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Specialization of a frame and its geometrical interpretation

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Abstract. If R_0 is a fixed frame in an n-dimensional homogeneous space (V_n, G) and if $x \to R_x$ is a frame field along an m-dimensional surface S: $V_m \to V_n$, then there exists exactly one element $a_x \in G$ of the Lie group such that $R_x = a_x R_0$. The mapping a: $V_m \to G$, a: $x \to a_x$, is called the *representation* of S in G. This paper contains a method of specialization of a frame on S (locally) by the means of a (local) representation of S in G.

A method of specialization of a frame in a homogeneous space was created by G. Darboux and extremely developed by E. Cartan about half century ago. Up to the present this method has been the subject of different investigations. In particular, in the recent years many papers devoted to this method have been published. It seems that the present interest in this method is caused by its formal character, namely E. Cartan's method consists in creating a formal sequence of operations on linear forms. Using formally this method, we have sometimes difficulties in order to give a geometrical interpretation for that sequence of operations and the obtained result. In spite of these inconveniences, it is necessary to say that Cartan's method is one of the greatest and most fruitful discoveries in the geometry. Recently some authors have published papers on a geometrical interpretation of special aspects of this method.

In this paper we present another method of specialization of a frame. Using a representation of a surface (i.e., an immersion of a manifold V_m in a homogeneous space V_n) in the Lie group we construct a sequence of operations which leads to a fixed frame field and which has a geometrical interpretation. Thus the sequence of consecutive operations may be more easily controled by its geometrical meaning. It seems that the method proposed is easier for practical purposes than Cartan's method, because all considerations are presented in terms of a natural coordinate system on a manifold and the isotropy subgroup is used in an explicit form.

We assume that all functions considered are of class C^∞.

1. The representation of a homogeneous space in the Lie group. Let (V_n, G) be a homogeneous space, where V_n is an n-dimensional manifold and G is an r-dimensional Lie group of transformations of V_n, i.e., there exists a mapping $f\colon G\times V_n\to V_n$, $f(g,x)=gx$, satisfying:

a) the mapping $x\to gx$ is a diffeomorphism of V_n:

b) $(g_1g_2)x=g_1(g_2x)$, $g_1,g_2\in G$;

c) $f\colon (g,x)\to gx$ is differentiable;

d) for each $x_1,x_2\in V_n$ there exists $g\in G$ such that $gx_1=x_2$;

e) G acts effectively on V_n, i.e., if $g_1x=g_2x$ for all x, then $g_1=g_2$.

If $p_0\in V_n$, then the set $H_0=\{g;\ gp_0=p_0, g\in G\}$ is a closed subgroup of G, called the *isotropy group* at p_0. The coset space G/H_0 is a differentiable manifold diffeomorphic with V_n and the diffeomorphism $\varphi\colon G/H_0\to V_n$ is given by the formula $\varphi\colon g/H_0\to gp_0$, $g/H_0\in G/H_0$. The projection $\pi\colon G\to G/H_0$, $\pi(g)=g/H_0$, is a differentiable mapping.

Let $\boldsymbol{q}_e$ and $\boldsymbol{h}_0$ be the Lie algebras of G and H_0, respectively, at the unity $e\in G$. If $\boldsymbol{m}_0$ a subspace of the vector space $\boldsymbol{q}_e$ satisfying

$$\boldsymbol{q}_e=\boldsymbol{h}_0+\boldsymbol{m}_0 \quad \text{(direct sum of vector spaces)}$$

and $\psi_{\boldsymbol{m}}=\exp_{\boldsymbol{m}_0}\colon \boldsymbol{m}_0\to G$, $\psi_{\boldsymbol{m}}(A)=\exp A$, is the exponential mapping, then there exists a neighbourhood M_0 of 0 in $\boldsymbol{m}_0$ such that $\psi_{\boldsymbol{m}}$ is a diffeomorphism of M_0 onto $\psi_{\boldsymbol{m}}(M_0)=\exp M_0=G_M\subset G$ and the projection π is a diffeomorphism of G_M on to a neighbourhood $\pi(G_M)\subset G/H_0$ of the point $e/H_0=\pi(e)$ (see [4]). Thus the mapping $\sigma=\varphi\cdot\pi\colon G_M\to V_n$ given by the formula

$$\sigma\colon g\to gp_0=x,\qquad g\in G_M,\qquad \sigma(G_M)=W(p_0),$$

is a diffeomorphism of G_M onto a neighbourhood $W(p_0)$ of p_0 in V_n. Thus for each $x\in W(p_0)$ there exists exactly one $g(x)\in G_M$ such that

$$g(x)p_0=x;$$

the mapping $g\colon W(p_0)\to G$ given by the above formula is called a (local) *representation* of V_n in G.

If $R_0=(p_0,p_1,\ldots,p_s)$ is a fixed frame (in the sense of E. Cartan, [5]) of G, $p_i\in W(p_0)$, and $x\to g(x)$ is a representation of $W(p_0)$ in G, then

$$x\to g(x)R_0=\big(g(x)p_0,\ldots,g(x)p_s\big)=\big(x,q_1(x),\ldots,q_s(x)\big)=R_x$$

is a frame field on $W(p_0)$.

The representation of $W(p_0)$ defined in this way is a special case of a more general construction. Namely, let $x\to R_x=\big(x,q_1(x),\ldots,q_s(x)\big)$ be a given frame field on $W(p_0)$, where $W(p_0)$ is a neighbourhood of $p_0\in V_n$. If $R_0=(p_0,p_1,\ldots,p_s)$ is a fixed frame in $W(p_0)$, then there exists exactly one $a(x)\in G$ satisfying

$$a(x)R_0=R_x;\qquad x\in W(p_0);$$

the function $a\colon W(p_0) \to G$ is called also a *representation* of V_n in G; if $R_0 = R_{p_0} = (p_0, q_1(p_0), \ldots, q_s(p_0))$, then $a(p_0) = e$.

If $a\colon W(p_0) \to G$ and $c\colon W(p_0) \to G$ are two representations with respect to R_0, then there exists $h(x) \in H_0$ satisfying

$$(1_0) \qquad a(x) = c(x)h(x), \quad h(x) \in H_0.$$

For two different initial frames R_0, $R_0' = b_0 R_0$, the representations $a(x)R_0 = R_x = a'(x)R_0' = a'(x)b_0R_0$ satisfy

$$(1_1) \qquad a(x) = a'(x)b_0,$$

but the isotropy groups H_0 and H_0' at points p_0 and p_0' respectively, $R_0' = (p_0', p_1', \ldots, p_s')$, satisfy

$$(1_2) \qquad H_0 = b_0^{-1}H_0'b_0.$$

Now, we consider the neighbourhood $N_0 \subset G/H_0$ consisting of the points $\hat{a}(x) = a(x)/H_0 = \pi(a(x))$ and the sets $a(x)H_0 = \{a(x)h;\, h \in H_0\}$, $a(x)/H_0 \in G/H_0$, $a(x)H_0 \subset G$. Here $a\colon W(p_0) \to G$ is a representation of V_n in G; then the surface $a(W(p_0)) = \{a(x);\, x \in W(p_0)\}$ is diffeomorphic with $W(p_0)$ and also with N_0. Each set gH_0, with $\pi(g) \in N_0$, contains exactly one point $a(x)$, $\pi(a(x)) = \pi(g) = \hat{a}(x)$. Thus the surface $a\colon W(p_0) \to G$ determines a cross section σ in $\pi^{-1}(N_0)$, $\sigma\colon N_0 \to \pi^{-1}(N_0)$, $\sigma(\hat{a}(x)) = a(x) \in \pi^{-1}(\hat{a}(x))$. The set $a(p_0)H_0$ contains the unity $e \in G$; hence $\hat{a}(p_0) = e/H_0 \in N_0$.

Conversely, each cross section $\sigma\colon N_0 \to \pi^{-1}(N_0)$, $\sigma(\hat{g}) = g \in \pi^{-1}(\hat{g}) = gH_0$, $\pi \cdot \sigma =$ identity, where N_0 is a neighbourhood of $e/H_0 = \pi(e)$, determines a frame field $x = gp_0 \to gR_0 = (gp_0, gp_1, \ldots, gp_s) = (x, q_1(x), \ldots, q_s(x))$ on a neighbourhood $W(p_0) \subset V_n$ of p_0. Hence we get a surface $a\colon W(p_0) \to G$, $(x, q_1(x), \ldots, q_s(x)) = a(x)R_0$, which is also a representation of $W(p_0)$ in G.

