega)$ with $\Gamma_{\varphi\psi} = \{\gamma \in Z^n_+ : \gamma = \alpha+\beta,\ \alpha \in \Gamma_\varphi,\ \beta \in \Gamma_\psi\}$. Moreover, for $1 \leqslant p \leqslant \infty$ let $f \in \mathscr{H}^p_\varphi$ and $g \in \mathscr{H}^p_\psi$. Then $fg \in \mathscr{H}^p_{\varphi\psi}$ with*

$$\|fg\|_{p,\varphi\psi} \leqslant \|f\|_{p,\varphi}\|g\|_{p,\psi}.$$

Equality holds if either $fg = 0$ or f and g are of the form $f = C_1 k_\varphi(\cdot, \zeta)$, $g = C_2 k_\psi(\cdot, \zeta)$ for some nonzero constants $C_1, C_2 \in C$ and for some $\zeta \in C^n$ such that $\zeta \in \bar{\Delta}^n$ when $p = \infty$ and $\varphi(|\zeta|^p) < \infty$, $\psi(|\zeta|^p) < \infty$ when $1 \leqslant p < \infty$. This condition is also necessary when $1 \leqslant p < \infty$ but not when $p = \infty$.

Proof. As in the previous theorem, the first assertion in this theorem is trivial and hence we turn to the second assertion. We denote by c_α and d_α $(\alpha \in Z^n_+)$ the coefficients of φ and ψ, respectively. Similarly, we let $a_\alpha = \{\partial^\alpha f(0)\}/\alpha!$ and $b_\alpha = \{\partial^\alpha g(0)\}/\alpha!$, $\alpha \in Z^n_+$. It follows that $a_\alpha = 0$ for $\alpha \in \Lambda_\varphi$ and $b_\alpha = 0$ for $\alpha \in \Lambda_\psi$, and, again, to avoid lengthy discussions in the proof we assume that c_α and d_α are positive for every $\alpha \in Z^n_+$. The proof given below will clearly show that only minor and trivial modifications are required to render it applicable to the fuller assertions of the theorem.

Under the above circumstances the coefficients M_α of $\varphi\psi$ are positive for every $\alpha \in Z^n_+$, and they are given by

$$M_\alpha = \sum_{\beta \leqslant \alpha} c_\beta d_{\alpha-\beta} \quad (\alpha \in Z^n_+).$$

Similarly, the coefficients A_α of fg are given by

$$A_\alpha = \sum_{\beta < \alpha} a_\beta b_{\alpha-\beta} \quad (\alpha \in Z^n_+).$$

We consider first the case $p = \infty$. In this case, for any $\alpha \in Z^n_+$

$$|A_\alpha| = \left|\sum_{\beta \leqslant \alpha} \frac{a_\beta}{c_\beta}\cdot\frac{b_{\alpha-\beta}}{d_{\alpha-\beta}}\cdot c_\beta d_{\alpha-\beta}\right| \leqslant \|f\|_{\infty,\varphi}\|g\|_{\infty,\psi} M_\alpha.$$

Thus

$$\|fg\|_{\infty,\varphi\psi} = \sup_\alpha M_\alpha^{-1}|A_\alpha| \leqslant \|f\|_{\infty,\varphi}\|g\|_{\infty,\psi},$$

and the inequality for $p = \infty$ follows. If $f = C_1 k_\varphi(\cdot, \zeta)$ and $g = C_2 k_\psi(\cdot, \zeta)$ for some $\zeta \in \bar{\Delta}^n$, then

$$\|f\|_{\infty,\varphi} = |C_1| \sup_{\alpha \in \Gamma_\varphi} |\zeta|^\alpha, \qquad \|g\|_{\infty,\psi} = |C_2| \sup_{\beta \in \Gamma_\psi} |\zeta|^\beta$$

$$\text{and} \qquad \|fg\|_{\infty,\varphi\psi} = |C_1 C_2| \sup_{\gamma \in \Gamma_{\varphi\psi}} |\zeta|^\gamma.$$

Since $\Gamma_{\varphi\psi} = \{\gamma \in \boldsymbol{Z}^n_+ : \gamma = \alpha + \beta,\ \alpha \in \Gamma_\varphi,\ \beta \in \Gamma_\psi\}$, the sufficient condition for equality for $p = \infty$ follows. To show that this condition is not necessary, we let $c_\alpha = d_\alpha = a_\alpha = 1$ for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{Z}^n_+$, $b_\alpha = 1$ for every $\alpha \in \boldsymbol{Z}^n_+ \setminus \{0\}$ and $b_0 = 2$. Then $M_\alpha = (\alpha_1 + 1) \ldots (\alpha_n + 1)$ and $A_\alpha = 1 + (\alpha_1 + 1) \ldots \ldots (\alpha_n + 1)$ for $\alpha \in \boldsymbol{Z}^n_+$, and thus $\|f\|_{\infty,\varphi} = 1$, $\|g\|_{\infty,\psi} = 2$ and $\|fg\|_{\infty,\varphi\psi} = 2$ with $\Omega = \Delta^n$. It follows that $\varphi(z) = \psi(z) = f(z) = [(1 - z_1) \ldots (1 - z_n)]^{-1}$ and $g(z) = \psi(z) + 1$ for $z = (z_1, \ldots, z_n) \in \Delta^n$, and $\|fg\|_{\infty,\varphi\psi} = \|f\|_{\infty,\varphi} \|g\|_{\infty,\psi}$. It is then clear that g is not of the form $C_2 k_\psi(\cdot, \zeta)$ for any $\zeta \in \boldsymbol{C}^n$.

We now assume that $1 \leqslant p < \infty$. For $r = (r_1, \ldots, r_n) \in [0, 1]^n$, we introduce

$$C(r) = \sum_\alpha c_\alpha^{1-p} |a_\alpha|^p r^\alpha, \qquad D(r) = \sum_\alpha d_\alpha^{1-p} |b_\alpha|^p r^\alpha, \qquad M(r) = \sum_\alpha M_\alpha^{1-p} |A_\alpha|^p r^\alpha.$$

By Hölder's inequality

$$|A_\alpha|^p = \Big|\sum_{\beta \leqslant \alpha} a_\beta b_{\alpha-\beta}\Big|^p = \Big|\sum_{\beta \leqslant \alpha} a_\beta b_{\alpha-\beta} (c_\beta d_{\alpha-\beta})^{-1/p'} (c_\beta d_{\alpha-\beta})^{1/p'}\Big|^p$$

$$\leqslant \Big(\sum_{\beta \leqslant \alpha} (|a_\beta|\,|b_{\alpha-\beta}|)^p (c_\beta d_{\alpha-\beta})^{1-p}\Big)\Big(\sum_{\beta \leqslant \alpha} c_\beta d_{\alpha-\beta}\Big)^{p/p'},$$

and so

$$M_\alpha^{1-p} |A_\alpha|^p \leqslant \sum_{\beta \leqslant \alpha} c_\beta^{1-p} |a_\beta|^p d_{\alpha-\beta}^{1-p} |b_{\alpha-\beta}|^p. \tag{2.5}$$

This shows that

$$M(r) \leqslant C(r) D(r), \qquad r \in [0, 1]^n,$$

as formal power series. Letting $r = \boldsymbol{1}$ in this inequality we obtain the desired inequality for $1 \leqslant p < \infty$. Moreover, by defining

$$G(r) = C(r) D(r) - M(r), \qquad r \in [0, 1]^n,$$

we find that G is nondecreasing on $[0, 1]^n$ with $G(0) = 0$. In particular, $G(r) \geqslant 0$ for $r \in [0, 1]^n$ with $G(\boldsymbol{1}) \geqslant 0$ as the above mentioned inequality. The equality $G(\boldsymbol{1}) = 0$, therefore, holds if and only if $G(r) = 0$ for every r in $[0, 1]^n$ which means that equality in (2.5) holds for every $\alpha \in \boldsymbol{Z}^n_+$. This is equivalent to an existence of $\lambda_\alpha \in \boldsymbol{C}$ for every $\alpha \in \boldsymbol{Z}^n_+$ such that

$$a_\beta b_{\alpha-\beta} = \lambda_\alpha c_\beta d_{\alpha-\beta} \qquad (\beta \leqslant \alpha,\ \alpha \in \boldsymbol{Z}^n_+). \tag{2.6}$$

Putting $\beta = 0$ and $\beta = \alpha$ in (2.6) results in

$$a_0 b_\alpha = \lambda_\alpha c_0 d_\alpha, \quad a_\alpha b_0 = \lambda_\alpha c_\alpha d_0 \quad (\alpha \in \mathbf{Z}_+^n). \tag{2.7}$$

On the other hand, summing up (2.6) from $\beta = 0$ through $\beta = \alpha$ gives

$$A_\alpha = \lambda_\alpha M_\alpha \quad (\alpha \in \mathbf{Z}_+^n). \tag{2.8}$$

If $a_0 b_0 = 0$, then by (2.7) $\lambda_\alpha = 0$ for every $\alpha \in \mathbf{Z}_+^n$. It follows from (2.8) that $A_\alpha = 0$ for all $\alpha \in \mathbf{Z}_+^n$ which means that $a_\alpha = 0$ or $b_\alpha = 0$ for every $\alpha \in \mathbf{Z}_+^n$. This covers the first part of the necessary and sufficient condition for equality when $1 \leqslant p < \infty$. We now assume that $a_0 b_0 \neq 0$ and define

$$C_1 = a_0 c_0^{-1}, \quad C_2 = b_0 d_0^{-1} \tag{2.9}$$

and

$$\zeta_k = c_0 \bar{a}_{e_k} (c_{e_k} \bar{a}_0)^{-1}, \quad 1 \leqslant k \leqslant n. \tag{2.10}$$

