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Some univalent compositions of polynomials with univalent functions

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Abstract. Let S denote the family of functions f, holomorphic and univalent in the open unit disk U, and normalized by $f(0) = 0$, $f'(0) = 1$.

MacGregor has proved the following: If $c \notin \operatorname{co} f(U)$ and $g(z) = f(z) - \frac{1}{2c} f(z)^2$, then $g \in S$ and $c/2 \notin g(U)$. This observation leads to the simplest proof of the $1/2$-theorem for convex functions.

The present paper extends the MacGregor idea to compositions $P \circ f$, where f is holomorphic and univalent (in some domain D), and where P is a polynomial of degree n. Under certain "modified convexity conditions" on a parameter c in the P-expression, the composition $P \circ f$ is also univalent in D and omits the value c/n.

Finally, some examples of application are briefly indicated.

1. Introduction. Let S denote the family of functions f,

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \tag{1.1}$$

holomorphic and univalent in the open unit disk U, and normalized by $f(0) = 0$, $f'(0) = 1$.

The following observation is due to MacGregor [2]:

If $c \notin \operatorname{co} f(U)$ and

$$g(z) = f(z) - \frac{1}{2c} f(z)^2, \tag{1.2}$$

then $g \in S$ and $c/2 \notin g(U)$.

MacGregor used this observation in combination with the $1/4$-theorem of Koebe–Bieberbach to give a new and very simple proof of the fact that the disk $|w| < \frac{1}{2}$ is contained in $f(U)$ for any $f \in S$ with convex $f(U)$.

This example indicates that it may be profitable to look for polynomials P such that $P \circ f \in S$ for $f \in S$. The purpose of the present paper is to construct examples of such polynomials. The method is in a certain sense a generalization of the MacGregor idea.

Following Schober [3] for notations, the MacGregor result may (in a slightly more general setting) be expressed as follows:

Let D be a domain in C, and let $H_u(D)$ denote the set of functions, holomorphic and univalent in D. If $f \in H_u(D)$ and P is the polynomial

$$P(w) = w - \frac{1}{2c} w^2, \tag{1.3}$$

where $c \notin \operatorname{co} f(D)$, then $P \circ f \in H_u(D)$, and $c/2 \notin (P \circ f)(D)$.

2. The idea. For the MacGregor polynomial the following holds when $u \neq v$:

$$\frac{P(u)-P(v)}{u-v} = 1 - \frac{u+v}{2c}. \tag{2.1}$$

If $(u+v)/2$ never takes the value c, the right-hand side $\neq 0$, and hence $P(u) \neq P(v)$. This is in particular the case if $f \in H_u(D)$, $u = f(z_1)$, $v = f(z_2)$, $z_1, z_2 \in D$, $z_1 \neq z_2$ and $c \notin \operatorname{co} f(D)$.

More generally, let P be a polynomial in one variable, with the property that for $u \neq v$ the two-variable polynomial $\frac{P(u)-P(v)}{u-v}$ is representable in the form

$$\frac{P(u)-P(v)}{u-v} = K \cdot \prod_{k=1}^{n-1} \left(1 - \frac{a_k u + (1-a_k) v}{c_k}\right), \tag{2.2}$$

where $K \in C$, $K \neq 0$ and $c_k \in C$, $c_k \neq 0$, $0 \leqslant a_k \leqslant 1$ for all natural numbers $k \leqslant n-1$. If $f \in H_u(D)$ and $c_k \notin \operatorname{co} f(D)$ for all natural $k \leqslant n-1$, then $P \circ f \in H_u(D)$. Hence, if we can find such polynomials, we will be able to construct new functions in $H_u(D)$ from a given $f \in H_u(D)$ and hopefully be able to obtain new information on $H_u(D)$ by using this technique.

The first thing to do is to attack the following problem:

PROBLEM 1. *Find corresponding to any natural number n a polynomial $P(w)$ of degree n, normalized by $P(0) = 0$, $P'(0) = 1$, and such that the expression $\frac{P(u)-P(v)}{u-v}$, $u \neq v$, is representable in the form*

$$\frac{P(u)-P(v)}{u-v} = \prod_{k=1}^{n-1} \left(1 - \frac{a_k u + (1-a_k) v}{c_k}\right), \tag{2.2'}$$

where $c_k \in C$, $c_k \neq 0$ and $0 \leqslant a_k \leqslant 1$ for all natural $k \leqslant n-1$.

Remark. The normalization is inessential, but saves us a few words when the results are to be formulated.

3. A discouraging result. As it will soon be seen Problem 1 represents a dead end in the sense that it does not produce any new polynomials of the kind we are looking for. More precisely:

PROPOSITION 1. *The only solutions of Problem* 1 *are the polynomials* w *and* $w-\frac{1}{2c}w^2$, $c \in \boldsymbol{C}$, $c \neq 0$.