We consider the direct sum $\boldsymbol{q}_e = \boldsymbol{h}_0 + \boldsymbol{m}_0$, where $\boldsymbol{q}_e$ is the Lie algebra of the group G and $\boldsymbol{h}_0$ is a subalgebra of $\boldsymbol{q}_e$. For each vector $Y_e \in \boldsymbol{q}_e$ we have $Y_e = \mathrm{pr}_{\boldsymbol{m}_0} Y_e + \mathrm{pr}_{\boldsymbol{h}_0} Y_e$, $\mathrm{pr}_{\boldsymbol{m}_0} Y_e \in \boldsymbol{m}_0$, $\mathrm{pr}_{\boldsymbol{h}_0} Y_e \in \boldsymbol{h}_0$; $\mathrm{pr}_{\boldsymbol{m}_0} Y_e$ is called the *projection of the vector* Y_e onto $\boldsymbol{m}_0$ and $\mathrm{pr}_{\boldsymbol{h}_0} Y_e$ is the projection onto $\boldsymbol{h}_0$. Analogously, if ω is a linear form on a vector space T with values in $\boldsymbol{q}_e$, then $\mathrm{pr}_{\boldsymbol{h}_0}\omega\colon T \to \boldsymbol{h}_0$ is the form given by the formula $(\mathrm{pr}_{\boldsymbol{h}_0}\omega)(v) = \mathrm{pr}_{\boldsymbol{h}_0}(\omega(v))$ and $\mathrm{pr}_{\boldsymbol{m}_0}\omega\colon T \to \boldsymbol{m}_0$ is given by $(\mathrm{pr}_{\boldsymbol{m}_0}\omega)(v) = \mathrm{pr}_{\boldsymbol{m}_0}(\omega(v))$.

Let $a\colon x \to a(x) = a_x$ be a representation of $W(p_0) \subset V_n$ in G and da_x the differential of a at $x \in W(p_0)$. If t_x is a vector tangent to V_n at $x \in W(p_0)$, then the value $da_x(t_x) = d_{t_x}a_x$ of da_x on t_x is a vector tangent to a at $a(x)$.

We introduce the following notation:

$$H_0^{-1}a_x^{-1}da_xH_0 = \{h^{-1}a_x^{-1}da_xh;\ h \in H_0\},$$
$$(2)\qquad H_0^{-1}a_x^{-1}d_{t_x}a_xH_0 = \{h^{-1}a_x^{-1}d_{t_x}a_xh;\ h \in H_0\},$$

where H_0 is the isotropy group at $p_0 \in W(p_0)$.

Now, we take another representation $c\colon W(p_0) \to G$, $c_x = a_xk_x$, $k_x \in H_0$, with respect to the same initial frame R_0. Then

$$dc_x = da_xk_x + a_xdk_x$$

and

$$c_x^{-1}dc_x = k_x^{-1}a_x^{-1}da_xk_x + k_x^{-1}dk_x,$$
$$H_0^{-1}c_x^{-1}dc_xH_0 = \{h^{-1}k_x^{-1}a_x^{-1}da_xk_xh + h^{-1}k_x^{-1}dk_xh;\ h \in H_0\},$$
$$\mathrm{pr}_{m_0}(H_0^{-1}c_x^{-1}dc_xH_0) = \{\mathrm{pr}_{m_0}(h^{-1}k_x^{-1}a_x^{-1}da_xk_xh),\ h \in H_0\}$$
$$= \mathrm{pr}_{m_0}(H_0^{-1}a_x^{-1}da_xH_0),$$

since $k_xh = h' \in H_0$. Thus we get

THEOREM 1. *The projection of the set* $H_0^{-1}a_x^{-1}da_xH_0$ *onto* $\boldsymbol{m}_0$, *where* $\boldsymbol{q}_c = \boldsymbol{h}_0 + \boldsymbol{m}_0$, *does not depend on the representation* $a\colon W(p_0) \to G$, *i.e., if* $c\colon W(p_0) \to G$ *is another representation with respect to the same initial frame* $R_0 = (p_0, \ldots, p_s)$, *then*

$$(3)\qquad \mathrm{pr}_{m_0}(H_0^{-1}c_x^{-1}dc_xH_0) = \mathrm{pr}_{m_0}(H_0^{-1}a_x^{-1}da_xH_0).$$

Now, let $a'\colon W(p_0) \to G$ be a representation of $W(p_0)$ with respect to an initial frame $R_0' = (p_0', \ldots, p_s')$ and $a\colon W(p_0) \to G$ a representation with respect to $R_0 = (p_0, \ldots, p_s)$, $R_0' = b_0R_0$, $a(x)R_0 = a'(x)R_0' = R_x$, $R_0' = b_0R_0$. We have

$$a(x) = a'(x)b_0, \qquad H_0 = b_0^{-1}H_0'b_0,$$

where H_0 and H_0' are the isotropy groups at p_0 and p_0', respectively. Thus we get

$$H_0^{-1}a_x^{-1}da_xH_0 = b_0^{-1}H_0'^{-1}a_x'^{-1}da_x'H_0'b_0.$$

For $m_0' = b_0m_0b_0^{-1}$, $h_0' = b_0h_0b_0^{-1}$ and $X = \mathrm{pr}_{m_0}X + \mathrm{pr}_{h_0}X \in q_c$, $X' = b_0Xb_0^{-1}$ we have

$$X' = b_0(\mathrm{pr}_{m_0}X)b_0^{-1} + b_0(\mathrm{pr}_{h_0}X)b_0^{-1}, \qquad \mathrm{pr}_{m_0'}X' = b_0(\mathrm{pr}_{m_0}X)b_0^{-1},$$

since $b_0(\mathrm{pr}_{m_0}X)b_0^{-1} \in m_0'$, $b_0(\mathrm{pr}_{h_0}X)b_0^{-1} \in h_0'$. Hence

$$(4)\qquad \mathrm{pr}_{m_0'}(H_0'^{-1}a_x'^{-1}da_x'H_0') = b_0[\mathrm{pr}_{m_0}(H_0^{-1}a_x^{-1}da_xH_0)]b_0^{-1}.$$

Let us consider a curve $\tau\colon \boldsymbol{R} \to W(p_0)$ in V_n and a vector $t_{x_0} = t_0$ tangent to τ at $x_0 = \tau(0)$. If $a\colon W(p_0) \to G$ is a (local) representation of V_n satisfying $a(p_0) = e$, then $a^\tau\colon s \to (\tau(s)) = a_{\tau(s)}$ is a representation of τ

in G and $d_{t_0}a_{x_0} = da_{x_0}(t_0)$ is the vector tangent to a^τ at a_{x_0}. Now, take the curve $\hat{\tau}$: $s \to s_{x_0}^{-1}\tau(s)$ and its representation $a^{\hat{\tau}}$: $s \to a(\hat{\tau}(s))$. Since $a^{\hat{\tau}}(0) = e \in G$, $a^{\hat{\tau}}(s) \in a(W(p_0))$, it follows that the vector $\hat{t}_0 = a_{x_0}^{-1}t_0$ is tangent to $\hat{\tau}$ at p_0 and the vector $d_{\hat{t}_0}a_e^{\hat{\tau}}$ tangent to $a^{\hat{\tau}}$ at e is contained in the vector space T_e^a tangent to a: $W(p_0) \to G$ at e. We have

$$a^\tau(s)p_0 = \tau(s), \qquad a^{\hat{\tau}}(s)p_0 = \hat{\tau}(s) = a_{x_0}^{-1}\tau(s) = a_{x_0}^{-1}a^\tau(s)p_0.$$

Thus there exists $h(s) \in H_0$ such that $a^{\hat{\tau}}(s)h(s) = a_{x_0}^{-1}a^\tau(s)$ and

$$da_e^{\hat{\tau}} + dh_e = a_{x_0}^{-1}da_{x_0}^\tau,$$

because $h(0) = e$.

Consider the representation b: $W(p_0) \to G$ given by $b(x) \in G_M$, $x \to b(x)R_0 = R_x^M = a(x)k(x)R_0$, $a(p_0) = b(p_0) = k(p_0) = e$, where $G_M = \exp M_0$, $M_0 \subset \boldsymbol{m}_0$, $k(x) \in H_0$. We get

$$b(x) = a(x)k(x), \qquad db_e = da_e + dk_e,$$

and since $d_{\hat{t}_0}b_e \in \boldsymbol{m}_0$, $d_{\hat{t}_0}k_e \in \boldsymbol{h}_0$, we have

$$\mathrm{pr}_{m_0} d_{\hat{t}_0}a_e \neq 0.$$

Therefore each vector $d_{\hat{t}_0}a_e^{\hat{\tau}} \in T_e^a$ is not contained in $\boldsymbol{h}_0$ and $\boldsymbol{q}_e = \boldsymbol{h}_0 + \boldsymbol{T}_e^a$, i.e., $\boldsymbol{T}_e^a$ can be taken as $\boldsymbol{m}_0$. Thus we get

THEOREM 2. *The Lie algebra $\boldsymbol{q}_e$ of G is the direct sum $\boldsymbol{q}_e = \boldsymbol{m}_0 + \boldsymbol{h}_0$, where $\boldsymbol{m}_0$ is the vector space tangent to a representation a: $W(p_0) \to G$ of the homogeneous space (V_n, G), $W(p_0) \subset V_n$, $a(x)R_0 = R_x$, $a(p_0) = e$, and $\boldsymbol{h}_0$ is the Lie algebra of the isotropy group $H_0 \subset G$ at the point p_0.*

If a^τ: $s \to a(\tau(s))$ is the representation of a curve τ: $\boldsymbol{R} \to W(p_0)$ and $a^{\hat{\tau}}$: $s \to a(\hat{\tau}(s))$ the representation of the curve $\hat{\tau}$: $s \to a_{\tau(0)}^{-1}\tau(s)$, then

$$da_e^{\hat{\tau}} = \mathrm{pr}_{m_0}(a_{\tau(0)}^{-1}da_{\tau(0)}^\tau),$$

where $\boldsymbol{m}_0$ is tangent to a: $W(p_0) \to G$ at e.