It follows from (2.7) that also

$$\zeta_k = d_0 \bar{b}_{e_k} (d_{e_k} \bar{b}_0)^{-1}, \quad 1 \leqslant k \leqslant n. \tag{2.11}$$

Clearly C_1, $C_2 \neq 0$. From (2.7), (2.8) and (2.9) we have

$$b_\alpha = C_2 C_1^{-1} (a_\alpha / c_\alpha) d_\alpha \quad (\alpha \in \mathbf{Z}_+^n) \tag{2.12}$$

and

(2.13)
$$a_\alpha b_0 (c_\alpha d_0 + c_0 d_\alpha) + c_\alpha d_0 \sum_{0 < \beta < \alpha} a_\beta b_{\alpha - \beta} = a_\alpha b_0 \sum_{\beta \leqslant \alpha} c_\beta d_{\alpha - \beta} \quad (\alpha \in \mathbf{Z}_+^n \setminus \{0\}).$$

We use induction on the weight $|\alpha|$ to show that with $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$,

$$a_\alpha = C_1 c_\alpha \bar{\zeta}^\alpha, \quad b_\alpha = C_2 d_\alpha \bar{\zeta}^\alpha \quad (\alpha \in \mathbf{Z}_+^n). \tag{2.14}$$

That this is true for $|\alpha| = 0$ and $|\alpha| = 1$ is immediate from (2.9)–(2.11). Assuming (2.14) is true for α with $0 \leqslant |\alpha| \leqslant m - 1$, $m \geqslant 2$, we find by (2.13) that for $|\alpha| = m$,

$$a_\alpha b_0 (c_\alpha d_0 + c_0 d_\alpha) + c_\alpha d_0 C_1 C_2 \bar{\zeta}^\alpha \sum_{0 < \beta < \alpha} c_\beta d_{\alpha - \beta} = a_\alpha b_0 \sum_{\beta \leqslant \alpha} c_\beta d_{\alpha - \beta},$$

and so, by (2.9),

$$a_\alpha b_0 \sum_{0 < \beta < \alpha} c_\beta d_{\alpha - \beta} = C_1 c_\alpha \bar{\zeta}^\alpha b_0 \sum_{0 < \beta < \alpha} c_\beta d_{\alpha - \beta} \quad (|\alpha| = m \geqslant 2).$$

This shows that $a_\alpha = C_1 c_\alpha \bar{\zeta}^\alpha$, and, by (2.12), also $b_\alpha = C_2 d_\alpha \bar{\zeta}^\alpha$, and (2.14) is proved. Finally, in this case we have $\|f\|_{p,\varphi} = |C_1| \{\varphi(|\zeta|^p)\}^{1/p}$ and $\|g\|_{p,\psi} = |C_2| \{\psi(|\zeta|^p)\}^{1/p}$. It follows that $\varphi(|\zeta|^p) < \infty$ and $\psi(|\zeta|^p) < \infty$. The proof is now complete.

COROLLARY 2.9. *Let $\varphi \in \mathscr{P}(\Omega)$ and $m \in \boldsymbol{Z}_+$ with $m \geqslant 2$. Then $\varphi^m \in \mathscr{P}(\Omega)$ with $\Gamma_{\varphi^m} = \{\gamma \in \boldsymbol{Z}_+^n:\ \gamma = \alpha_1 + \ldots + \alpha_m,\ \alpha_j \in \Gamma_\varphi\ (1 \leqslant j \leqslant m)\}$. Moreover, if f is in $\mathscr{H}_\varphi^p$ for $1 \leqslant p \leqslant \infty$, then $f^m \in \mathscr{H}_{\varphi^m}^p$ with*

$$\|f^m\|_{p,\varphi^m} \leqslant \|f\|_{p,\varphi}^m.$$

Equality for $1 \leqslant p < \infty$ holds if and only if either $f = 0$ or f is of the form $f = Ck_\varphi(\cdot, \zeta)$ for some nonzero constant $C \in \boldsymbol{C}$ and for some point ζ in $\boldsymbol{C}^n$ with $\varphi(|\zeta|^p) < \infty$. For $p = \infty$, equality holds if either $f = 0$ or f is of the form $f = Ck_\varphi(\cdot, \zeta)$ for some nonzero constant C and some $\zeta \in \bar{\Delta}^n$, but the converse is, in general, not true.

Proof. The only thing, if any, that is now required to be shown is that the sufficient condition for equality when $p = \infty$ is not necessary. To this end, we take $n = 1$, $\Omega = \Delta$ and $\varphi(\lambda) = (1-\lambda)^{-q}$, $\lambda \in \Delta$, where q is a fixed number with $q > 1$. Let $f(\lambda) = (1-\lambda)^{-1}$, $\lambda \in \Delta$. Then $\|f^2\|_{\infty,\varphi^2} = \|f\|_{\infty,\varphi}^2 = 1$, but f is not of the form $f = Ck_\varphi(\cdot, \zeta)$ for any $\zeta \in \boldsymbol{C}$.

A function $\varphi \in \mathscr{P}(\Omega)$ is said to *belong to* $\mathscr{P}_\infty(\Omega)$ if $\varphi(|\zeta|^2) = \infty$ for every ζ in $\partial\Omega$. This means that Ω is the domain of convergence of the power series $\varphi(|z|^2)$ $(z \in \Omega)$. Since Ω is a complete Reinhardt domain, we deduce that any $\varphi \in \mathscr{P}_\infty(\Omega)$, as a power series, satisfies $\varphi(|z|^2) < \infty$ if and only if $z \in \Omega$. An immediate consequence of this observation is the following proposition:

PROPOSITION 2.10. *Let $\varphi \in \mathscr{P}_\infty(\Omega)$ and $0 \leqslant p < \infty$. Then $\Omega(p) = \{z \in \boldsymbol{C}^n: \varphi(|z|^p) < \infty\}$.*

3. Decompositions and operators. Let φ be a fixed function in $\mathscr{P}(\Omega)$ with $\Gamma_\varphi = \boldsymbol{Z}_+^n$, i.e., $c_\alpha = c_\alpha(\varphi)$ is positive for every $\alpha \in \boldsymbol{Z}_+^n$. For any subset Γ of $\boldsymbol{Z}_+^n$ we define φ_Γ by

$$\varphi_\Gamma(z) = \sum_{\alpha \in \Gamma} c_\alpha z^\alpha \quad (z \in \Omega(1)).$$

Then $\varphi_\Gamma \in \mathscr{P}(\Omega)$ with $\Gamma_{\varphi_\Gamma} = \Gamma$, and for any $1 \leqslant p \leqslant \infty$, $\mathscr{H}_{\varphi_\Gamma}^p$ is a closed subspace of $\mathscr{H}_\varphi^p$ with

$$\mathscr{H}_{\varphi_\Gamma}^p = \{f \in \mathscr{H}_\varphi^p:\ \partial^\alpha f(0) = 0,\ \alpha \in \Lambda\},$$

where

$$\Lambda = \boldsymbol{Z}_+^n \setminus \{0\}.$$

We then have the obvious decompositions $\varphi = \varphi_\Gamma + \varphi_\Lambda$ and $k_\varphi = k_{\varphi_\Gamma} + k_{\varphi_\Lambda}$, where k_{φ_Γ} and k_{φ_Λ} are the reproducing kernels of $\mathscr{H}_{\varphi_\Gamma}^2$ and $\mathscr{H}_{\varphi_\Lambda}^2$, respectively. In this way we may express φ as the direct sum

$$\varphi = \sum_\alpha \varphi_\alpha$$

where $\varphi_\alpha(z) = c_\alpha z^\alpha$, $\alpha \in \boldsymbol{Z}^n_+$. It follows that $k_{\varphi_\alpha}(z, \zeta) = \varphi_\alpha(z \cdot \bar{\zeta}) = c_\alpha z^\alpha \bar{\zeta}^\alpha$ gives the reproducing kernel for the one-dimensional span of φ_α in $\mathscr{H}^2_\varphi$. In particular, $\varphi_0 = c_0$ reproduces the constant functions which, obviously, belong to $\mathscr{H}^2_\varphi$.

The proofs of the following simple propositions are straightforward.

Proposition 3.1. *Let $\varphi \in \mathscr{P}(\Omega)$ with $\Gamma_\varphi = \boldsymbol{Z}^n_+$ and $1 \leqslant p \leqslant \infty$. Then:*

(i) *For any subset Γ of $\boldsymbol{Z}^n_+$ with $\Lambda = \boldsymbol{Z}^n_+ \setminus \Gamma$, we have $\mathscr{H}^p_\varphi = \mathscr{H}^p_{\varphi,\Gamma} \oplus \mathscr{H}^p_{\varphi,\Lambda}$ and $\mathscr{H}^p_{\varphi,\Gamma} = (\mathscr{H}^{p'}_{\varphi,\Lambda})^\perp$ with respect to the pairing $\langle\ ,\ \rangle_\varphi$;*

(ii) *$\{c_\alpha^{1/p'} z^\alpha\}$, $\alpha \in \boldsymbol{Z}^n_+$, is an orthonormal sequence in $\mathscr{H}^p_\varphi$ with respect to $\langle\ ,\ \rangle_\varphi$ and is a basis when $1 \leqslant p < \infty$.*

Proposition 3.2. *Let $\varphi \in \mathscr{P}(\Omega)$ with $\Gamma_\varphi = \boldsymbol{Z}^n_+$ and $c_\alpha = c_\alpha(\varphi)$, $\alpha \in \boldsymbol{Z}^n_+$. Let $\beta \in \boldsymbol{Z}^n_+$, $c > 0$ and $1 \leqslant p \leqslant \infty$. For a function $f \in H(\Omega(p'))$, define $\{M_\beta f\}(z) = z^\beta f(z)$. The following conditions are equivalent:*