Remark. It turns out that the proof, in addition to establishing a useless result, also indicates a side-track, which to a certain extent re-establishes the value of the problem (somewhat modified).

Proof of Proposition 1. Let P be a solution of Problem 1, and assume that the degree of P is $\geqslant 2$,

$$P(w) = w + A_2 w^2 + \ldots + A_n w^n. \tag{3.1}$$

Then (for $u \neq v$)

$$\frac{P(u)-P(v)}{u-v} = 1 + A_2(u+v) + \ldots + A_n(u^{n-1}+u^{n-2}v + \ldots + v^{n-1}). \tag{3.2}$$

From (2.2′) follows, by letting $u \to v = w$:

$$P'(w) = \prod_{k=1}^{n-1}\left(1-\frac{w}{c_k}\right) = 1 + 2A_2 w + \ldots + nA_n w^{n-1}, \tag{3.3}$$

and hence

$$A_n = \frac{(-1)^{n-1}}{n}\prod_{k=1}^{n-1}\frac{1}{c_k}. \tag{3.4}$$

On the other hand it follows from (2.2′) by putting $v = 0$, $u = w$:

$$P(w) = w\prod_{k=1}^{n-1}\left(1-\frac{a_k w}{c_k}\right). \tag{3.5}$$

This yields

$$A_n = (-1)^{n-1}\prod_{k=1}^{n-1}\frac{a_k}{c_k}. \tag{3.6}$$

Comparing the two expressions for A_n we find

$$\prod_{k=1}^{n-1} a_k = \frac{1}{n}. \tag{3.7}$$

From (2.2′) and (3.2) it follows that

$$(-1)^{n-1}\left(\prod_{k=1}^{n-1}\frac{1}{c_k}\right)\prod_{k=1}^{n-1}[a_k u + (1-a_k)v] = A_n(u^{n-1}+u^{n-2}v + \ldots + v^{n-1}).$$

Since $a_k \neq 0$ for all $k \leqslant n-1$ it follows by using (3.6):

$$\prod_{k=1}^{n-1}\left[u-\left(1-\frac{1}{a_k}\right)v\right] = u^{n-1}+u^{n-2}v+\ldots+v^{n-1}$$

$$= \prod_{k=1}^{n-1}\left[u-\exp\left(\frac{2k\pi i}{n}\right)\cdot v\right].$$

From this we may conclude

$$1-\frac{1}{a_k} = \exp\left(\frac{2k\pi i}{n}\right),$$

and hence

$$a_k = \tfrac{1}{2}\left[1+i\cot\text{an}\frac{k\pi}{n}\right], \tag{3.8}$$

$k = 1, 2, \ldots, n-1$.

This shows that for any $n \geqslant 3$ there must be at least one a_k with $\operatorname{Im} a_k \neq 0$. Hence Problem 1 cannot have any solution with $n \geqslant 3$. For $n = 1$ the polynomial $P(w) = w$ is obviously the only solution (in fact the only polynomial with correct normalization). For $n = 2$ we have, with

$$P(w) = w+A_2w^2, \qquad A_2 \neq 0,$$

that

$$\frac{P(u)-P(v)}{u-v} = 1+A_2(u+v), \qquad u \neq v.$$

Hence $P(w)$ is always a solution of Problem 1 with $a_1 = \frac{1}{2}$ and $A_2 = -1/2c$, where $c \neq 0$. This concludes the proof of Proposition 1.

4. Modification of Problem 1. If we drop the condition that the numbers a_k be real, there is still a possibility of finding polynomials of arbitrary degree satisfying condition (2.2′).

PROBLEM 2. *Problem* 1 *without the restriction* $0 \leqslant a_k \leqslant 1$ *on the numbers* a_k.

This modification has no effect in the cases $n = 1$ and $n = 2$, i.e. the solutions of Problem 2 and Problem 1 are the same for $n = 1$ and $n = 2$.

For $n = 3$ a straightforward computation shows that the polynomials

$$P(w) = \frac{c}{3}\left[1-\left(1-\frac{w}{c}\right)^3\right], \qquad c \neq 0, \tag{4.1}$$

are the only solutions of Problem 2, and that $c_1 = c_2 = c$, $a_{1,2} = \frac{1}{2}[1 \pm i/\sqrt{3}]$.

In a similar manner it is easily proved that the only solutions of Problem 2 in the case $n = 4$ are

$$P(w) = \frac{c}{4}\left[1-\left(1-\frac{w}{c}\right)^4\right], \quad c \neq 0, \tag{4.2}$$

and that $c_1 = c_2 = c_3 = c$, $a_1 = \frac{1}{2}$, $a_{2,3} = \frac{1}{2}(1 \pm i)$.