EXAMPLE 1. Let $G(\mathscr{A}_n)$ be the affine group of transformations of the n-dimensional affine space $\mathscr{A}_n$, given in affine coordinates x^i by means of the equations

$$f_a\colon\ y^i = a_j^i x^j + a^i, \quad i, j = 1, \ldots, n, \ \det(a_j^i) \neq 0.$$

The coordinates $a_\beta^\alpha, \alpha, \beta = 1, \ldots, n+1$, of $f_a \in G(\mathscr{A}_n)$ form the matrix $(a_\beta^\alpha) = a$, $a_{n+1}^i = a^i$, $a_\beta^{n+1} = \vartheta_\beta^{n+1}$, where ϑ_β^{n+1} is the Kronecker symbol. Such matrices form a group G, isomorphic with $G(\mathscr{A}_n)$, when the rule of multiplication in G is given by the formula $(a_\beta^\alpha)(b_\beta^\alpha) = (a_\gamma^\alpha b_\beta^\gamma)$. A frame in $\mathscr{A}_n$ consists of a point x and n linearly independent vectors $v_j = v_j^i e_i$, where e_i is a base of the vector space of free vectors (i.e., equivalence classes with respect to parallel translations of the vectors tangent to $\mathscr{A}_n$). Each frame $R_x = (x, v_1, \ldots, v_n)$ is the image of $R_0 = (0, e_1, \ldots, e_n)$ by f_a, i.e., $R_x = f_a(R_0) = \big(f_a(0), f_a(e_1), \ldots, f_a(e_n)\big)$, where $f_a^i(0) = x^i = a^i$, $f_a^i(e_j) = v_j^i = a_j^i$, and that is why we may write $R_x = aR_0$.

We see that the isotropy subgroup H_0 at the origin 0 is formed by the matrices

$$H_0 = \{(v_\beta^a)\} = \{h\}, \quad v_{n+1}^\beta = v_\beta^{n+1} = \vartheta_{n+1}^\beta, \quad \beta = 1, \ldots, n+1,$$
$$h = (v_1, \ldots, v_n, 0),$$

(i.e., such matrices h that $h(0, e_1, \ldots, e_n) = hR_0 = (0, v_1, \ldots, v_n)$), and the surface $G_M \subset G$ is formed by the matrices

$$G_M = \{(p_\beta^a)\} = \{a_0\}, \quad p_j^i = \vartheta_j^i, \quad p_\beta^{n+1} = \vartheta_\beta^{n+1}, \quad p_{n+1}^i = x^i,$$
$$a_0 = (I_1, \ldots, I_n, x)$$

(i.e., such matrices a_0 that $a_0R_0 = (x, e_1, \ldots, e_n) = R_x^0$). G_M is a subgroup of G; therefore $G_M = \exp \boldsymbol{m}_0$, where $\boldsymbol{m}_0$ is the vector space tangent to G_M at e.

The surface G_M is the representation of the frame field $R_x^0 = (x, e_1, \ldots, e_n) = a_0(x)R_0$, $a_0(x) \in G_M$. For another frame field $R_x = \big(x, c_1^i(x)e_i, \ldots, c_n^i(x)e_i\big) = a_0(x)c(x)R_0$, $c(x) \in H_0$, we obtain another surface a_0': $x \to a_0(x)c(x)$ as a representation of A_n, where $c_j^i = c_j^i(x)$ are functions of points $x \in A_n$. The vector spaces $T_{a_0}(x)$ and $T_{a_0'}(x)$ tangent to a and a', respectively, have the representations in $\boldsymbol{q}_e$

$$a_{0x}^{-1}da_{0x} = dx^iE_i, \quad a_{0x}'^{-1}da_{0x}' = \tilde{c}_j^i(x)dx^jE_i + \tilde{c}_j^i(x)dc_{kx}^jE_i^k, \quad \tilde{c}_k^ic_j^k = \vartheta_j^i,$$

where E_i, E_i^j are the vectors of the natural base in G.

The vector space $\boldsymbol{m}_0$ is spanned by the vectors E_i and $\boldsymbol{h}_0$ by E_i^j. Since

$$H_0^{-1}a_0^{-1}da_0H_0 = \{\tilde{h}_k^idx^kE_i, (h_\beta^a) \in H_0\},$$
$$H_0^{-1}a_0'^{-1}da_0'H_0 = \{\tilde{h}_k^i\tilde{c}_j^kdx^jE_i + \tilde{h}_k^i\tilde{c}_j^kdc_s^jh_p^sE_i^p, (h_\beta^a) \in H_0\},$$
$$\tilde{h}_k^ih_j^k = \vartheta_j^i, \ (c_k^ah_\beta^k) \in H_0,$$

we have

$$\mathrm{pr}_{\boldsymbol{m}_0}(H_0^{-1}a_0^{-1}da_0H_0) = \mathrm{pr}_{\boldsymbol{m}_0}(H_0^{-1}a_0'^{-1}da_0'H_0) = H_0^{-1}dx^iE_iH_0$$
$$= \{\tilde{h}_k^idx^kE_i, (h_\beta^a) \in H_0\}$$

at the point $x \in A_n$.

If we take another initial frame $R_q = (q, w_1, \ldots, w_n) = b_0R_0$, $b_0 = (w_\beta^a)$, $w_i = w_i^je_j$, $w_{n+1}^i = q^i$, $w_\beta^{n+1} = \delta_\beta^{n+1}$, then $R_x^0 = (x, e_1, \ldots, e_n) = a_0(x)R_0 = \hat{a}_0(x)R_q, \hat{a}_0(x)b_0R_0, \hat{a}_0(x) = \big(\hat{a}_\beta^a(x)\big) = a_0(x)b_0^{-1} = (\tilde{w}_j^i, -\tilde{w}_k^iq^k + x^i)$. Thus

$$H_q^{-1}\hat{a}_0^{-1}d\hat{a}_0H_q = H_q^{-1}w_j^idx^jE_iH_q = \{w_j^i\tilde{h}_k^jdx^kE_i, (h_\beta^a) \in H_0\}$$
$$= b_0H_0^{-1}a_0^{-1}da_0H_0b_0^{-1},$$

because $H_q = b_0H_0b_0^{-1}$.

In this case we have $\hat{\boldsymbol{m}}_0 = b_0 \boldsymbol{m}_0 b_0^{-1} = \boldsymbol{m}_0$, $H_0^{-1} a_0^{-1} da_0 H_0 \subset \boldsymbol{m}_0$, $H_a^{-1} \hat{a}_0^{-1} d\hat{a}_0 H_a \subset \boldsymbol{m}_0$.

2. A specialization of a frame. Let a: $W(p_0) \to G$ be a local representation of a homogeneous space (V_n, G), $W(p_0) \subset V_n$, and let S: $V_l \to V_n$, $l < n$, be an immersion (surface) of a manifold V_l (l-dimensional) into V_n. We take a local chart (U, Φ) on V_l such that $S(U) \subset W(p_0)$, where $(W(p_0), \hat{\Phi})$ is a local chart on V_n containing p_0. Then a_0: $U \to G$ given by the formula $a_0(u) = a(S(u))$ is a representation of the surface S. If $g_0 \in G$, then $g_0 a_0$ is a representation of the surface $g_0 S$ in G. For each point $u_0 \in V_l$ there exist a neighbourhood $U \subset V_l$, a point $u_0 \in V_l$ and $g_0 \in G$ such that $g_0 S(U) \subset W(p_0)$. Since the process of specialization of a frame is constructed in a way invariant with respect to left translations, it follows that using local representations we can specialize a frame on the whole surface S, i.e., on a neighbourhood of each point of V_l.

If v_u is a vector tangent to V_l at u, then $d_{v_u} S_u = dS_u(v_u)$ is a vector tangent to S at $S(u)$ and $d_{v_u} a_{0u} = da_{0u}(v_u)$ is tangent to the representation a_0: $U \to G$ at $a_0(u)$. In this way we get the mapping

$$a_0^{-1} d_v a_0 \colon\ U \to \boldsymbol{q}_e, \qquad a_0^{-1} d_v a_0 \colon\ u \to a_0^{-1}(u)^{-1} d_{v_u} a_{0u}, \qquad a_0(u) = a_{0u},$$

and the set

$$H_0^{-1} a_0(u)^{-1} d_{v_u} a_{0u} H_0 = \{h_0^{-1} a_{0u}^{-1} d_{v_u} a_{0u} h_0,\ h_0 \in H_0\},$$

where H_0 is the isotropy subgroup of G at $p_0 \in V_n$ and v is a vector field on $U \subset V_l$. In Section 1 we proved that $\mathrm{pr}_{\boldsymbol{m}_0}(h_0^{-1} a_{0u}^{-1} d_{v_u} a_{0u} h_0) \neq 0$ for $d_{v_u} a_{0_u} \neq 0$.