(i) *M_β is a continuous linear transformation from $\mathscr{H}^p_\varphi$ into $\mathscr{H}^p_\varphi$ with norm $\|M_\beta\| \leqslant c^{1/p'}$;*

(ii) *$c_\alpha / c_{\alpha+\beta} \leqslant c$ for every $\alpha \in \boldsymbol{Z}^n_+$.*

Proposition 3.3. *Let $\varphi \in \mathscr{P}(\Omega)$, $1 \leqslant p < \infty$ and $E \subset \Omega(p)$. Then the following conditions are equivalent:*

(i) *The set $\{k_\varphi(\cdot, \zeta) \colon \zeta \in E\}$ spans $\mathscr{H}^p_\varphi$;*

(ii) *E is a set of uniqueness for $\mathscr{H}^{p'}_\varphi = \mathscr{H}^{p'}_\varphi(\Omega(p))$, i.e., the restriction map $f \mapsto f|_E$ is injective on $\mathscr{H}^{p'}_\varphi$.*

Let φ and ψ be in $\mathscr{P}(\Omega)$ with $c_\alpha = c_\alpha(\varphi)$ and $d_\alpha = c_\alpha(\psi)$, $\alpha \in \boldsymbol{Z}^n_+$, and let $\{A_{\alpha\beta}\}$ be a sequence (matrix) of complex numbers with $(\alpha, \beta) \in \Gamma_\varphi \times \Gamma_\psi$. Let P be a sufficiently small polydisk about the origin in $\boldsymbol{C}^n$, and let $f \in H(P \colon \Lambda_\psi)$, $g \in H(P \colon \Lambda_\varphi)$ with $a_\beta = \{\partial^\beta f(0)\}/\beta!$, $b_\alpha = \{\partial^\alpha g(0)\}/\alpha!$, α, $\beta \in \boldsymbol{Z}^n_+$. We then define formally

$$\{Af\}(z) = \sum_{\alpha \in \Gamma_\varphi} \Big(\sum_{\beta \in \Gamma_\psi} d_\beta^{-1} A_{\alpha\beta} a_\beta \Big) z^\alpha \tag{3.1}$$

and

$$\{A^* g\}(z) = \sum_{\beta \in \Gamma_\psi} \Big(\sum_{\alpha \in \Gamma_\varphi} c_\alpha^{-1} \bar{A}_{\alpha\beta} b_\alpha \Big) z^\alpha. \tag{3.2}$$

With this notation we shall prove the following theorem. Its proof is based on a slight variation of a technique which is due essentially to Schur [12].

Theorem 3.4. *Let φ and ψ be in $\mathscr{P}(\Omega)$ with $c_\alpha = c_\alpha(\varphi)$ and $d_\alpha = c_\alpha(\psi)$, $\alpha \in \boldsymbol{Z}^n_+$, and let $\{A_{\alpha\beta}\}$ be a sequence (matrix) of complex numbers with $(\alpha, \beta) \in \Gamma_\varphi \times \Gamma_\psi$. Let $1 \leqslant p \leqslant \infty$ be fixed and assume that there exist constants c, $d \geqslant 0$ such that*

$$\sum_{\alpha \in \Gamma_\varphi} |A_{\alpha\beta}| \leqslant c d_\beta \qquad (\beta \in \Gamma_\psi,\ p = 1),$$

when $p = 1$, *and*

$$\begin{aligned} &\sum_{\alpha\in\Gamma_\varphi} |A_{\alpha\beta}|\, c_\alpha \leqslant c d_\beta^p \qquad (\beta\in\Gamma_\psi) \\ &\sum_{\beta\in\Gamma_\psi} |A_{\alpha\beta}|\, d_\beta \leqslant d c_\alpha^{p'} \qquad (\alpha\in\Gamma_\varphi) \end{aligned} \qquad (1 < p < \infty)$$

when $1 < p < \infty$, *and*

$$\sum_{\beta\in\Gamma_\psi} |A_{\alpha\beta}| \leqslant d c_\alpha \qquad (\alpha\in\Gamma_\varphi,\ p=\infty)$$

when $p = \infty$. *Then the operator* A *defined in* (3.1) *is a continuous linear transformation of* $\mathscr{H}_\psi^p$ *into* $\mathscr{H}_\varphi^p$ *with norm* $\|A\| \leqslant c^{1/p} d^{1/p'}$. *Moreover, the operator* A^* *defined in* (3.2) *is the adjoint of* A *and is a continuous linear transformation of* $\mathscr{H}_\varphi^{p'}$ *into* $\mathscr{H}_\psi^{p'}$ *with* $\|A^*\| = \|A\|$.

Proof. Let $f\in\mathscr{H}_\psi^p$ with $a_\beta = \{\partial^\beta f(0)\}/\beta!$, $\beta\in\boldsymbol{Z}_+^n$. Thus $a_\beta = 0$ for $\beta\in\Lambda_\psi$. For $p = 1$, we have

$$\begin{aligned} \|Af\|_{1,\varphi} &\leqslant \sum_{\alpha\in\Gamma_\varphi}\sum_{\beta\in\Gamma_\psi} d_\beta^{-1} |A_{\alpha\beta}|\,|a_\beta| = \sum_{\beta\in\Gamma_\psi} d_\beta^{-1}|a_\beta| \sum_{\alpha\in\Gamma_\varphi} |A_{\alpha\beta}| \\ &\leqslant c \sum_{\beta\in\Gamma_\psi} |a_\beta| = c\,\|f\|_{1,\psi}. \end{aligned}$$

Similarly, for $p = \infty$,

$$\|Af\|_{\infty,\varphi} \leqslant \sup_{\alpha\in\Gamma_\varphi} \{c_\alpha^{-1} \sum_{\beta\in\Gamma_\psi} d_\beta^{-1}|A_{\alpha\beta}|\,|a_\beta|\} \leqslant \|f\|_{\infty,\psi} \sup_{\alpha\in\Gamma_\varphi} \{c_\alpha^{-1}\sum_{\beta\in\Gamma_\psi} |A_{\alpha\beta}|\},$$

and thus $\|Af\|_{p,\varphi} \leqslant c^{1/p} d^{1/p'} \|f\|_{p,\psi}$ for $p = 1, \infty$. We now assume that $1 < p < \infty$. By Hölder's inequality and Fubini's theorem,

$$\begin{aligned} \|Af\|_{p,\varphi}^p &\leqslant \sum_{\alpha\in\Gamma_\varphi} c_\alpha^{1-p} \Big(\sum_{\beta\in\Gamma_\psi} d_\beta^{-1}|A_{\alpha\beta}|\,|a_\beta|\Big)^p \\ &\leqslant \sum_{\alpha\in\Gamma_\varphi} c_\alpha^{1-p}\Big(\sum_{\beta\in\Gamma_\psi} |A_{\alpha\beta}|\, d_\beta\Big)^{p/p'} \Big(\sum_{\beta\in\Gamma_\psi} d_\beta^{1-2p}|A_{\alpha\beta}|\,|a_\beta|^p\Big) \\ &\leqslant d^{p/p'} \sum_{\alpha\in\Gamma_\varphi} c_\alpha \sum_{\beta\in\Gamma_\psi} d_\beta^{1-2p}|A_{\alpha\beta}|\,|a_\beta|^p \\ &= d^{p/p'} \sum_{\beta\in\Gamma_\psi} |a_\beta|^p d_\beta^{1-2p} \sum_{\alpha\in\Gamma_\varphi} |A_{\alpha\beta}|\, c_\alpha \\ &\leqslant c d^{p/p'} \sum_{\beta\in\Gamma_\psi} d_\beta^{1-p}|a_\beta|^p = c d^{p/p'} \|f\|_{p,\psi}^p. \end{aligned}$$

It follows that $\|Af\|_{p,\varphi} \leqslant c^{1/p} d^{1/p'} \|f\|_{p,\psi}$ for every $f\in\mathscr{H}_\psi^p$, $1\leqslant p\leqslant\infty$, and hence, by Theorem 2.1, A is a continuous linear operator from $\mathscr{H}_\psi^p$ into $\mathscr{H}_\varphi^p$ with norm $\|A\| \leqslant c^{1/p} d^{1/p'}$. It also follows that A^* is indeed the adjoint of A and is a continuous linear operator from $\mathscr{H}_\varphi^{p'}$ into $\mathscr{H}_\psi^{p'}$ with norm $\|A^*\| = \|A\| \leqslant c^{1/p} d^{1/p'}$. This concludes the proof.

4. Composition functions. For a function $\varphi \in \mathscr{P}(\Omega)$ we let $c_\alpha(j) = c_\alpha(\varphi^j)$, $j \in \mathbf{Z}_+$, $\alpha \in \mathbf{Z}_+^n$. Thus $c_\alpha(0) = \delta_{\alpha 0}$ and $c_\alpha(1) = c_\alpha$. Moreover, $\Gamma_{\varphi^0} = \Gamma_1 = \{0\}$ and

$$\Gamma_{\varphi^j} = \{\gamma \in \mathbf{Z}_+^n:\ \gamma = \alpha_1 + \ldots + \alpha_j,\ \alpha_k \in \Gamma_\varphi\ (1 \leqslant k \leqslant j)\}.$$

For a subset J of $\mathbf{Z}_+$, whose elements m_k are ordered as $0 \leqslant m_1 < m_2 < \ldots$, we let $\operatorname{card}(J)$ stand for the number of its elements.