As an illustration of the argument we shall prove the last statement. The idea in the proof is the same as in the proof of Proposition 1, and we shall use the same reference numbers for the equalities:

$$P(w) = w + A_2 w^2 + A_2 w^3 + A_4 w^4, \tag{3.1}$$

$$\begin{aligned} P'(w) &= \left(1-\frac{w}{c_1}\right)\cdot\left(1-\frac{w}{c_2}\right)\cdot\left(1-\frac{w}{c_3}\right) \\ &= 1 + 2A_2 w + 3A_3 w^2 + 4A_4 w^3, \end{aligned} \tag{3.3}$$

$$P(w) = w\left(1-\frac{a_1}{c_1}w\right)\left(1-\frac{a_2}{c_2}w\right)\left(1-\frac{a_3}{c_3}w\right). \tag{3.5}$$

In this case we have, from the proof of Proposition 1, that $a_1 = \frac{1}{2}$, $a_{2,3} = \frac{1}{2}(1 \pm i)$.

Comparing coefficients we find for w^2:

$$\frac{a_1}{c_1} + \frac{a_2}{c_2} + \frac{a_3}{c_3} = \frac{1}{2}\left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right) = -A_2.$$

This gives

$$\frac{i}{2c_2} - \frac{i}{2c_3} = 0$$

and hence $c_2 = c_3 = c$.

For w^3 we find

$$\frac{\frac{1}{4}(1+i)}{c_1 c} + \frac{\frac{1}{2}(1-i)}{c_1 c} + \frac{\frac{1}{2}}{c^2} = \frac{1}{3}\left(\frac{1}{c_1 c} + \frac{1}{c_1 c} + \frac{1}{c^2}\right),$$

and hence

$$\frac{1}{2c_1} + \frac{1}{2c} = \frac{2}{3c_1} + \frac{1}{3c},$$

from which follows $c_1 = c$. From (3.3) with $c_1 = c_2 = c_3 = c$ follows that

$$P(w) = \frac{c}{4}\left[1-\left(1-\frac{w}{c}\right)^4\right],$$

and the statement is thus proved.

Quite generally the following holds:

PROPOSITION 2. *The polynomials*

$$P(w) = \frac{c}{n}\left[1-\left(1-\frac{w}{c}\right)^n\right], \quad c \neq 0, \tag{4.3}$$

are solutions of Problem 2, with $c_1 = c_2 = \ldots = c_{n-1} = c$ *and*

$$a_k = \left(1-\exp\left(\frac{2k\pi i}{n}\right)\right)^{-1}, \quad k = 1, 2, \ldots, n-1.$$

Proof. From the proof of Proposition 1 we know that the only possible a_k-values of a solution of Problem 2 are the values in the statement of Proposition 2.

For a given natural number n let $Q(w)$ be the polynomial

$$Q(w) = \frac{P(cw)}{c} = \frac{1}{n}\left(1-(1-w)^n\right),$$

and let

$$\varepsilon = \exp\left(\frac{2k\pi i}{n}\right).$$

Then for all U, V

$$Q(U)-Q(V) = (U-V)\prod_{k=1}^{n-1}\left(1-\frac{U}{1-\varepsilon^k}-\frac{V}{1-\varepsilon^{-k}}\right).$$

This follows immediately from the fact that for any fixed V both sides are polynomials of degree n in U with the same zeroes

$$U = 1-(1-V)\varepsilon^k, \quad k = 1, 2, \ldots, n,$$

and the same constant term

$$-Q(V) = -V\prod_{k=1}^{n-1}\left(1-\frac{V}{1-\varepsilon^{-k}}\right).$$

From this follows immediately, that

$$P(u)-P(v) = (u-v)\prod_{k=1}^{n-1}\left(1-\frac{a_k u+(1-a_k)v}{c}\right),$$

with $a_k = (1-\varepsilon^k)^{-1}$ (and $1-a_k = (1-\varepsilon^{-k})^{-1}$).

This concludes the proof of Proposition 2.

Remark 1. The polynomials in Proposition 2 seem to be the only solutions of Problem 2. We know that this is true for $n \leqslant 4$.

Remark 2. For any fixed n we have: If $w \neq c$, then $P(w) \neq c/n$.

5. Geometric interpretation. We shall need the following lemma:

LEMMA 1. *Let p and q be distinct points in C, and let a be an arbitrary non-real complex number. Then the circular arc from p through $ap+(1-a)q$ to q has angular measure*

$$2\left(\pi-\left|\arg\frac{a}{a-1}\right|\right), \tag{5.1}$$

where $\arg\varphi$ *is chosen in* $(-\pi, +\pi)$.