Our method of specialization of a frame consists in the following general idea. Let $\boldsymbol{q}_e = \boldsymbol{h}_0 + \boldsymbol{m}_0$ be the direct sum of vector spaces $\boldsymbol{h}_0$ and $\boldsymbol{m}_0$, where $\boldsymbol{h}_0$ is the Lie algebra of a closed subgroup H_0 of the Lie group G and $\boldsymbol{q}_e$ is the Lie algebra of G. The subgroup H_0 acts on $\boldsymbol{m}_0$ as the group of transformations h_0: $Y_e \to \mathrm{pr}_{\boldsymbol{m}_0}(h_0^{-1} Y_e h_0)$, $Y_e \in \boldsymbol{m}_0$. If b: $U \to G$ is a cross section in G (i.e., $\pi \circ b$; $U \to G/H_0$ is an imbedding, where π: $G \to G/H_0$ is the natural projection) and $Y_{b(u)}$ is a vector tangent to b at $b(u)$, then the vector $b(u)^{-1} Y_{b(n)}$ is contained in $\boldsymbol{q}_e$. We consider the orbit of $b(u)^{-1} Y_{b(u)}$ in $\boldsymbol{m}_0$, i.e.,

$$\mathrm{pr}_{\boldsymbol{m}_0}\big(H_0^{-1} b(u)^{-1} Y_{b(u)} H_0\big) = \big\{\mathrm{pr}_{\boldsymbol{m}_0}\big(h_0^{-1} b(u)^{-1} Y_{b(u)} h_0\big),\ h_0 \in H_0\big\}.$$

In order to distinguish in H_0 a subgroup and an element $h_0(u)$, we put on the set $\mathrm{pr}_{\boldsymbol{m}_0}\big(H_0^{-1} b(u)^{-1} Y_{b(u)} H_0\big)$ an invariant condition, f.i. we choose a fixed subspace $\boldsymbol{P} \subset \boldsymbol{m}_0$ or a fixed vector $Y_e \in \boldsymbol{m}_0$ and take such $h_0(u) \in H_0$ that

$$h_0(u)^{-1} b(u)^{-1} Y_{b(u)} h_0(u) \in \boldsymbol{P} + \boldsymbol{h}_0 \quad \text{(i.e., } \mathrm{pr}_{\boldsymbol{m}_0}\big(h_0(u)^{-1} b(u)^{-1} Y_{b(u)} h_0(u)\big) \in \boldsymbol{P}),$$

or

$h_0(u)^{-1}b(u)^{-1}Y_{b(u)}h_0(u) \in Y_e+\boldsymbol{h}_0 \qquad (\mathrm{pr}_{\boldsymbol{m}_0}(h_0(u)^{-1}b(u)^{-1}Y_{b(u)}h_0(u) = Y_e)),$

respectively, for all $u \in U$.

Then the vector field $g \to gY_{b(u)}h(u)$, $g \in G$, is contained in the left invariant distribution $g \to g(\boldsymbol{P}+\boldsymbol{h}_0)$ or in the left invariant set field $g \to g(Y_e+\boldsymbol{h}_0)$ defined by the left invariant vector field and the left invariant distribution for all $u \in U$. We have got $h_0(u) \in H_0$ by the restriction of $H_0^{-1}b(u)^{-1}Y_{b(u)}H_0$ to the set $\boldsymbol{P}+\boldsymbol{h}_0$ or $Y_e+\boldsymbol{h}_0$, which is also a left invariant condition.

A subgroup $H_1 \subset H_0$ can be obtained from the condition (as maximal set)

$$H_1^{-1}(\boldsymbol{P}+\boldsymbol{h}_0)H_1 = \boldsymbol{P}+\boldsymbol{h}_0 \quad \text{or} \quad H_1^{-1}Y_eH_1 \subset Y_e+\boldsymbol{h}_0.$$

Thus the surface

$$b_1\colon u \to b_1(u) = b_0(u)h_0(u), \quad b_0(u) = b(u),$$

is determined by means of left invariant conditions and we may repeat with respect to b_1, H_1 the same which has been done with respect to a_0, H_0. We get b_2, H_2 and so on. When we need no constant subspace $\boldsymbol{P}_k$ to determine H_{k+1}, then we may take $P_k \subset \boldsymbol{q}_e$ as a function of $u \in U$, i.e., we use the restriction of $H_k^{-1}b_k(u)Y_{b_k}^{(k)}(u)H_k$ to $\boldsymbol{P}_k(u)$ having possibly small dimension as an invariant condition.

There exist many different variants of this idea, but we present two of them in details.

(A) Let $a_0\colon U \to G$ be a representation of a surface $S_U\colon U \to V_n$, $U \subset V_l$, let T_u be the vector space tangent to V_l at u, $T_{a_0(u)}$ the space tangent to a_0 at $a_0(u)$, and let $\boldsymbol{q}_e = \boldsymbol{h}_0+\boldsymbol{m}_0$ be the decomposition considered above, where $\boldsymbol{h}_0$ is the Lie algebra of the isotropy subgroup H_0 at $p_0 \in W(p_0)$. We put on H_0 the condition:

$$(5_0) \quad \mathrm{pr}_{\boldsymbol{m}_0}(H_0^{-1}a_{0u}^{-1}T_{a_0(u)}H_0) \subset \boldsymbol{P}_{n_0} \subset \boldsymbol{m}_0 \quad (\text{i.e., } H_0^{-1}a_{0u}^{-1}T_{a(u)}H_0 \subset \boldsymbol{P}_{n_0}+\boldsymbol{h}_0)$$

for all $u \in U$. This condition may be written in a more explicit form. Namely, if ω_u is a linear form on T_u with values in $\boldsymbol{q}_e$ and if $v_u \in T_u$, then $\omega_u \subset \boldsymbol{P}_{n_0}$ means $\omega_u(v_u) \in P_{n_0}$ for all $v_u \in T_u$. Since

$$a_{0u}^{-1}T_{a_0(u)} = \{a_{0u}^{-1}da_{0u}(v_u),\ v_u \in T_u\}, \quad a_{0u} = a_0(u),\ da_{0u}(v_u) = d_{v_u}a_{0u},$$

the above condition may be written in the form

$$(5) \quad \mathrm{pr}_{\boldsymbol{m}_0}(H_0^{-1}a_{0u}^{-1}da_{0u}H_0) \subset \boldsymbol{P}_{n_0}, \quad \text{i.e.,} \quad H_0^{-1}a_{0u}^{-1}da_{0u}H_0 \subset \boldsymbol{P}_{n_0}+\boldsymbol{h}_0,$$
$$\boldsymbol{P}_{n_0} \subset \boldsymbol{m}_0,$$

where $\boldsymbol{P}_{n_0}+\boldsymbol{h}_0$ is the direct sum of the vector subspaces P_{n_0} and $\boldsymbol{h}_0$ of $\boldsymbol{q}_e$.

It follows from (3) that (5) does not depend on the choice of the representation a_0 of S.

If there exists an $h_0(u) \in H_0$ and a subgroup $H_1 \subset H_0$ such that

$$(5_1)\qquad \mathrm{pr}_{m_0}\big(h_0(u)^{-1}a_{0u}^{-1}da_{0u}h_0(u)\big) \in P_{n_0} \quad \text{and} \quad \mathrm{pr}_{m_0}(H_1^{-1}P_{n_0}H_1) = P_{n_0},$$

then we get the surface $a_1(u) = a_0(u)h_0(u)$ which is also a representation of S.

Assume that there exists another $\hat{h}_0(u) \in H_0$ satisfying (5_1). We take such $k(u) \in H_0$ that $\hat{h}_0(u) = h_0(u)k(u)$. Then

$$\begin{aligned}\mathrm{pr}_{m_0}[k(u)^{-1}h_0(u)^{-1}a_{0u}^{-1}da_{0u}h(u)k(u)]&\\ &= \mathrm{pr}_{m_0}\big[k(u)^{-1}\big[\mathrm{pr}_{m_0}\big(h_0(u)^{-1}a_{0u}^{-1}da_{0u}h(u)\big)\big]k(u)\big]\\ &\subset \mathrm{pr}_{m_0}[k(u)^{-1}P_{n_0}k(u)] \subset P_{n_0}\end{aligned}$$

and $k(u) \in H_1$, $\hat{h}_0(u)H_1 = h_0(u)H_1$. Thus the set $a_0(u)h_0(u)H_1$ is independent on the choice of $h_0(u)$ satisfying (5_1) and for $H_1 = \{e\}$ we have $\hat{h}_0(u) = h_0(u)$.