We now fix a function F of one complex variable λ, given by

$$F(\lambda) = \sum_{j \in J} a_j \lambda^j \qquad (\lambda \in \Delta_R),$$

where $a_j > 0$ for every $j \in J \subset \mathbf{Z}_+$ and $\Delta_R = \{\lambda \in \mathbf{C}:\ |\lambda| < R\}$, $R > 0$, is the disk of convergence of the above power series. We also assume that the elements m_k of J are ordered as $0 \leqslant m_1 < m_2 < \ldots$, and note that J may be finite, in which case F is a polynomial and $R = \infty$. Under these circumstances, the next proposition is an immediate consequence of Theorem 2.7 and Corollary 2.9.

Proposition 4.1. *Let $\varphi \in \mathscr{P}(\Omega)$ with $\varphi(|z|^2) < R$ for every $z \in \Omega$. Then $F \circ \varphi \in \mathscr{P}(\Omega)$ with*

$$\Gamma_{F\circ\varphi} = \bigcup_{j \in J} \Gamma_{\varphi^j}.$$

We also prove:

Theorem 4.2. *Let $\varphi \in \mathscr{P}(\Omega)$ with $\varphi(|z|^2) < R$, and let $f \in \mathscr{H}_\varphi^\infty$ with $\sup_{j \in J} \|f\|_{\infty,\varphi}^j < \infty$. Then $F \circ f \in \mathscr{H}_{F\circ\varphi}^\infty$ with*

$$\|F \circ f\|_{\infty, F\circ\varphi} \leqslant \sup_{j \in J} \|f\|_{\infty,\varphi}^j.$$

Equality holds if either $m_1 > 0$ and $f = 0$ or one of the following five cases is valid:

(i) *When $J = \{0\}$ and $f \neq 0$ is arbitrary;*

(ii) *When $J = \{1\}$ and $f \in \mathscr{H}_\varphi^\infty$ is arbitrary;*

(iii) *When $J = \{0, 1\}$ and f is of the form $f(z) = \sum_\alpha \varkappa_\alpha c_\alpha z^\alpha$, $z \in \Omega(1)$, where $\varkappa_\alpha$, $\alpha \in \mathbf{Z}_+^n$, are complex numbers with $\varkappa_0 = 1$ and $\sup_{\alpha \in \Gamma_\varphi} |\varkappa_\alpha| < \infty$;*

(iv) *When $J = \{m\}$, $m \geqslant 2$, and f is of the form $f = Ck_\varphi(\cdot, \zeta)$, where $\zeta \in \bar{\Delta}^n$ and C is a nonzero constant;*

(v) *When $\operatorname{card}(J) \geqslant 2$ and J contains an integer $m \geqslant 2$, and f is of the form $f = Ck_\varphi(\cdot, \zeta)$, where $\zeta \in \bar{\Delta}^n$ and C is a nonzero constant such that $\sup_{j \in J} |C|^j < \infty$, and there exists a sequence $\{\varkappa_\alpha\}$, $\alpha \in \mathbf{Z}_+^n$, of complex numbers with $\varkappa_0 = 1$ if $0 \in J$ and $\varkappa_\alpha = C^j \bar{\zeta}^\alpha$ for every $\alpha \in \Gamma_{\varphi^j}$, where $j \in J \setminus \{0\}$. In particular, if*

$\zeta = 0$ *then* $0 \in \Gamma_\varphi$ *and* $f \equiv Cc_0$, *where* C *is a root of unity whose order divides* $(j-k)$ *for every* j, $k \in J$.

Proof. Since $\sup_{j \in J} \|f\|^j_{\infty,\varphi} < \infty$, we deduce that if J is infinite then $\|f\|_{\infty,\varphi} \leqslant 1$. If J is finite then it is clear that $F \circ f \in H(\Omega(1)\colon \Lambda_{F\circ\varphi})$, where $\Lambda_{F\circ\varphi} = \boldsymbol{Z}^n_+ \setminus \Gamma_{F\circ\varphi} = \bigcap_{j \in J} \Lambda_{\varphi^j}$. This is also true when J is infinite. Indeed, if $\zeta \in \Omega(1)$ then $\varphi(|\zeta|) < R$ and, by Theorem 2.3, $|f(\zeta)| \leqslant \|f\|_{\infty,\varphi}\, \varphi(|\zeta|) < R\|f\|_{\infty,\varphi} \leqslant R$ which shows that $F \circ f \in H(\Omega(1)\colon \Lambda_{F\circ\varphi})$ when J is infinite.

We now use Theorem 2.7, Proposition 2.5 and Corollary 2.9 to obtain

$$\|F \circ f\|_{\infty,F\circ\varphi} = \Big\|\sum_{j \in J} a_j f^j\Big\|_{\infty,\sum_{j \in J} a_j \varphi^j} \leqslant \sup_{j \in J} \|a_j f^j\|_{\infty,a_j\varphi^j}$$

$$= \sup_{j \in J} a_j \|f\|_{\infty,a_j\varphi^j} = \sup_{j \in J} \|f^j\|_{\infty,\varphi^j} \leqslant \sup_{j \in J} \|f\|_{\infty,\varphi},$$

and the inequality statement follows. As for the quality statement, we assume that $f \neq 0$ and consider the above-mentioned five cases. Cases (i) and (ii) are trivial, case (iii) follows from Theorem 2.7 and case (iv) is a reformulation of the equality statement for $p = \infty$ in Corollary 2.9. We now verify case (v). We first observe that .

$$c_\alpha(F \circ \varphi) = \sum_{j \in J} a_j c_\alpha(j) \qquad (\alpha \in \boldsymbol{Z}^n_+),$$

and that the coefficients $b_\alpha(f)$ of $f = Ck_\varphi(\cdot, \zeta)$ are given by

$$b_\alpha(f) = \sum_{j \in J} a_j c_\alpha(j) C^j \bar{\zeta}^\alpha \qquad (\alpha \in \boldsymbol{Z}^n_+).$$

It follows that

$$\|F \circ f\|_{\infty,F\circ\varphi} = \sup_{\alpha \in \Gamma_{F\circ\varphi}} \left\{\frac{b_\alpha(f)}{c_\alpha(F \circ \varphi)}\right\} = \sup_{\alpha \in \Gamma_{F\circ\varphi}} |\varkappa_\alpha| = \sup_{j \in J} \sup_{\alpha \in \Gamma_{\varphi^j}} |\varkappa_\alpha|$$

$$= \sup_{j \in J} |C|^j \sup_{\alpha \in \Gamma_{\varphi^j}} |\bar{\zeta}^\alpha| = \sup_{j \in J} (\sup_{\alpha \in \Gamma_\varphi} |C\bar{\zeta}^\alpha|)^j = \sup_{j \in J} \|f\|^j_{\infty,\varphi}.$$

Since also $\zeta \in \bar{\Delta}^n$, we find that $\sup_{j \in J} \|f\|^j_{\infty,\varphi} \leqslant \sup_{j \in J} |C|^j < \infty$, and the theorem follows.

THEOREM 4.3. *Let* $\varphi \in \mathscr{P}(\Omega)$ *with* $\varphi(|z|^2) < R$, *and let* $f \in \mathscr{H}^p_\varphi$ *with* $1 \leqslant p < \infty$ *and* $F(\|f\|^p_{p,\varphi}) < \infty$. *Then* $F \circ f \in \mathscr{H}^p_{F\circ\varphi}$ *with*

$$\|F \circ f\|^p_{p,F\circ\varphi} \leqslant F(\|f\|^p_{p,\varphi}).$$

Equality holds if and only if either $m_1 > 0$ *and* $f = 0$ *or one of the following five cases is valid*:

(i) *When* $J = \{0\}$ *and* $f \neq 0$ *is arbitrary*;

(ii) *When* $J = \{1\}$ *and* $f \in \mathscr{H}^p_\varphi$ *is arbitrary*;

(iii) *When* $J = \{0, 1\}$ *and* f *is of the form* $f(z) = \sum_\alpha \varkappa_\alpha c_\alpha z^\alpha$, $z \in \Omega(p')$, *where* $\varkappa_\alpha$, $\alpha \in \mathbf{Z}^n_+$, *are complex numbers with* $\varkappa_0 = 1$ *and* $\sum_\alpha c_\alpha |\varkappa_\alpha|^p < \infty$;

(iv) *When* $J = \{m\}$, $m \geqslant 2$, *and* f *is of the form* $f = Ck_\varphi(\cdot, \zeta)$ *for some nonzero constant* C *and some point* $\zeta \in \mathbf{C}^n$ *with* $\varphi(|\zeta|^p) < \infty$;

(v) *When* $\operatorname{card}(J) \geqslant 2$ *and* J *contains an integer* $m \geqslant 2$, *and* f *is of the form* $f = Ck_\varphi(\cdot, \zeta)$ *where* $\zeta \in \mathbf{C}^n$ *and* C *is a nonzero constant such that* $F(|C|^p \varphi(|\zeta|^p)) < \infty$, *and there exists a sequence* $\{\varkappa_\alpha\}$, $\alpha \in \mathbf{Z}^n_+$, *of complex numbers with* $\varkappa_0 = 1$ *if* $0 \in J$ *and* $\varkappa_\alpha = C^j \bar{\zeta}^\alpha$ *for every* $\alpha \in \Gamma_{\varphi^j}$, *where* $j \in J \setminus \{0\}$. *In particular, if* $\zeta = 0$ *then* $0 \in \Gamma_\varphi$ *and* $f \equiv Cc_0$, *where* C *is a root of unity whose order divides* $(j-k)$ *for every* $j, k \in J$.

