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Continuous solutions of finite variation of a linear functional equation

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Abstract. In this paper there are investigated continuous solutions φ in $J = (a, b\rangle$ of equation (1) which have a finite limit $\lim\limits_{x\to a^+} \varphi(x)$ and are of finite variation in J. The given functions g and F are of bounded variation in J while f is a function strictly increasing in J.

In the present paper we shall consider the functional equation

$$\varphi\big(f(x)\big) = g(x)\varphi(x)+F(x), \tag{1}$$

where φ is the unknown function and f, g, F are given. Let J be an interval of the form $J = (a, b\rangle$, $-\infty \leqslant a < b < +\infty$.

We introduce the following function classes:

A real-valued function φ is said to *belong to* $BV[J]$ iff φ is defined on J and $\sup\limits_{P\subset J} \mathrm{Var}\,\varphi|P < \infty$, the supremum being taken over all finite intervals $P \subset J$.

A real-valued function φ is said to *belong to* $B_1V[J]$ iff φ is defined on J, $\mathrm{Var}\,\varphi|P < \infty$ for every closed and finite interval $P \subset J$ and there exists a finite limit $\lim\limits_{x\to a^+} \varphi(x)$.

We shall assume the following hypotheses regarding the functions f, g, and F.

(i) f is continuous and strictly increasing in the interval $J = (a, b\rangle$ and $a < f(x) < x$ for $x \in J$ (we admit $a = -\infty$),

(ii) g, $F \in BV[J]$, $\inf\limits_{J} g > 0$, $\lim\limits_{x\to a^+} g(x) = 1$ and $\lim\limits_{x\to a^+} F(x) = 0$,

(iii) g and F are continuous in J.

Let f^n denote the n-th iterate of f. Condition (i) implies that for every $x \in J$ the sequence $f^n(x)$ is strictly decreasing and $\lim\limits_{n\to\infty} f^n(x) = a$ (cf. [3], Theorem 0.4, p. 21).

From Theorem 2.1 in [3] and Lemma 1 in [6] we immediately obtain the following

LEMMA 1. *Let hypotheses* (i), (ii), (iii) *be fulfilled and suppose that* φ *satisfies equation* (1) *in* J. *If* $\varphi | I_0$ *is continuous and* $\varphi | I_0 \in BV[I_0]$, *where* $I_0 = \langle f(x_0), x_0 \rangle$ *for an* $x_0 \in J$, *then* φ *is continuous in* J *and* $\operatorname{Var} \varphi | P < \infty$ *for every finite and closed interval* $P \subset J$.

LEMMA 2. *Let hypothesis* (i) *be fulfilled,* $F \in BV[J]$ *and* $\lim_{x \to a+} F(x) = 0$. *If the series* $\sum_{n=0}^{\infty} F(f^n(x))$ *converges* (*absolutely*) *for an* $x \in J$, *then it converges* (*absolutely and*) *uniformly in every compact set* $K \subset J$.

Proof. From Lemma 3 in [4] and the proofs of Lemma 2 in [4] and Theorem 5.3 in [3] it is easy to observe that the series $\sum_{n=0}^{\infty} F(f^n(x))$ converges almost uniformly in J.

Put

$$G_0(x) \equiv 1, \quad G_n(x) = \prod_{i=0}^{n-1} g(f^i(x)), \quad x \in J, \ n = 1, 2, \ldots \tag{2}$$

In paper [5] (Lemma 1 and Corollary 2) we have proved the following facts (which we now call Lemmas 3 and 4):

LEMMA 3. *Let hypotheses* (i) *and* (ii) *be fulfilled* ($F \equiv 0$); *then there exists a positive constant* L, *such that for any* $x_0 \in J$ *the following inequality holds*:

$$\frac{1}{L} G_k(x_0) \leqslant G_k(x) \leqslant L G_k(x_0) \quad \text{for } x \in \langle f(x_0), x_0 \rangle, \ k = 0, 1, \ldots \tag{3}$$

LEMMA 4. *Let hypotheses* (i) *and* (ii) *be fulfilled* ($F \equiv 0$); *then there exists an* $M > 0$ *such that*

$$\operatorname{Var} G_n | \langle f(x_0), x_0 \rangle \leqslant M G_n(x_0) \tag{4}$$

for any $x_0 \in J$.

There are the following three posibilities of the behaviour of sequence (2):

(A) The product $\prod_{i=0}^{\infty} g(f^i(x))$ converges for an $x \in J$.

(B) There exists an $x \in J$ such that $\lim_{n \to \infty} G_n(x) = 0$.

(C) Neither of cases (A) and (B) occurs.

1. First we consider case (A).

THEOREM 1. *Let hypotheses* (i), (ii), (iii) *and* (A) *be fulfilled, and suppose that the series* $\sum_{n=0}^{\infty} \frac{F(f^n(x))}{G_{n+1}(x)}$ *converges absolutely for an* $x \in J$.