Now, let a_0' be another representation of S with respect to the same initial frame R_0. If a_0' and $h_0'(u)$ satisfy (5_1) and $a_0'(u) = a_0(u)b(u)$, $b(u) \in H_0$, then

$$\mathrm{pr}_{m_0}[h_0'(u)^{-1}a_{0u}'^{-1}da_{0u}'h_0'(u)] = \mathrm{pr}_{m_0}[h_0'(u)^{-1}b(u)^{-1}a_{0u}^{-1}da_{0u}b(u)h_0'(u)] \subset P_{n_0}$$

and $b(u)h_0'(u) = \hat{h}_0(u)$ satisfies (5_1). Hence

$$(6)\qquad \begin{aligned} b(u)h_0'(u)H_1 = h_0(u)H_1 \quad \text{and} \quad & a(u)b(u)h_0'(u)H_1 = a(u)h_0(u)H_1\\ \text{or} \quad & a_0'(u)h_0'(u)H_1 = a_0(u)h_0(u)H_1.\end{aligned}$$

Thus we get

THEOREM 3. *The set $a_0(u)h_0(u)H_1$ satisfying (5_1) is independent of the choice of $h_0(u) \subset H_0$ and of the choice of a representation a_0 of the surface S_U: $U \to V_n$, $U \subset V_l$, with respect to a fixed frame R_0 (the same for all representations considered).*

This is the important fact because now, as the second step in this process, we repeat everything for the surface a_1: $u \to a_0(u)h_0(u)$ and the subgroup H_1 which we have constructed at the first step starting from the surface a_0 and the subgroup H_0. In this way we get the sequence:

I. $a_1(u) = a_0(u)h_0(u)$, $\mathrm{pr}_{m_1}[H_1^{-1}a_{1u}^{-1}da_{1u}H_1] \subset P_{n_1} \subset m_1$; $m_0 \subset m_1$, $q_e = h_1+m_1$, $\mathrm{pr}_{m_1}[h_1(u)^{-1}a_{1u}^{-1}da_{1u}h_1(u)] \subset P_{n_1}$, $\mathrm{pr}_{m_1}(H_2^{-1}P_{n_1}H_2) = P_{n_2}$, $h_1(u) \in H_1$, where h_1 is the Lie algebra of H_1, P_{n_1} is a subspace of m_1 having possibly small dimension and H_2 is a subgroup of H_1;

II. $a_2(u) = a_1(u)h_1(u)$, $\mathrm{pr}_{m_2}[H_2^{-1}a_{2u}^{-1}da_{2u}H_2] \subset P_{n_2} \subset m_2$, $m_1 \subset m_2$, $q_e = h_2+m_2$, $\mathrm{pr}_{m_2}[h_2(u)^{-1}a_{2u}^{-1}da_{2u}h_2(u)] \subset P_{n_2}$, $\mathrm{pr}_{m_2}[H_3^{-1}P_{n_2}H_3] = P_{n_2}$, $h_2(u) \in H_2$;

III. $a_3(u) = a_2(u)h_3(u), \ldots$

The specialization of a frame will be finished when we get such a sequence $H_0, H_1, \ldots, H_k$ that $K_{k+1} = \{e\}$.

Here we take $\boldsymbol{P}_i = \text{const}$ or $\boldsymbol{P}_i = \boldsymbol{P}_i(u)$, $i = 1, \ldots, k-1$, in order to get the maximal subgroup H_{i+1} satisfying $H_{i+1}^{-1}(\boldsymbol{P}_i+\boldsymbol{h}_i)H_{i+1} = \boldsymbol{P}_i+\boldsymbol{h}_i$. Thus if $H_{k+1} = \{e\}$, we may take $\boldsymbol{P}_k$ as a function of u, $\boldsymbol{P}_k = \boldsymbol{P}_k(u)$.

Let $E_1, \ldots, E_r$ be a fixed base of $\boldsymbol{q}_e$ and let $E_1, \ldots, E_{n_i}$ be these vectors of this base on which the subspace $\boldsymbol{m}_i$ is spanned. Since

$$\mathrm{pr}_{\boldsymbol{m}_i}[H_{i+1}^{-1}h_i(u)^{-1}a_i(u)^{-1}da_{iu}h_i(u)H_{i+1}] \subset \boldsymbol{P}_{n_i},$$

it follows that

$$\mathrm{pr}_{\boldsymbol{m}_i}[h_{i+1}^{-1}h_i(u)^{-1}a_{iu}^{-1}da_{iu}h_i(u)h_{i+1}] = f^j(u, h_{i+1})E_j, \qquad j = 1, \ldots, n_i.$$

The forms f^j (or a function of these forms) independent on h_{i+1} are invariant forms of the surface S_U with respect to the action of G.

If the subgroup H_i is such that

$$\mathrm{pr}_{\boldsymbol{m}_i}(H_i^{-1}\boldsymbol{Q}_lH_i) = \boldsymbol{Q}_l \quad \text{for every an } l\text{-dimensional subspace } \boldsymbol{Q}_l \subset \boldsymbol{m}_i,$$

then this variant is stopped at this step and the specialization of a frame cannot be realized by means of this method.

The same happens in the case where

$$\mathrm{pr}_{\boldsymbol{m}_i}(a_i^{-1}da_i) \subset \boldsymbol{P}_l \quad \text{and} \quad \mathrm{pr}_{\boldsymbol{m}_i}(H_i^{-1}\boldsymbol{P}_lH_i) = \boldsymbol{P}_l$$

for a fixed l-dimensional $\boldsymbol{P}_l \subset \boldsymbol{m}_i$.

The described method of specialization of a frame is also invariant with respect to a change of an initial frame R_0 for $R_0' = b_0R_0$. Then $a_0'(u) = a_0(u)b_0^{-1}$ is another representation of S_U and $H_0' = b_0H_0b_0^{-1}$ is the isotropy subgroup at $p_0' = b_0p_0$. If we take $\boldsymbol{m}_0' = b_0\boldsymbol{m}_0b_0^{-1}$ and $\boldsymbol{P}_{n_0}' = b_0\boldsymbol{P}_{n_0}b_0^{-1}$, then $h_0'(u) = b_0h_0(u)b_0^{-1} \in H_0'$ satisfies

$$\begin{aligned}\mathrm{pr}_{\boldsymbol{m}_0'}&[h_0'(u)^{-1}a_0'(u)^{-1}da_{0u}'h_0'(u)] \\ &= \mathrm{pr}_{\boldsymbol{m}_0'}[b_0h_0(u)^{-1}a_{0u}^{-1}da_{0u}h_0(u)b_0^{-1}] \subset b_0\boldsymbol{P}_{n_0}b_0^{-1} = \boldsymbol{P}_{n_0}'\end{aligned}$$

and

$$a_1'(u) = a_0'(u)h_0'(u) = a_0(u)h_0(u)b_0^{-1} = a_1(u)b_0^{-1}$$

or

$$a_1'(u)R_0' = a_1(u)R_0.$$

Repeating these considerations we get

$$a_i'(u)R_0' = a_i(u)R_0 \quad \text{for } i = 0, 1, \ldots, k, \tag{7}$$

and

THEOREM 4. *The frame field $u \to a_i(u)R_0$ on S_U, where $a_i\colon V \to G$ is a cross section obtained from the representation $a_0\colon U \to G$ by the method presented above, is independent on the choice of an initial frame R_0.*

If we apply successively Theorem 3 with respect to the cross sections $a_0, a_1, \ldots, a_k$, $a_k = a_0 h_0 h_1 \ldots h_{k-1}$, $H_{k+1} = \{e\}$, then $a_{k+1}(u) = a_0(u) h_0 \times \times (u) h_1(u) \ldots h_k(u)$ depends only on the choice of the sequence $\boldsymbol{P}_{n_0}, \ldots, \boldsymbol{P}_{n_k}$. Theorems 3 and 4 say that the final frame field $u \to a_{k+1}(u) R_0$, obtained by means of this process, depends only on the sequence $\boldsymbol{P}_{n_0}, \ldots, \boldsymbol{P}_{n_k}$ of subspaces of $\boldsymbol{q}_e$.

EXAMPLE 2. In the Euclidean space $\mathscr{E}_2$, let x^i be the orthonormal coordinates and $R_0 = (0, e_1, e_2)$ be an orthonormal frame, where 0 is the origin of the coordinate system and e_i are the vectors of the base in $\mathscr{E}_2$. The isometry group is given by the equations F_a: $y^i = a^i_j x^j + a^i$, i.e., in these coordinates the isometry group is isomorphic with the group G of matrices $a = (a^\alpha_\beta)$, where $a^i_3 = a^i$, $a^3_\beta = \vartheta^3_\beta$, and (a^i_j) is an orthonormal matrix, $\alpha, \beta = 1, 2, 3$; $i, j = 1, 2$.

We consider the field $x = (x^1, x^2) \to R_x = (x, e_1, e_2) = a(x) R_0$ and the representation

$$a\colon x \to \big(a^\alpha_\beta(x)\big), \quad a^i_j(x) = \vartheta^i_j, \quad a^i(x) = a^i_3(x) = x^i, \quad a^3_\beta = \vartheta^3_\beta,$$

of $\mathscr{E}_2$ in the group G. Thus the representation of a curve S: $x^i = x^i(u)$ is given as a curve a_0: $u \to \big(a^\alpha_\beta(x(u))\big) = a_0(u)$ in G, $u \in \boldsymbol{R}$.

We denote by E^i_j, $E^i_j = -E^j_i$, E_j the vectors of the natural base at $e \in G$ and by $h_0 = (h^\alpha_\beta) \in G$, $h^i_3 = 0$, the elements of isotropy subgroup H_0 at 0. Hence the Lie algebra $\boldsymbol{h}_0$ of H_0 is spanned on E^1_2 and we take as $\boldsymbol{m}_0$ the vector space spanned on E_1, E_2. We have

$$a_0(u)^{-1} da_{0u} = dx^i_u E_i, \quad h_0^{-1} a_{0u}^{-1} da_{0u} h_0 = \tilde{h}^i_j dx^j_u E_i \in \boldsymbol{m}_0, \quad \tilde{h}^i_k h^k_j = \vartheta^i_j.$$

Let us take the subspace of $\boldsymbol{m}_0$ spanned on E_2 as $\boldsymbol{P}_1$. Then the condition $\tilde{h}^i_j dx^j E_i \subset \boldsymbol{P}_1$ for H_0 has the form

$$\tilde{h}^1_j dx^j = 0, \quad \text{i.e.,} \quad \vartheta_{ij} h^i_1 dx^j = 0.$$

Thus the vector (column) $\big(h^i_1(u)\big)$ is orthogonal to the column (dx^i_u). Since (h^i_2) is orthogonal to (h^i_1), hence we may take $\big(h^1_2(u), h^2_2(u)\big) = (dx^1_u/|dx|_u, dx^2_u/|dx|_u)$, where $|dx| = (\vartheta_{ij} dx^i dx^j)^{1/2} = ds$. Now $h_{0u}^{-1} a_{0u}^{-1} da_{0u} h_{0u} = ds\, E_2$ and $H_1 = \{e\}$, hence ds is an invariant form.