Proof. For $z \in \Omega(p')$ we have, by Theorem 2.3, $|f(z)| \leqslant \|f\|_{p,\varphi} R^{1/p'}$ with equality only when f is a constant and $R < \infty$ (when $1 < p < \infty$). It follows, since $F(\|f\|^p_{p,\varphi}) < \infty$, that $F \circ f \in H(\Omega(p'))$ and hence also $F \circ f \in H(\Omega(p')\colon \Lambda_{F\circ\varphi})$. We now, as before, use Theorem 2.7, Proposition 2.5 and Corollary 2.9 to obtain

$$\|F \circ f\|^p_{p,F\circ\varphi} = \Big\|\sum_{j\in J} a_j f^j\Big\|_{p,\sum_{j\in J} a_j\varphi^j} \leqslant \sum_{j\in J} \|a_j f^j\|^p_{p,a_j\varphi^j} = \sum_{j\in J} a_j^p \|f^j\|^p_{p,a_j\varphi^j}$$

$$= \sum_{j\in J} a_j \|f^j\|^p_{p,\varphi^j} \leqslant \sum_{j\in J} a_j \|f\|^{pj}_{p,\varphi} = F(\|f\|^p_{p,\varphi}),$$

and the inequality statement follows. To prove the equality statement we assume that $f \neq 0$ and consider the above mentioned five cases. Once again, cases (i) and (ii) are trivial, case (iii) follows from Theorem 2.7, and case (iv) is a reformulation of the equality statement for $1 \leqslant p < \infty$ in Corollary 2.9. We now establish case (v). From Theorem 2.7 and Corollary 2.9 we deduce that the above inequality becomes an equality if and only if (1) there exists a sequence $\{\varkappa_\alpha\}$ of complex numbers so that, for any $j \in J$, $\partial^\alpha f(0) = \alpha!\, \varkappa_\alpha c_\alpha(j)$ for every $\alpha \in \mathbf{Z}^n_+$, and such that

$$\sum_{j\in J} a_j \Big(\sum_\alpha c_\alpha(j) |\varkappa_\alpha|^p\Big) < \infty,$$

and (2) f is of the form $f = Ck_\varphi(\cdot, \zeta)$, where C is a nonzero constant and $\zeta \in \mathbf{C}^n$ with $F(|C|^p \varphi(|\zeta|^p)) < \infty$. These two conditions are completely equivalent to the main assertion in case (v). This concludes the proof.

As a specialization of the last two theorems we obtain the following corollary:

COROLLARY 4.4. *Let $\varphi \in \mathscr{P}(\Omega)$ with $\Gamma_\varphi = \{\alpha \in \mathbf{Z}_+^n : |\alpha| \geqslant m\}$, $m = 0, 1, \ldots,$ and let F be an entire function on $\mathbf{C}$ with $F^{(j)}(0) > 0$ for $j = 0, 1, \ldots$ Then $F \circ \varphi \in \mathscr{P}(\Omega)$ with $\Gamma_{F\circ\varphi} = \{0\} \cup \Gamma_\varphi$. Moreover, for $f \in \mathscr{H}_\varphi^\infty$ with $\|f\|_{\infty,\varphi} \leqslant 1$ we have $F \circ f \in \mathscr{H}_{F\circ\varphi}^\infty$ with*

$$\|F \circ f\|_{\infty, F\circ\varphi} \leqslant 1.$$

Equality holds if f is of the form $f = k_\varphi(\cdot, \zeta)$ for some $\zeta \in \bar{\Delta}^n$. Similarly, for $f \in \mathscr{H}_\varphi^p$ with $1 \leqslant p < \infty$ we have $F \circ f \in \mathscr{H}_{F\circ\varphi}^p$ with

$$\|F \circ f\|_{p, F\circ\varphi}^p \leqslant F(\|f\|_{p,\varphi}^p).$$

Equality holds if and only if f is of the form $f = k_\varphi(\cdot, \zeta)$ for some $\zeta \in \mathbf{C}^n$ with $\varphi(|\zeta|^p) < \infty$. In particular, if also $\varphi \in \mathscr{P}_s(\Omega)$ then $\zeta \in \Omega(p)$.

Proof. The first part of the corollary follows from Proposition 4.1. The inequality in the second part follows from the inequality statement of Theorem 4.2. Here the convention of $0^0 = 1$ has been employed. The equality statement follows from case (v) of the same theorem. In this case if $f \neq 0$ then $C = 1$, and hence f is of the form $f = k_\varphi(\cdot, \zeta)$ where $\zeta \in \bar{\Delta}^n$. We now observe that the possibility of $f = 0$ can only occur when $m \geqslant 1$, in which case f is also of the above form with $\zeta = 0$.

We now prove the assertions for $1 \leqslant p < \infty$. Again, the inequality statement follows from Theorem 4.3 while the equality statement is equivalent to its counterpart, with case (v), in the same theorem. Finally, if also $\varphi \in \mathscr{P}_s(\Omega)$ then by Proposition 2.10, $\varphi(|\zeta|^p) < \infty$ is equivalent to $\zeta \in \Omega(p)$. This concludes the proof.

A special case of this corollary, namely when $n = 1$, $m = 1$ and $1 \leqslant p < \infty$ was proved first by Milin (see [9], pp. 27–32) by using different methods. This result has been extended in Kwak's dissertation [7] to include the case $n > 1$, with a proof which is essentially similar to that of Milin. When $p = 2$ these results admit an extension to abstract functional Hilbert spaces of functions which are not necessarily holomorphic (see Burbea [2], [4], [5]). The simplest example for the function F in Corollary 4.4 is, of course, when F is the exponential function $F(\lambda) = \exp\lambda$, $\lambda \in \mathbf{C}$. We record this special instance in the following corollary:

COROLLARY 4.5. *Let $\varphi \in \mathscr{P}(\Omega)$ with $\Gamma_\varphi = \{\alpha \in \mathbf{Z}_+^n : |\alpha| \geqslant m\}$, $m = 0, 1, \ldots$ Then $\exp\varphi \in \mathscr{P}(\Omega)$, $\Gamma_{\exp\varphi} = \{0\} \cup \Gamma_\varphi$. Moreover, for $f \in \mathscr{H}_\varphi^\infty$ with $\|f\|_{\infty,\varphi} \leqslant 1$ we have $\exp f \in \mathscr{H}_{\exp\varphi}^\infty$ with*

$$\|\exp f\|_{\infty, \exp\varphi} \leqslant 1.$$

Equality holds if f is of the form $f = k_\varphi(\cdot, \zeta)$ for some $\zeta \in \bar{\Delta}^n$. Similarly, for $f \in \mathscr{H}_\varphi^p$ with $1 \leqslant p < \infty$ we have $\exp f \in \mathscr{H}_{\exp\varphi}^p$ with

$$\|\exp f\|_{p,\exp\varphi}^p \leqslant \exp\|f\|_{p,\varphi}^p.$$

Equality holds if and only if f is of the form $f = k_\varphi(\cdot, \zeta)$ for some $\zeta \in C^n$ with $\varphi(|\zeta|^p) < \infty$. In particular, if also $\varphi \in \mathscr{P}_\infty(\Omega)$ then $\zeta \in \Omega(p)$.

5. Applications. We shall discuss briefly some applications that arise from Examples 1.1, 1.2 and 1.3 by using the general theory. We also refer to [2]–[5] for additional details when $p = 2$.

We begin with $\varphi = \varphi_{r,q}$ $(r, q > 0)$ of Example 1.1. The corresponding space $\mathscr{H}_\varphi^p$ and the norm $\|\cdot\|_{p,\varphi}$ are denoted by $\mathscr{H}_{r,q}^p$ and $\|\cdot\|_{p;r,q}$, respectively. Similarly, we let $\langle\ ,\ \rangle_{r,q}$ stand for the pairing $\langle\ ,\ \rangle_\varphi$. For $1 \leqslant p \leqslant \infty$, $\mathscr{H}_{r,p}^p$ is a Banach space of holomorphic functions f, on C^n if $1 < p \leqslant \infty$ and on Δ^n if $p = 1$, such that $\|f\|_{p;r,q} < \infty$. When $p = 2$, $\mathscr{H}_{r,q}^2$ is a Hilbert space with the quadratic norm

$$\|f\|_{2;r,q}^2 = \pi^{-n} \int_{C^n} |f(z)|^2 \|z\|^{2(r-1)} e^{-q\|z\|^2} dv(z) \qquad (f \in \mathscr{H}_{r,q}^2).$$

One shows easily that for $f \in \mathscr{H}_{r,q}^p$ and $g \in \mathscr{H}_{r,q}^p$ with $1 < p < \infty$, we have

$$\langle f, g\rangle_{r,q} = \pi^{-n} \lim_{R \to \infty} \int_{\Delta_R^n} f(z)\overline{g(z)}\, \|z\|^{2(r-1)} e^{-q\|z\|^2} dv(z),$$

where $\Delta_R^n = \{z \in C^n\colon \|z\|_\infty < R\}$, $R > 0$. We also recall that $\Gamma_{\varphi_{r,q}} = Z_+^n$ and that the reproducing kernel $k_{r,q}$ of $\mathscr{H}_{r,q}^2$ is as in (1.5), and thus $k_{1,q}(z, \zeta) = e^{q\langle z,\zeta\rangle}$ $(z, \zeta \in C^n)$. In particular, $\varphi_{r,q} \in \mathscr{P}_\infty(C^n)$. This, together with Theorem 2.8 and Proposition 2.10, gives the following sharp norm inequality:

THEOREM 5.1. *Let $1 \leqslant p \leqslant \infty$ and let $f_j \in \mathscr{H}_{1,q_j}^p$, where $q_j > 0$ for $j = 1, \ldots, m$, $m \geqslant 2$. Then $\prod_{j=1}^m f_j \in \mathscr{H}_{1,q_1+\ldots+q_m}^p$ with*

$$\Big\|\prod_{j=1}^m f_j\Big\|_{p;1,q_1+\ldots+q_m} \leqslant \prod_{j=1}^m \|f_j\|_{p;1,q_j}.$$

Equality, when $1 \leqslant p < \infty$, holds if and only if either $\prod_{j=1}^m f_j = 0$ or each f_j is of the form $f_j = C_j k_{1,q_j}(\cdot, \zeta)$ for some $\zeta \in C^n$ and some nonzero constants C_j $(1 \leqslant j \leqslant m)$. When $p = \infty$, equality holds if either $\prod_{j=1}^m f_j = 0$ or each f_j is of the form $f_j = C_j k_{1,q_j}(\cdot, \zeta)$ for some $\zeta \in \bar{\Delta}^n$ and some nonzero constants C_j $(1 \leqslant j \leqslant m)$.