The proof is a simple application of a theorem in school geometry, and shall be omitted here.

Remark. If $\operatorname{Re} a$ has a fixed value in $(0,1)$ and we let $\operatorname{Im} a \to 0$, then the angular measure $\to 0$. If $\operatorname{Re} a$ has a fixed value not in $[0,1]$ and we let $\operatorname{Im} a \to 0$, then the angular measure $\to 2\pi$. The two cases correspond to the open line segment from p to q and to the complement of the closed segment with respect to the line through p and q.

We shall use the term *β-arc* from p to q to denote an arc of angular measure β from p to q. We shall accept the term 0-arc for the line segment from p to q.

The a-values occurring in our cases are

$$a = a_k = \frac{1}{1-\varepsilon^k}, \quad \text{where } \varepsilon = \exp\left(\frac{2\pi i}{n}\right),$$

and hence

$$\frac{a}{a-1} = \varepsilon^{\,k}, \quad k = 1, 2, \ldots, (n-1).$$

For even n the a-value $\frac{1}{2}$ occurs (for $k = n/2$). All other a-values are non-real, and the corresponding arcs have angular measures

$$2\pi\left(1-\frac{2k}{n}\right), \quad k = 1, 2, \ldots, \left[\frac{n-1}{2}\right],$$

according to Lemma 1.

From this, Proposition 2 and Remark 2 it follows immediately:

THEOREM 1. *Let D be a domain in the complex plane, let $f \in H_u(D)$ and let c be a point not lying on any $2\pi(1-2k/n)$-arc between points of $f(D)$, $k = 1, 2, \ldots, [n/2]$. Let furthermore*

$$g(z) = \frac{c}{n}\left[1-\left(1-\frac{f(z)}{c}\right)^n\right]. \tag{5.2}$$

Then $g \in H_u(D)$, and $c/n \notin g(D)$.

For $n = 2$ this reduces to the result of MacGregor.

6. Examples. The main difficulty in using this method is to determine which c-values are permitted and which are not. In some cases there is *no* permitted c-value. This is e.g. the case for the Koebe functions $z/(1-\eta z)^2$, $|\eta| = 1$, for arbitrary degree of P and for the functions $z/(1-\eta z)$, $|\eta| = 1$, when the degree of P is $\geqslant 3$.

We shall here briefly indicate some special cases, where the c-problem is easily treated.

One such case is the case when (in addition to $f \in H_u(D)$) we require $f(D)$ to be contained in a fixed disk. In this case we shall need the following lemma:

LEMMA 2. *Let the disk*

$$|w-a| < R \tag{6.1}$$

and the number $\beta \in (0, 2\pi)$ be given. Then the set of β-arcs between points of the disk covers exactly the disk

$$|w-a| < \frac{R}{\cos\frac{\beta}{4}}. \tag{6.2}$$

The proof is quite elementary and shall be omitted here.

The simplest example is obtained by taking $f(z) = z$, in which case $a = 0$ and $R = 1$ in the lemma. From Theorem 1 and Lemma 2 it follows in this example:

The functions

$$g(z) = \frac{c}{n}\left[1-\left(1-\frac{z}{c}\right)^n\right] \tag{6.3}$$

are in S for all c with $|c| \geqslant 1/\sin\frac{\pi}{n}$.

For $n = 2$ this actually gives the whole family of univalent polynomials of degree 2. For $n = 3$ it shows that the polynomials

$$z - \frac{\sqrt{3}}{2}az^2 + \frac{1}{4}a^2z^3$$

are univalent for all a with $|a| \leqslant 1$. An interesting observation is that for $a = \pm 1$ the polynomials are extreme points of the family of polynomials of degree 3 in S with real coefficients [1].

Another example of application of Lemma 2 in combination with Theorem 1 is to functions in S_M, i.e. functions in S for which $|f(z)| < M$ for some $M \geqslant 1$. For such functions we have:

If $f \in S_M$ and $|c| \geqslant M/\sin\dfrac{\pi}{n}$, then the function

$$g(z) = \frac{c}{n}\left(1-\left(1-\frac{f(z)}{c}\right)^n\right)$$

is in S, and $g(z) \neq c/n$.

Related results may be obtained for functions $f \in S$, where $f(U)$ is contained in some fixed disk, not necessarily centered at the origin.

A second special case where the determination of permitted c-values is easy, is when $f(D)$ is contained in an angular domain.

In a subsequent paper we shall take a closer look at the cases briefly mentioned above and at some other cases.

7. Final remark. As remarked to me by Don Blevins in an oral communication Theorem 1 also has an extension to arbitrary *rational* n-values, in which case there will be a *finite* number of a-values. (A non-rational n, however, would give an a-set, dense on the line $\operatorname{Re} a = \frac{1}{2}$.)

References

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