The curve a_1: $u \to a_1(u) = a_0(u) h_0(u) = \big(h^\alpha_1(u), h^\alpha_2(u), x^\alpha(u)\big)$, $x^3(u) = 1$, $h^i_2(u) = dx^i(u)/ds$, is a representation of the specialized frame field $u \to R_u = a_1(u) R_0$ because $H_1 = \{e\}$. The components of the form

$$\begin{aligned} a_1^{-1} da_1 &= h^i_k dh^k_j E^j_i + h^i_k dx^k E_i = 2(h^2_2 dh^1_2 - h^1_2 dh^2_2) E^1_2 + ds\, E_2 \\ &= 2\left(\frac{dx^1}{ds}\frac{d^2x^2}{ds^2} - \frac{dx^2}{ds}\frac{d^2x^1}{ds^2}\right) E^1_2 ds + ds\, E_2 = 2k\, ds\, E^1_2 + ds\, E_2, \end{aligned}$$

where k is the curvature of S, are invariant forms of S. If $k = \text{const}$, then all vectors $a_1^{-1} da_1(v)$ are contained in the constant subspace spanned on the vectors $2kE_2^1+E_2$. Thus the surface $u \to a_1(u_0)^{-1}a_1(u)$ is a 1-dimensional subgroup tangent to $2kE_2^1+E_2$ at e because it is tangent to the left invariant vector field generated by $2kE_2^1+E_2$. That means that, in the case $k = \mathrm{const}$, the frame field $u \to a_1(u_0)^{-1}a_1(u)R_0$ (and the curve $a_1(u_0)^{-1}S$) is invariant with respect to this subgroup.

EXAMPLE 3. Let $\mathscr{E}_{n+1}$ be the $(n+1)$-dimensional Euclidean space. We repeat the considerations of Examples 1 and 2. Thus we get the group G of matrices $a = (a_\beta^\alpha)$, $a_\beta^{n+2} = \vartheta_\beta^{n+2}$, $a_{n+2}^i = a^i$, $\vartheta_{ij}a_k^i a_l^j = \vartheta_{kl}$, $i, j, k, l = 1, \ldots, n+1$; $\alpha, \beta = 1, \ldots, n+2$.

A surface $S_U\colon U \to \mathscr{E}_{n+1}$, where $U \subset V_n$ is a chart on a differentiable manifold V_n and S_U is a restriction of $S\colon V_n \to \mathscr{E}_{n+1}$, has a representation $a_0\colon u \to a(S(u))$, $a_0(u) = (a_\beta^\alpha(u))$, $a_j^i(u) = \vartheta_j^i$, $a_{n+2}^i(u) = x^i(u)$.

Identically as in Example 2 we get

$$h_0^{-1}a_0^{-1}da_0h_0 = \tilde{h}_j^i dx^j E_i,$$

and if we take P_1 spanned on $E_1, \ldots, E_n$, then

$$\tilde{h}_j^{n+1}dx^j = 0,$$

i.e., the vector (column) $(h_{n+1}^i(u))$ of $h_0(u)$ must be orthogonal to dx. We take such functions $X_j^i\colon u \to X_j^i(u)$ that for $X_{n+1}^i(u) = h_{n+1}^i(u)$ the matrix $(X_j^i(u))\big((X_\beta^\alpha(u)) = h_0(u),\ X_{n+2}^\beta(u) = X_\beta^{n+2}(u) = \vartheta_\beta^{n+2}\big)$ is orthogonal. Now we have

$$a_1(u) = a_0(u)h_0(u) = (X_1(u), \ldots, X_{n+1}(u), x(u)),$$

where $X_j(u) = (X_j^\alpha(u))$ denotes a column.

The subgroup $H_1 \subset H_0$, where $H_1^{-1}\boldsymbol{P}_1H_1 = \boldsymbol{P}_1$, contains the orthogonal matrices $h_1 = (k_\beta^\alpha)$, where $k_j^{n+1} = k_{n+1}^j = \vartheta_j^{n+1}$, $k_\beta^{n+2} = k_{n+2}^\beta = \vartheta_\beta^{n+2}$. Hence

$$h_1^{-1}a_1^{-1}da_1h_1 = \tilde{k}_p^i\tilde{X}_s^p dX_r^s h_j^r E_i^j + \tilde{k}_p^i\tilde{X}_j^p dx^j E_i, \qquad \tilde{X}_p^i X_j^p = \vartheta_j^i, \qquad \tilde{k}_p^i k_j^p = \vartheta_j^i.$$

We put

$$\text{(a)} \quad \tilde{k}_B^A\tilde{X}_i^B dx_j^i k_{n+1}^j = \lambda^A\tilde{k}_B^A\tilde{X}_j^B dx^j, \quad \text{i.e.,} \quad \tilde{k}_B^A\tilde{X}_j^B dX_{n+1}^j = \lambda^A\tilde{k}_B^A\tilde{X}_j^B dx^j;$$
$$A, B = 1, \ldots, n,\ \lambda^A \in \boldsymbol{R};$$

this is possible because h_1 is spanned on E_B^A and m_1 is spanned on $E_1, \ldots, E_{n+1}, E_{n+1}^1, \ldots, E_{n+1}^n$. The above condition means that we take $\boldsymbol{P}_{n_1}(u)$ spanned on $\lambda^1(u)E_1^{n+1}+E_1, \ldots, \lambda^n(u)E_n^{n+1}+E_n$. Since $\mathrm{pr}_{m_1}(h_1^{-1}a_{1u}^{-1}d_v a_{1u}h_1)$ takes its values in the $2n$-dimensional subspace spanned on $E_1, \ldots, E_n, E_{n+1}^1, \ldots, E_{n+1}^n$, then condition (a) gives the restriction upon H_1 that these values are contained in the n-dimensional space $P_{n_1}(u)$ at each $u \in U$.

We introduce the linear forms given by $dX^i_j = X^i_k \omega^k_j$, $dx^i = X^i_j \omega^j$. Then (a) takes the form

$$\tilde{k}^A_B \omega^B_{n+1} = \lambda^A \tilde{k}^A_B \omega^B. \tag{b}$$

Using Cartan's lemma $(\vartheta_{ij}\omega^i_{n+1} \wedge \omega^j = 0)$, we get $\omega^B_{n+1} = f^B_A \omega^A$, where f^B_A are symmetric with respect to A and B. Hence from (b) we get

$$\tilde{k}^A_B f^B_C = \lambda^A \tilde{k}^A_C \quad \text{or} \quad f^C_B k^B_A = \lambda_A k^C_A, \quad \lambda_A = \lambda^A. \tag{c}$$

We see that the column k_A is an eigen-vector and λ_A is a (characteristic) eigen-value of the matrix (f^A_B). If the matrix $(f^A_B(u))$ has n different eigen-values for each u, then there exist n orthonormal different eigen-vectors $k_A(u)$ and the specialization of a frame is finished. Then $a_2(u) = a_1(u)h_1(u)$, $h_1(u) = (k^\alpha_\beta(u))$ and $R_{x(u)} = a_2(u)R_0$, $x(u) = S(u)$, is an invariant frame field on a surface $S_U\colon U \to \mathscr{E}_{n+1}$.

The components of the vector form

$$\tilde{h}^{-1}_{1u} a^{-1}_{1u} da_{1u} h_{1u} = \tilde{k}^A_B(u)\,\omega^B_{Cu} k^C_{Du} E^D_A + \tilde{k}^A_{Bu}\omega^B_u \lambda^A_u 2E^{n+1}_A + \tilde{k}^A_{Bu}\omega^B_u E_A,$$

$$E^i_j = -E^j_i,$$

are second rank invariant forms of a surface $S\colon V_n \to \mathscr{E}_{n+1}$. The functions λ^A are the coefficients of proportionality for the pairs of invariant forms; hence they are invariant scalar functions.

Let $v_{Au} \in T_u(V_n)$ be vectors such that $dx^i(v_A)e_i = X^i_A e_i = \hat{X}_A$. Then $\omega^A(v_B) = \vartheta^A_B$ and it is easy to verify that for $\hat{v}_A = v_B k^B_A$ we have

$$dX^i_{n+1}(\hat{v}_A) = \lambda_A dx^i(\hat{v}_A).$$

i.e., the vector $dx^i(\hat{v}_A)e_i$ has the principal direction on the surface $S\colon V_n \to \mathscr{E}_{n+1}$ and λ_A is the principal curvature of this surface.

The case when the matrix $(f^A_B(u))$ has p, $p < n$, different eigen-values will not be considered here.

The form $h^{-1}_1 a^{-1}_{0u} da_{0u} h_1 = \tilde{k}^A_B \omega^B E_A$ gives

$$\vartheta_{AC} \tilde{k}^A_B \omega^B \tilde{k}^C_D \omega^D = \vartheta_{BD}\omega^B\omega^D = ds^2$$

as the first rank invariant quadratic form.