We now turn to $\varphi = \varphi_q$, $\boldsymbol{q} = (q_1, \ldots, q_n) \in \boldsymbol{R}_+^n \setminus \{0\}$, of Example 1.2. The corresponding space and norm, for $1 \leqslant p \leqslant \infty$, will be denoted by H_q^p and $\|\cdot\|_{p,q}$, respectively. In this case, the reproducing kernel k_q is given by

$$k_q(z, \zeta) = \varphi_q(z, \bar{\zeta}) = \prod_{j=1}^n (1 - z_j\bar{\zeta}_j)^{-q_j} \qquad (z \cdot \bar{\zeta} \in \Delta^n),$$

and $\Omega(p) = \Delta^n(p) = \Delta^n$ for every $0 < p \leqslant \infty$. Thus, for $1 \leqslant p \leqslant \infty$, H_q^p is a Banach space of holomorphic functions f on Δ^n with $\|f\|_{p,q} < \infty$. The quadratic norm $\|\cdot\|_{2,q}$ of the Hilbert space $\mathscr{H}_q^2$ admits an integral representation provided $\boldsymbol{q} \geqslant \boldsymbol{1} = (1, \ldots, 1)$. In this case, for $f \in \mathscr{H}_q^2$ we have

$$\|f\|_{2,q}^2 = \pi^{-n}\Big(\prod_{j=1}^{n} (q_j - 1)\Big) \int_{\Delta^n} |f(z)|^2 \Big(\prod_{j=1}^{n} (1-|z_j|^2)^{q_j-2}\Big) dv(z) \qquad (\boldsymbol{q} > \boldsymbol{1})$$

and

$$\|f\|_{2,q}^2 = (2\pi)^{-n} \int_{T^n} |f(z)|^2 \, d\sigma(z) \qquad (\boldsymbol{q} = \boldsymbol{1}),$$

where in the last integral, f stands for the nontangential distinguished boundary-values of the holomorphic function f in Δ^n. Here, of course, $d\sigma(z) = |dz_1| \ldots |dz_n|$ is the surface measure on the distinguished boundary $T^n = \partial_0 \Delta^n$. It follows that $\mathscr{H}_{\boldsymbol{1}}^2$ is the customary *Hardy space* $H^2(\Delta^n)$, $\mathscr{H}_q^2$ for $\boldsymbol{q} > \boldsymbol{1}$ is the *weighted Bergman space* $A_q^2(\Delta^n)$ and $\mathscr{H}_{\boldsymbol{2}}^2$, $\boldsymbol{2} = (2, \ldots, 2)$, is the ordinary *Bergman space* $A^2(\Delta^n)$. The natural pairing $\langle\ ,\ \rangle_\varphi$ is denoted by $\langle\ ,\ \rangle_q$, $\boldsymbol{q} \in \boldsymbol{R}_+^n \setminus \{0\}$, and we note that for $\boldsymbol{q} \geqslant \boldsymbol{1}$ this pairing admits an alternative representation which is induced by the integral representation of the quadratic norm $\|\cdot\|_{2,q}$. We also note that $\varphi_q \in \mathscr{P}_\infty(\Delta^n)$ for any $\boldsymbol{q} \in \boldsymbol{R}_+^n \setminus \{0\}$.

As before, Theorem 2.8 and Proposition 2.10 imply the following sharp norm inequality:

THEOREM 5.2. *Let* $1 \leqslant p \leqslant \infty$ *and let* $f_j \in \mathscr{H}_{q_j}^p$, *where* $\boldsymbol{q}_j \in \boldsymbol{R}_+^n \setminus \{0\}$ *for* $j = 1, \ldots, m$, $m \geqslant 2$. *Then* $\prod_{j=1}^{m} f_j \in \mathscr{H}_{q_1 + \ldots + q_m}^p$ *with*

$$\Big\|\prod_{j=1}^{m} f_j\Big\|_{p,q_1+\ldots+q_m} \leqslant \prod_{j=1}^{m} \|f_j\|_{p,q_j}.$$

Equality, when $1 \leqslant p < \infty$, *holds if and only if either* $\prod_{j=1}^{m} f_j = 0$ *or each* f_j *is of the form* $f_j = C_j k_{q_j}(\cdot, \zeta)$ *for some* $\zeta \in \Delta^n$ *and some nonzero constants* C_j $(1 \leqslant j \leqslant m)$. *When* $p = \infty$, *equality holds if either* $\prod_{j=1}^{m} f_j = 0$ *or each* f_j *is of the form* $f_j = C_j k_{q_j}(\cdot, \zeta)$ *for some* $\zeta \in \bar{\Delta}^n$ *and some nonzero constants* C_j $(1 \leqslant j \leqslant m)$.

In order to apply Corollary 4.4 to the present setting of Example 1.2, we introduce another function ψ_q in $\mathscr{P}_\infty(\Delta^n)$. This function is defined as follows: For $\boldsymbol{q} = (q_1, \ldots, q_n) \in \boldsymbol{R}_+^n \setminus \{0\}$ and $z = (z_1, \ldots, z_n) \in \Delta^n$, we let

$$\psi_q(z) = -\sum_{j=1}^{n} q_j \log(1 - z_j) = \sum_{m=1}^{\infty} m^{-1} \sum_{j=1}^{n} q_j z_j^m,$$

and thus $\Gamma_{\psi_q} = \bigcup_{j=1}^{n} \{m_j e_j\}_{m_j=1}^{\infty}$. For $1 \leqslant p \leqslant \infty$, we let $\mathcal{S}_q^p$ and $|||\cdot|||_{p,q}$ be the associated Banach space $\mathcal{H}_{\psi_q}^p$ and norm $\|\cdot\|_{p,\psi_q}$, respectively. Let $A(\Delta^n)$ be the subclass of $H(\Delta^n)$ consisting of all functions f of the form

$$f(z) = \sum_{j=1}^{n} f_j(z_j); \quad f_j \in H(\Delta),\ j = 1, \ldots, n,\ z = (z_1, \ldots, z_n) \in \Delta^n.$$

It follows that $\mathcal{S}_q^p = \{f \in A(\Delta^n): f_j(0) = 0,\ 1 \leqslant j \leqslant n,\ |||f|||_{p,q} < \infty\}$ and that $\mathcal{S}_q^2$ is a Hilbert space whose quadratic norm can be realized as

$$|||f|||_{2,q}^2 = \pi^{-1} \sum_{j=1}^{n} q_j^{-1} \int_{\Delta} |f_j'(\lambda)|^2 \, dA(\lambda),$$

where $dA = dv_1$ is the Lebesgue measure on $\boldsymbol{C}$. This quadratic norm induces an alternative expression for the pairing $\langle\ ,\ \rangle_{\psi_q}$ in a manner similar to that discussed previously. The reproducing kernel is now

$$K_q(z, \zeta) = \psi_q(z \cdot \bar{\zeta}) \quad (z \cdot \bar{\zeta} \in \Delta^n),$$

and Corollary 4.4 gives:

THEOREM 5.3. *Let $f \in \mathcal{S}_q^p$ with $\boldsymbol{q} \in \boldsymbol{R}_+^n \setminus \{0\}$ and $1 \leqslant p \leqslant \infty$. For $1 \leqslant p < \infty$, we have $\exp f \in \mathcal{H}_q^p$ and*

$$\|\exp f\|_{p,q}^p \leqslant \exp |||f|||_{p,q}^p.$$

Equality holds if and only if f is of the form $f = K_q(\cdot, \zeta)$ for some $\zeta \in \Delta^n$. If also $|||f|||_{\infty,q} < 1$, then for $p = \infty$ we have $\exp f \in \mathcal{H}_q^{\infty}$ and

$$\|\exp f\|_{\infty,q} \leqslant 1.$$

Equality holds if f is of the form $f = K_q(\cdot, \zeta)$ for some $\zeta \in \bar{\Delta}^n$.

Finally, we now turn to $\varphi = \varphi_q$, $q \geqslant 0$, of Example 1.3. The corresponding space and norm, for $1 \leqslant p \leqslant \infty$, will be denoted by $\mathcal{H}_q^p$ and $\|\cdot\|_{p,q}$, respectively. Similarly, we let $\langle\ ,\ \rangle_q$ stand for the pairing $\langle\ ,\ \rangle_\varphi$ and let k_q be the reproducing kernel k_φ. It follows that $\mathcal{H}_q^p = \{f \in H(B(p')): \|f\|_{p,q} < \infty\}$, $q > 0$, and $\mathcal{H}_0^p = \{f \in H(B(p')): f(0) = 0,\ \|f\|_{p,0} < \infty\}$. When $q \geqslant n$, the quadratic norm $\|\cdot\|_{2,q}$ admits an integral representation. To show this, we introduce a family dv_s, $s \geqslant 0$, of probability measures on $\bar{B}$ by letting $dv_0 = d\sigma_B$ when $s = 0$ and

$$dv_s(z) = \pi^{-n} (s)_m (1 - \|z\|^2)^{s-1} \, dv(z)$$

when $s > 0$. It can be easily verified by a calculation based on polar coordinates that as a measure on $\bar{B}$, $dv_s \to dv_0$ as $s \to 0^+$. In particular, if f is a continuous function on $\bar{B}$, then

$$\int f \, dv_0 = \int_{\partial B} f \, d\sigma_B = \lim_{s \to 0^+} \int_B f \, dv_s.$$

On the other hand,

$$\int f\,dv_s = \int_B f\,dv_s \qquad (s > 0)$$

if f is integrable with respect to dv_s. With this notation we have

$$\|f\|_{2,q}^2 = \int |f|^2\,dv_{q-n} \qquad (f \in \mathscr{H}_q^2,\ q \geqslant n), \tag{5.1}$$

where for $q = n$, the integration is carried over the nontangential boundary values of $f \in \mathscr{H}_n^2$. Moreover, for $q \geqslant n$ the quadratic norm $\|\cdot\|_{2,q}$ induces an alternative expression for the pairing $\langle\ ,\ \rangle_q$ in the usual way. We also observe that $\mathscr{H}_n^2$ is the *Hardy space* $H^2(B)$, that $\mathscr{H}_{n+q}^2$, $q > 0$, is the *weighted Bergman space* A_q^2 and that $\mathscr{H}_{n+1}^2$ is the ordinary *Bergman space* (see [1], [4]). It is also clear that $\varphi_q \in \mathscr{P}_\infty(B)$ for every $q \geqslant 0$.

By Theorem 2.8 and Proposition 2.10 we obtain:

THEOREM 5.4. *Let* $1 \leqslant p \leqslant \infty$ *and let* $f_j \in \mathscr{H}_{q_j}^p$, *where* $q_j > 0$ *for* $j = 1, \ldots, m$, $m \geqslant 2$. *Then* $\prod_{j=1}^m f_j \in \mathscr{H}_{q_1+\ldots+q_m}^p$ *and*

$$\Big\|\prod_{j=1}^m f_j\Big\|_{p,q_1+\ldots+q_m} \leqslant \prod_{j=1}^m \|f_j\|_{p,q_j}.$$

Equality, when $1 \leqslant p < \infty$, *holds if and only if either* $\prod_{j=1}^m f_j = 0$ *or each* f_j *is of the form* $f_j = C_j k_{q_j}(\cdot, \zeta)$ *for some* $\zeta \in B(p)$ *and some nonzero constants* C_j $(1 \leqslant j \leqslant m)$. *When* $p = \infty$, *equality holds if either* $\prod_{j=1}^m f_j = 0$ *or each* f_j *is of the form* $f_j = C_j k_{q_j}(\cdot, \zeta)$ *for some* $\zeta \in \bar{\Delta}^n$ *and some nonzero constants* C_j $(1 \leqslant j \leqslant m)$.

Similarly, by Corollary 4.4 we have:

THEOREM 5.5. *Let* $f \in \mathscr{H}_0^p$ *with* $1 \leqslant p \leqslant \infty$, *and let* $q > 0$. *For* $1 \leqslant p < \infty$, *we have* $\exp f \in \mathscr{H}_q^p$ *and*

$$\|\exp f\|_{p,q}^p \leqslant \exp\{q^{1-p}\|f\|_{p,0}^p\}.$$

Equality holds if and only if f *is of the form* $f = qk_0(\cdot, \zeta)$ *for some* $\zeta \in B(p)$. *If also* $\|f\|_{\infty,0} \leqslant q$, *then for* $p = \infty$ *we have* $\exp f \in \mathscr{H}_q^\infty$ *and*

$$\|\exp f\|_{\infty,q} \leqslant 1.$$

Equality holds if f *is of the form* $f = qk_0(\cdot, \zeta)$ *for some* $\zeta \in B(p)$.

There are other consequences that arise from the general theory. For brevity we shall only discuss those that arise from the last theorem. For $q \geqslant 0$ and $0 < p < \infty$, we let A_q^p stand for the space of all $f \in H(B)$ such that

$$|||f|||_{p,q} \equiv \Big\{\int |f|^p\,dv_q\Big\}^{1/p} < \infty.$$

It follows that for $1 \leqslant p < \infty$, A_q^p is a functional Banach space of holomorphic functions on B with norm $|||\cdot|||_{p,q}$, while for $0 < p < \infty$, A_q^p is a functional Fréchet space of holomorphic functions on B with the metric $\varrho(f, g) = |||f-g|||_{p,q}^p$ $(f, g \in A_q^p)$. Moreover, A_q^2 is precisely $\mathscr{H}_{n+q}^2$ with $|||\cdot|||_{2,q} = \|\cdot\|_{2,n+q}$ and thus we may extend the definition of A_q^2 to include every q with $q \geqslant -n$. Evidently, A_0^p is the *Hardy space* H^p, A_q^p for $q > 0$ is a *weighted Bergman space* and A_1^p is the ordinary *Bergman space*.

We now introduce some weighted Sobolev spaces of holomorphic functions on B. For this purpose we consider the radial derivative operator

$$\mathscr{R} = \sum_{j=1}^{n} z_j \partial_j \qquad (z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n)$$

and we set

$$D_l = \mathscr{R} + l \qquad (l \in \boldsymbol{C})$$

with $D = D_1$. Thus, for any $f \in H(B)$ with

$$f(z) = \sum_{\alpha} a_\alpha z^\alpha$$

we have

$$\{D_l^s f\}(z) = \sum_{\alpha} (|\alpha| + l)^s a_\alpha z^\alpha \qquad (l \in \boldsymbol{C})$$

for any $s \in \boldsymbol{Z}_+$. This shows that $D_l^s f$ may be defined for every $s \in \boldsymbol{R}$, provided $-l \notin \boldsymbol{Z}_+$. When $l = -m$, $m = 0, 1, \ldots$, $D_l^s f$ is well-defined provided $\partial^\alpha f(0) = 0$ for all $\alpha \in \boldsymbol{Z}_+^n$ with $|\alpha| = m$. We fix a number $l > 0$ and define, for $q \geqslant 0$, $0 < p < \infty$ and $s \in \boldsymbol{R}$, the space $A_{q,s}^p$ as the space of all $f \in H(B)$ such that $|||f|||_{p,q;s} = |||D_l^s f|||_{p,q} < \infty$. This definition is independent of $l > 0$ in the sense that the spaces resulting from different choices of l are equivalent to one another, and hence it is sufficient to take $l = 1$. Moreover, if $f \in A_{q,s}^p$ satisfies $f(0) = 0$, then the norm $|||D_l^s f|||_{p,q}$ is equivalent to $|||\mathscr{R}^s f|||_{p,q}$ for any $l > 0$. We should also observe that as far as the functional analytic properties of the Sobolev space $A_{q,s}^p$ are concerned, they are identical with those described before for the space $A_q^p = A_{q,0}^p$. In particular, $A_{q,s}^2$ may be defined for all $q \geqslant -n$. Moreover, when also $s \in \boldsymbol{Z}_+$ then the norm $|||D_l^s f|||_{p,q}$ $(l \geqslant 0,\ s \in \boldsymbol{Z}_+,\ q \geqslant 0,\ 0 < p < \infty)$ is equivalent to the norm

$$\left\{\sum_{|\alpha| \leqslant s} \frac{|\alpha|!}{\alpha!} |||\partial^\alpha f|||_{p,q}^p\right\}^{1/p} \qquad (f \in A_{q,s}^p).$$

For this and related properties we refer to [1].

The proof of the next lemma appears in [1].

Lemma 5.6. *Let* $q \geqslant 0$ *and* $s \in \boldsymbol{R}$ *such that* $s < (n+q)/2$. *Then* $A_{q,s}^2 \subset A_q^{2(n+q)/[n+q-2s]}$, *and the inclusion is continuous.*

LEMMA 5.7. *Let q and s be real numbers with $n+q \geqslant \max(2s, 0)$. Then $A^2_{q,s} = A^2_{q-2s}$, and the norms are equivalent. In particular, the space $\mathscr{H}^2_0 = A^2_{-n}$ is equivalent to $\{f \in A^2_{1,(n+1)/2}\colon f(0) = 0\}$. Moreover, for any $f \in \mathscr{H}^2_0$ we have*

$$\|f\|^2_{2,0} \leqslant (n+1)\,|||\mathscr{R}^{(n+1)/2} f|||^2_{2,1} \leqslant (n+1)!\,\|f\|^2_{2,0}.$$

Proof. The first part of the lemma follows from Theorem 2.6 and Stirling's formula. The last inequalities are easily verified by a straightforward calculation, and the proof is complete.

We turn now to the above mentioned consequences of Theorem 5.5.

COROLLARY 5.8. *Let $f \in \mathscr{H}^2_0$. Then, for any $q \geqslant 0$ and $0 < p < \infty$, we have $\exp f \in A^p_q$, and*

$$\int |\exp f|^p \, dv_q \leqslant \exp\left\{\frac{p^2}{4(n+q)}\|f\|^2_{2,0}\right\}.$$

Equality holds if and only if f is of the form $f = 2(n+q)\,p^{-1} k_0(\cdot, \zeta)$ for some $\zeta \in B$. Moreover, for $n = 1$ the above sharp inequality is equivalent to the sharp inequality

$$\int |\exp f|^p \, dv_q \leqslant \exp\left\{\frac{p^2}{4(1+q)} \int |f'|^2 \, dv_1\right\},$$

and, in general,

$$\int |\exp f|^p \, dv_q \leqslant \exp\left\{\frac{p^2}{4}\cdot\frac{n+1}{n+q} \int |\mathscr{R}^{(n+1)/2} f|^2 \, dv_1\right\}.$$

Proof. The first part of the corollary follows from Theorem 5.5 (with $p = 2$ there) by considering $(p/2) f$ instead of f and observing that $A^2_q = \mathscr{H}^2_{n+q}$. The second part follows from Lemma 5.7, and the proof is complete.