The calculation of the third rank invariants is left to the reader.

EXAMPLE 4. We repeat the considerations of Example 1 for the subgroup of the affine group given by the matrices $a = (a^\alpha_\beta)$, $a^i_j = \lambda\vartheta^i_j$, $\alpha, \beta = 1, \ldots, n+1$; $i, j = 1, \ldots, n$. Now for each 1-dimensional subspace $\boldsymbol{P}_{n_0} \subset \boldsymbol{m}_0$ we have $H^{-1}_0 \boldsymbol{P}_{n_0} H_0 = \boldsymbol{P}_{n_0}$; thus the specialization of a frame is not possible by means of our method.

(B) We now present a second variant of the given idea of the specialization of a frame on a surface $S\colon V_l \to V_n$ in a homogeneous space (V_n, G). All notation is the same as in variant (A).

The main difference between this variant and that given in (A) is that here we specialize a frame with respect to certain vector fields, i.e., for given vector fields $S_1, \ldots, S_p$; $S_i = dS(v_i)$, $v_i \in T(V_l)$, on $S_U\colon U \to V_n$, $U \subset V_l$, $S_U(u) = S(u)$, we find a certain frame field determined invariantly by S and G.

Let $a_0\colon u \to a(x(u)) = a_0(u)$, $x(u) = S(u)$, be a representation of a surface S in G. The differential da_{0u} maps a vector $v_u = v(u) \in T_u(V_l)$ onto a vector $d_v a_{0u} = da_{0u}(v_u)$ tangent to a_0 at $a_0(u)$. Thus we get a vector $a_{0u}^{-1} d_v a_{0u}$ in $\boldsymbol{q}_e$ and a set

$$\mathrm{pr}_{\boldsymbol{m}_0}(H_0^{-1} a_{0u}^{-1} d_v a_{0u} H_0) = \{\mathrm{pr}_{\boldsymbol{m}_0}(h_0^{-1} a_{0u}^{-1} d_v a_{0u} h_0);\ h_0 \in H_0\},$$

where $\boldsymbol{q}_e = \boldsymbol{m}_0 + \boldsymbol{h}_0$.

If $E_1, \ldots, E_n$ form a fixed base of $\boldsymbol{m}_0$, then

$$\mathrm{pr}_{\boldsymbol{m}_0}\big(h_0^{-1} a_{0u}^{-1} da_{0u}(v_u) h_0\big) = f^i(h_0, v_u, u) E_i, \quad i = 1, \ldots, n.$$

We find $h_0 = h_0(u, v_u)$ so as to obtain possibly great number of components f^i equal to constant numbers for all $u \in U$, i.e., we take such vectors $d_v a_{0u} h_0(u, v_u)$ that their components f^i, $i = 1, \ldots, p$, with respect to the left invariant base field $g \to gE_\beta$, $\beta = 1, \ldots, r$, $E_\beta \in q_e$, are constant along $u \to a_0(u) h_0(u, v_u)$. This condition is invariant with respect to the action of the group G on S. If there exists exactly one such $h_0(u, v_u)$, then the specialization of a frame is finished and the frame field $u \to a_1(u) R_0 = a_0(u) h_0(u, v_u) R_0$ is invariant for the fixed vector field v. In this case the components of the vector $a_1^{-1} d_w a_1$ with respect to the base E_β are invariant functions depending on some vector fields v and w; the components $f^{p+1}\big(h_0(u, v_u), u\big), \ldots, f^n\big(h_0(u, v_u), u\big)$ are also invariant functions.

If for $j = 1, \ldots, p$ all the vectors $f^1\big(h_0(u, v_u), u\big) E_1, \ldots, f^p\big(h_0(u, v_u), u\big) E_p$ are constant and there exists a maximal subgroup H_{01} of H_0 satisfying $\mathrm{pr}_{\boldsymbol{m}_0}(H_{01}^{-1} A_1 H_{01}) = A_1$, where $A_1(u) = f^i\big(h_0(u, v_u), u\big) E_i$, $i = 1, \ldots, n$, then we take a vector field v_2 on V_l and we repeat with respect to v_2 and H_{01}, $\boldsymbol{m}_0$ all the same what we have done for $v = v_1$, H_0, $\boldsymbol{m}_0$. That means, we find such $h_{01}(u) = h_{01}(u, v_{1u}, v_{2u}) \in H_{01}$ that $\mathrm{pr}_{\boldsymbol{m}_0}\big(h_{01}(u)^{-1}\big)$, $\mathrm{pr}_{\boldsymbol{m}_0}\big(h_{01}(u)^{-1} a_{0u}^{-1} d_{v_2} a_{0u} h_{01}(u)\big)$ easy possibly great number of components equal to constant numbers (i.e., independing of u). In such a way we get vector fields $v_1, \ldots, v_k$ and vectors $A_1, \ldots, A_k$, $k \leqslant l$, $l = \dim V_l$. Naturally, the vectors $v_1, \ldots, v_k$ must be chosen so that this process can be realized, i.e., the vectors $\mathrm{pr}_{\boldsymbol{m}_0}(a_{0u}^{-1} d_{v_j} a_{0u})$, $j = 1, \ldots, k$, must form a part of a frame of the group of transformations ad h_0^{-1}: $W \to \mathrm{pr}_{\boldsymbol{m}_0}(h_0^{-1} W h_0)$, $W \in \boldsymbol{m}_0$, $h_0 \in H_0$, at each $u \in U$.

It is easy to see that we can get the same when for $v_1, \ldots, v_k$ and $A_1, \ldots, A_k$ we find an $h_0(u) \in H_0$ such that

$$\mathrm{pr}_{\boldsymbol{m}_0}\big(h_0(u)^{-1} a_{0u}^{-1} da_{0u} h_0(u)\big) = A_j, \quad j = 1, \ldots, k.$$

Assume that there esists such $h_0(u) = h_{01}(u)h_{02}(u)\ldots h_{0l}(u)$ and $H_1 = H_{0l} \subset H_0$ that $\mathrm{pr}_{m_0}(h_0(u)^{-1}a_{0u}^{-1}da_{0u}h_0(u)) = A_j$ and $\mathrm{pr}_{m_0}(H_1^{-1}A_jH_1) = A_j$, respectively, for $j = 1, \ldots, l$, where A_j are linearly independent vectors. Then we take the cross section $a_1(u) = a_0(u)h_0(u)$ and some vector fields $w_1, \ldots, w_l$ on V_l and we repeat with respect to a_1 and w_j all the same what we have done for a_0 and v_j. In such a way we get some sequences $H_0, H_1, \ldots, H_s$, $H_{s+1} = \{e\}$; $h_0(u, v_1(u), \ldots)$, $h_1(u, v_1(u), \ldots \ldots, w_1(u), \ldots), \ldots, h_j(u, v_1(u), \ldots, w_1(u), \ldots, z_1(u), \ldots)$; $a_0(u), a_1(u, v_1(u), \ldots)$, $\ldots$, i.e., we repeat the considerations of variant (A). The invariant frame field $u \to R_{x(u)} = a_{s+1}(u)R_0$ depends on the vector fields $v_i, w_i, \ldots$ The obtained invariants will be invariants of the vector fields on S_U. Identically as in variant (A), the vectors A_i may be taken as functions of u if they determine a subgroup H_{0j}, $\mathrm{pr}_{m_0}(H_{0j}^{-1}A_jH_{0j}) = A_j$. Usually we take $v_i = w_i = z_i = \ldots$

Another variant of this type results when we take a field v_1 on V_l and find $h_{01}(u) \in H_0$ and $H_{01} \subset H_0$ as above and, next, for a vector field v_2 we find $h_{02}(u) \in H_{01}$ such that the vector

$$\mathrm{pr}_{m_{01}}(h_{02}(u)^{-1}a_{01u}^{-1}d_{v_2}a_{01u}h_{02}(u)),$$

where $a_{01}(u) = a_0(u)h_{01}(u)$, $\boldsymbol{q}_e = \boldsymbol{m}_{01} + \boldsymbol{h}_{01}$, has possibly great number of components equal to constant numbers for all $u \in U$. In this way we get another sequence of invariants and an invariant frame field depending on some vector fields $v_1, v_2, \ldots$

From above considerations we see that the theoretical foundations of variant (B) are the same as these of variant (A).

Variants (A) and (B) may be applied simultaneously.

EXAMPLE 5. Let S_U: $x^i = x^i(u)$ be a surface in the Euclidean space $\mathscr{E}_3$. Identically as in Example 3, we get a representation

$$a_0(u) = (a_\beta^\alpha(u)), \quad a_j^i(u) = \vartheta_j^i, \quad a_4^i(u) = x^i(x),$$
$$\alpha, \beta = 1, \ldots, 4; \ i, j = 1, 2, 3.$$

We take two vector fields v_1, v_2 on $U \subset V_2$, $v_A = v_A^i e_i$, $A = 1, 2$. Then we have

$$a_{0u}^{-1}d_{v_A}a_{0u} = dx_u^i(v_A)E_i = v_A^i(u)E_i \in \boldsymbol{m}_0.$$

If $(h_\beta^\alpha) \in \boldsymbol{H}_0$, we can put the conditions

$$\tilde{h}_i^3 v_A^i(u) = 0, \quad \tilde{h}_i^1 v_1^i(u) = 1, \quad \tilde{h}_i^2 v_2^i(u) = 1,$$

i.e., $h_0^{-1}a_{0u}^{-1}d_{v_A}a_{0u}h_0 = E_A$. Hence

$$h_1^i(u) = v_1^i(u), \quad h_2^i(u) = v_2^i(u)$$

and v_1, v_2 must satisfy

$$\vartheta_{ij}v_A^i v_B^j = \vartheta_{AB}, \quad A, B = 1, 2.$$

Assume that such vector fields v_A are given.