COROLLARY 5.9. *The mapping $f \mapsto \exp f$ is continuous from the space $A^2_{1,(n+1)/2}$ into the spaces $A^2_{q,s}$ and $A_q^{2(n+q)/[n+q-2s]}$ for any $q \geqslant 0$ and any $s \in \mathbf{R}$ such that $s < (n+q)/2$. In particular, this mapping is continuous from the space $A^2_{1,(n+1)/2}$ into the space $A^2_{1,(n+1-t)/2}$ for any $t > 0$.*

Proof. This follows from Theorem 5.5 and Lemmas 5.6 and 5.7.

We should remark that $A^2_{1,(n+1)/2}$ in the last corollary can be replaced by any equivalent space $A^2_{r,(n+r)/2}$ $(r \geqslant 0)$. We also remark that, as in the previous corollary, the last corollary also shows that the mapping $f \mapsto \exp f$ is continuous from $A^2_{1,(n+1)/2}$ into A^p_q for any $q \geqslant 0$ and any $0 < p < \infty$.

Finally, we shall prove the following result:

THEOREM 5.10. *Let F be an entire function in C such that $F^{(m)}(0) > 0$ for $m = 0, 1, \ldots$, and such that $\lim_{m\to\infty} F^{(m+1)}(0)/F^{(m)}(0) \equiv L$ with $L \in [0, \infty)$, and let $q \geqslant 0$. Let $0 < M < \sqrt{(n+q)/eL}$. Then there exists a constant $C = C(M, n, q)$, $0 < C < \infty$, such that*

$$\int F(|f|^2)\,dv_q \leqslant C$$

for every $f \in \mathscr{H}_0^2$ *with* $\|f\|_{2,0} \leqslant M$.

Proof. Let $a_m = F^{(m)}(0)/m!$ $(m = 0, 1, \ldots)$. By assumption $a_m > 0$, $\lim\limits_{m\to\infty} (m+1)\,a_{m+1}/a_m = L$ and

$$F(t) = \sum_{m=0}^{\infty} a_m t^m \qquad (t \in \mathbf{C}).$$

It follows from (5.1) and Theorem 5.4 that

$$\begin{aligned}\int F(|f|^2)\,dv_q &= \sum_{m=0}^{\infty} a_m \int |f|^{2m}\,dv_q = a_0 + \sum_{m=1}^{\infty} a_m \|f^m\|_{2,n+q}^2 \\ &= a_0 + \sum_{m=1}^{\infty} a_m \|f^m\|_{2,m\cdot(n+q)/m}^2 \leqslant a_0 + \sum_{m=1}^{\infty} a_m \|f\|_{2,(n+q)/m}^{2m}.\end{aligned}$$

Since $f \in \mathscr{H}_0^2$, f is of the form

$$f(z) = \sum_{\alpha>0} b_\alpha z^\alpha \qquad (z \in B)$$

with

$$\|f\|_{2,0}^2 = \sum_{\alpha>0} \frac{\alpha!}{\Gamma(|\alpha|)} |b_\alpha|^2 < \infty.$$

Moreover,

$$\|f\|_{2,Q}^2 = \sum_{\alpha>0} \frac{\alpha!}{(Q)_{|\alpha|}} |b_\alpha|^2$$

for any $Q > 0$. It follows, since for $\alpha \in \mathbf{Z}_+^n \setminus \{0\}$

$$\frac{\Gamma(|\alpha|)}{(Q)_{|\alpha|}} = \frac{\Gamma(Q)\,\Gamma(|\alpha|)}{\Gamma(Q+|\alpha|)} = \int_0^1 x^{Q-1}(1-x)^{|\alpha|-1}\,dx \leqslant \int_0^1 x^{Q-1}\,dx = Q^{-1},$$

that

$$\|f\|_{2,Q}^2 \leqslant Q^{-1}\,\|f\|_{2,0}^2 \qquad (Q > 0).$$

We deduce from the last inequality that

$$\int F(|f|^2)\,dv_q \leqslant a_0 + \sum_{m=1}^{\infty} a_m \left(\frac{m}{n+q}\|f\|_{2,0}^2\right)^m \leqslant a_0 + \sum_{m=1}^{\infty} a_m \left(\frac{m}{n+q} M^2\right)^m$$

for every $f \in \mathscr{H}_0^2$ with $\|f\|_{2,0} \leqslant M$. Since, by assumption,

$$\lim_{m\to\infty} (m+1)\frac{a_{m+1}}{a_m}\left(1+\frac{1}{m}\right)^m \cdot \frac{M^2}{n+q} = \frac{eLM^2}{n+q} < 1,$$

we conclude from the ratio-test that the last series converges to, say, $C = C(M, n, q)$, and the proof is complete.

COROLLARY 5.11. *Let $q \geq 0$ and $0 < M < \sqrt{(n+q)/e}$. Then there exists a constant $C = C(M, n, q)$, $0 < C < \infty$, such that*

$$\int e^{|f|^2} dv_q \leq C$$

for every $f \in \mathscr{H}_0^2$ with $\|f\|_{2,0} \leq M$.

Proof. This follows from the theorem by letting $F(t) = e^t$ and observing that $\lim_{m \to \infty} F^{(m+1)}(0)/F^{(m)}(0) = 1$.

When $n = 1$ and $q = 0$ this corollary admits a stronger version in that the restriction $0 < M < e^{-1/2}$ may be replaced by $0 < M \leq 1$. The proof of this one dimensional stronger version is rather involved and is due to Chang and Marshall [6] who use methods of potential theory associated with Dirichlet integrals. In any case, the mere fact that $f \in \mathscr{H}_0^2$ does not guarantee that e^f will have *bounded mean oscillation* (BMO) on ∂B, and this is in spite of Corollary 5.8 which asserts, in particular, that $e^f \in A_0^p = H^p$ for every $0 < p < \infty$. To see this we let $f = h_t \circ u_1$ where $u_1(z) = z_1$ for $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ and

$$h_t(\lambda) = \left\{\frac{1}{\lambda} \log \frac{1}{1-\lambda}\right\}^t \qquad (\lambda \in \Delta,\ t > 0).$$

Then for $a_m(t) = h_t^{(m)}(0)/m!$, we have $a_0(t) = 1$ and

$$c^{-1} \leq (m+1)\{\log(m+1)\}^{1-t} |a_m(t)| \leq c \qquad (m = 1, 2, \ldots)$$

for some positive constant $c = c(t)$ (see [8], p. 93). It follows that h_t is in $\mathscr{H}_0^2(\Delta)$ if and only if $t < 1/2$. Moreover,

$$(e^{h_t(\lambda)})' = te^{h_t(\lambda)} h_{t-1}(\lambda) \frac{1}{\lambda(1-\lambda)} \{1 - (1-\lambda) h_1(\lambda)\}$$

and thus, for $0 < t < 1/2$, $(1 - |\lambda|)\{e^{h_t(\lambda)}\}'$ is not bounded on Δ. This means that for $0 < t < 1/2$, the function $f = h_t \circ u_1$ is in $\mathscr{H}_0^2$ while e^f is not in the *Bloch space*, i.e., $(1 - \|z\|^2)\{De^f\}(z)$ is not bounded on B. Since, as is well-known, holomorphic functions on B with bounded mean oscillation on ∂B are contained in the Bloch space, we deduce that e^f (for $0 < t < 1/2$) has no bounded mean oscillation on ∂B.

References

[1] F. Beatrous and J. Burbea, *Sobolev spaces of holomorphic functions in the ball*, Dissertationes Math. (to appear).

[2] J. Burbea, *A Dirichlet norm inequality and some inequalities for reproducing kernel spaces*, Proc. Amer. Math. Soc. 83 (1981), 279–285.

[3] —, *Norm inequalities of exponential type of holomorphic functions*, Kōdai Math. J. 5 (1982), 339–354.

[4] —, *Inequalities for reproducing kernel spaces*, Illinois Math. J. 27 (1983), 130–137.

[5] —, *Inequalities for holomorphic functions of several complex variables*, Trans. Amer. Math. Soc. 276 (1983), 247–266.

[6] S. Y. Chang and D. E. Marshall, *A sharp inequality concerning the Dirichlet integral*, Amer. J. Math., 107 (1985), 1015–1033.

[7] D. Y. Kwak, *Norm estimates of holomorphic functions in the ball of C^n in terms of their Taylor coefficients*, Dissertation, Univ. of Pittsburgh, Pittsburgh 1985.

[8] J. E. Littlewood, *Lectures on the Theory of functions*, Oxford Univ. Press, London 1944.

[9] I. M. Milin, *Univalent Functions and Orthogonal Systems*, Translations Math. Monographs 49, Amer. Math. Soc., Providence, R. I., 1977.

[10] S. Saitoh, *Some inequalities for analytic functions with a finite Dirichlet integral on the unit disc*, Math. Ann. 246 (1979), 69–77.

[11] —, *Some inequalities for entire functions*, Proc. Amer. Math. Soc. 80 (1980), 254–258.

[12] J. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, J. reine angew. Math. 140 (1911), 1–28.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA, U.S.A.

Reçu par la Rédaction le 20.03.1987