(For an arbitrary vector field v we have $h_0^{-1}a_{0u}^{-1}d_v a_{0u}h_0 = h_j^i v^j(u)E_i$. Thus the function $F(u) = \vartheta_{ks}h_j^k v^j(u)h_i^s v^i(u) = \vartheta_{ij}v^i(u)v^j(u)$ of components is independent on h_0 and therefore it is an invariant of a vector field on S.)

The column $h_3^i(u)$, $\vartheta_{ij}h_3^i(u)v_A^i(u) = 0$ is denoted by $v_3^i(u)$. The matrix group H_1 satisfying $H_1^{-1}E_A H_1 = E_A$ is trivial, and so the specialization of a frame is finished.

Now we have $h_0(u) = \big(v_\beta^a(u)\big)$, $a_1(u, v_{1u}, v_{2u}) = a_0(u)h_0(u) = \big(v_{1u}, v_{2u}, v_{3u}, x(u)\big)$,

$$a_{1u}^{-1}da_{1u} = v_{ju}^i dv_{ku}^j E_i^k + v_{ku}^i dx_u^k E_i = 2\vartheta_{ij}v_{2u}^i dv_{1u}^j E_2^1 + 2\vartheta_{ij}v_{3u}^i dv_{1u}^j E_3^1 + \\ + 2\vartheta_{ij}v_{3u}^i dv_{2u}^j E_3^2 + \vartheta_{ij}v_{1u}^i dx_u^j E_1 + \vartheta_{ij}v_{2u}^i dx_u^j E_2.$$

Thus we get second rank invariant forms as the components of the above vector form. For instance, $\vartheta_{ij}v_2^i d_{v_1}v_1^j$ is the geodesic curvature of the vector field $S_1 = dS(v_1)$ on S_U, $\vartheta_{ij}v_3^i d_{v_1}v_1^j$ is the normal curvature of S_1 (i.e., of an integral curve given by this field), $\vartheta_{ij}v_1^i d_v x^j = g(v_1, v)$ is the value of the metric tensor g.

Example 6. Consider once more Example 4. For a vector field v on. $U \subset V_{n-1}$, $S\colon V_{n-1} \to A_n$, we have

$$h_0^{-1}a_{0u}^{-1}d_v a_{0u}h_0 = v^i(u)E_i/\lambda$$

and the conditon $v^n(u) = \lambda(u)$ finishes the specialization of a frame. The functions v^i/v^n are first rank invariants.

Example 7. We take a subgroup G' of the isometry group acting on $\mathscr{E}_3$. Namely, let G' be isomorphic to a group of matrices $a = (a_\beta^a)$, where (a_B^A) is orthogonal, $a_\beta^3 = a_3^\beta = \vartheta_\beta^3$, $a_\beta^4 = \delta_\beta^4$, $a_4^i = a^i$, $A, B = 1, 2$; $i = 1, 2, 3$; $\beta = 1, 2, 3, 4$. It is easy to verify that variant (B) does not specialize a frame in the case of a curve given by the equations: $x^1 = 0$, $x^2 = 0$, $x^3 = u$, $u \in \boldsymbol{R}$.

3. A formal theory. Let ω be a linear form on a differentiable manifold V_l with values in the Lie algebra $\boldsymbol{q}_e$ of a Lie group G and let ω_U be the restriction of ω to a chart $U \subset V_l$. We assume that H_0 is a closed subgroup of G, $\boldsymbol{h}_0$ is its Lie algebra and $\boldsymbol{m}_0$ is a subspace of $\boldsymbol{q}_e$ satisfying $\boldsymbol{q}_e = \boldsymbol{h}_0 + \boldsymbol{m}_0$, $\dim \boldsymbol{m}_0 > l$. We assume also that

$$\mathrm{pr}_{\boldsymbol{m}_0}[\omega_u(v_u)] \neq 0 \quad \text{for each } u \in U,$$

where $v\colon u \to v_u = v(u)$ is a vector field on $U \subset V_l$.

If there exists a surface $a_0\colon U \to G$ such that $a_0^{-1}da_0 = \omega_U$, i.e., $da_0 = a_0\omega_U$, then it is easy to verify that the integrability condition of this Pfaff system has the form

$$d\omega_U = -\tfrac{1}{2}[\omega_U, \omega_U],$$

where [,] denotes the product in q_e. Thus we get the structure equation for such ω.

If such a form ω_U is given, then we can repeat the process given in (A) with respect to $a_0^{-1}da_0$ as follows. We choose a subspace $P_{n_1} \subset m_0$ having a possibly small dimension and we put on H_0 the condition

$$\mathrm{pr}_{m_0}(H_0^{-1}\omega_U H_0) \subset P_{n_1},$$

which determines $h_0(u) \in H_0$ and a maximal subgroup $H_1 \subset H_0$ satisfying

$$\mathrm{pr}_{m_0}\big(h_0(u)^{-1}\omega_u h_0(u)\big) \subset P_{n_1} \quad \text{and} \quad \mathrm{pr}_{m_0}(H_1^{-1}P_{n_1}H_1) = P_{n_1}$$

for all $u \in U$.

We get a decomposition $q_e = h_1 + m_1$, where h_1 is the Lie algebra of H_1 and $m_1 \supset m_0$. Next, we consider the form

$$\omega_u^1 = h_0(u)^{-1}\omega_u h_0(u) + h_0(u)^{-1}dh_{0u}$$

and choose a subspace $P_{n_2} \subset m_1$ such that the condition

$$\mathrm{pr}_{m_1}(H_1^{-1}\omega_U^1 H_1) \subset P_{n_2}$$

determines $h_1(u) \in H_1$ and $H_2 \subset H_1$ satisfying

$$\mathrm{pr}_{m_1}\big(h_1(u)^{-1}\omega_u^1 h_1(u)\big) \subset P_{n_2} \quad \text{and} \quad \mathrm{pr}_{m_1}(H_2^{-1}P_{n_2}H_2) = P_{n_2},$$

i.e., we repeat the considerations we have done for ω_U. We get a form

$$\omega_u^2 = h_1(u)^{-1}\omega_u^1 h_1(u) + h_1(u)^{-1}dh_{1u}.$$

In this way we construct the sequences: $h_0(u), \ldots, h_s(u)$;

$$\omega_u^{k+1} = h_k(u)^{-1}\omega_u^k h_k(u) + h_k(u)^{-1}dh_{ku}, \qquad h_k(u) \in H_k,$$

$$H_0 \supset H_1 \supset \ldots \supset H_s \supset H_{s+1} = \{e\},$$

$$\mathrm{pr}_{m_k}(H_k^{-1}P_{n_k}H_k) = P_{n_k} \subset m_k \subset m_{k+1}, \qquad q_e = m_k + h_k,$$

and after $s+1$ steps we get a form ω_U^{s+1}, which gives a cross section $u \to b(u)$ by the integration of the equation $b^{-1}db = \omega_U^{s+1}$.

If $a = cb$ is another solution of this equation, then

$$b^{-1}c^{-1}(dcb + cdb) = b^{-1}db, \quad \text{i.e.,} \quad dc = 0 \text{ and } c = \text{const}.$$

This means that two solutions of the equation $b^{-1}db = \omega_U^{s+1}$ differ by a constant left-hand side factor.

Let a_0: $u \to a_0(u)$ be a maximal solution of the equation $a_0^{-1}da_0 = \omega_U$ and $\omega_U^1 = h_0^{-1}\omega_U h_0 + h_0^{-1}dh_0$, h_0: $U \to H_0$. Then the equation

$$a_1^{-1}da_1 = \omega_U^1$$

can be written as

$$a_1^{-1}da_1 = h_0^{-1}a_0^{-1}da_0 h_0 + h_0^{-1}dh_0, \quad \text{i.e.,} \quad a_1^{-1}da_1 = (a_0h_0)^{-1}d(a_0h_0)$$

and $a_1 = a_0h_0$ is a maximal solution of the equation $a_1^{-1}da_1 = \omega_U^1$.

Thus if the equation $a_0^{-1}da_0 = \omega_U$ is integrable, then each equation $a_i^{-1}da_i = \omega_U^i$ is also integrable.

In such a way we get a formal theory of variant (A). Now we may consider directly a form ω_U satisfying the above conditions instead of a surface in G/H_0. Two forms $\hat{\omega}_U$ and ω_U are equivalent (i.e., corresponding to the same surface $S\colon V_l \to G/H_0$) if there exists a function $u \to h(u) \in H_0$ such that $\omega_u = h(u)^{-1}\omega_u h(u) + h(u)^{-1}dh_u$.

Variant (B) may be analogously repeated if we determine the functions $u \to h(u)$ with the aid of the values $\omega_U(v_i)$ of ω_U on vectors tangent to V_l.

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