Then equation (1) *has exactly a one-parameter family of solutions* $\varphi \in B_1 V[J]$. *These solutions are continuous in* J *and are given by the formula*

$$\varphi(x) = \frac{\eta}{G(x)} - \sum_{n=0}^{\infty} \frac{F(f^n(x))}{G_{n+1}(x)}, \tag{5}$$

where

$$G(x) = \lim_{n\to\infty} G_n(x) = \prod_{i=0}^{\infty} g(f^i(x)) \quad \textit{for } x \in J. \tag{6}$$

Proof. Condition (A) implies that the product (6) converges uniformly in every compact $K \subset J$, G is continuous in J and $\lim_{x\to a^+} G(x) = 1$ (see Theorem 1 in [5]).

Let φ fulfil equation (1) and suppose there exists $\lim_{x\to a^+} \varphi(x) = \eta$. By assumption (A) it follows that φ has the form (5) (see Theorem 5 in [1](1)).

We shall show that the series

$$\sum_{n=0}^{\infty} \frac{F(f^n(x))}{G_{n+1}(x)} \stackrel{\text{df}}{=} \varphi_0(x) \tag{7}$$

converges uniformly in $I_0 = \langle f(x_0), x_0\rangle$ for an $x_0 \in J$ and that there exists the limit $\lim_{x\to a^+} \varphi_0(x) = 0$.

From Lemma 3 it follows that there exists an $L > 0$ such that

$$\frac{1}{L} G_n(x_k) \leqslant G_n(x) \leqslant L G_n(x_k)$$

for $x \in I_k = \langle x_{k+1}, x_k\rangle$, where $x_k = f^k(b)$ and $n, k = 0, 1, \ldots$ Hence by the identity

$$G_n(f^k(x)) = \frac{G_{n+1}(x)}{G_k(x)} \quad \text{for } n, k = 0, 1, \ldots, \ x \in J, \tag{8}$$

we obtain

$$\frac{G_{n+k}(x_0)}{L G_k(x_0)} \leqslant G_n(x) \leqslant \frac{L G_{n+k}(x_0)}{G_k(x_0)} \quad \text{for } x \in I_k. \tag{9}$$

Hence

$$\frac{1}{L} \prod_{i=k}^{n+k-1} g(x_i) \leqslant G_n(x) \leqslant L \prod_{i=k}^{n+k-1} g(x_i), \quad x \in I_k.$$

(1) It is easy to observe that all results of [1] remain true for continuous functions in $(a, b\rangle$ which have a finite right-hand side limit at a.

From the convergence of product (6) at x_0 it follows that there exists an m' and M' such that $0 < m' \leqslant \prod_{i=k}^{m} g(x_i) \leqslant M'$ for any $m > k > 0$. Thus

$$\frac{1}{L} m' \leqslant G_n(x) \leqslant LM' \quad \text{for } x \in I_k,\ n, k = 0, 1, \ldots,$$

whence

$$0 < \overline{m} \leqslant \frac{1}{G_n(x)} \leqslant \overline{M} \quad \text{for } x \in J, \tag{10}$$

where $\overline{m}$ and $\overline{M}$ are some constants.

Series (7) converges absolutely at a point $\bar{x} \in J$ and the sequence $1/G_n(\bar{x})$ is bounded. Consequently the series

$$\sum_{n=0}^{\infty} F\big(f^n(x)\big) \tag{11}$$

converges absolutely at the point $\bar{x}$. Then Lemma 2 implies the absolute and uniform convergence of series (11) in I_0. Hence and from inequality (10) it follows that also series (7) converges absolutely and uniformly in I_0. Moreover, we have the estimation

$$|\varphi_0(x)| = \Big|\sum_{n=0}^{\infty} \frac{F\big(f^n(x)\big)}{G_{n+1}(x)}\Big| \leqslant \sum_{n=0}^{\infty} \frac{\big|F\big(f^n(x)\big)\big|}{G_{n+1}(x)} \leqslant \overline{M} \sum_{n=0}^{\infty} \big|F\big(f^n(x)\big)\big| \overset{\mathrm{df}}{=} \psi(x).$$

From Theorem 5 in [4] it follows that $\lim_{x\to a^+} \psi(x) = 0$. Thus $\lim_{x\to a^+} \varphi(x) = 0$.

Now we shall estimate $\operatorname{Var}\varphi_0|I_0$. Let $s_n = \operatorname{Var}F|I_n$. By Lemma 4 (applied to the function $1/g$) there exists an $M > 0$ such that $\operatorname{Var}1/G_n|I_0 \leqslant M/G_n(x_0)$ for $n = 1, 2, \ldots$ Then from Lemma 3 and inequality (10) we obtain

$$\begin{aligned}\operatorname{Var}\varphi_0|I_0 &\leqslant \sum_{n=0}^{\infty} \operatorname{Var}\frac{F\circ f^n}{G_{n+1}}\Big| I_0 \leqslant \sum_{n=0}^{\infty} \{\operatorname{Var}F\circ f^n|I_0 \sup_{I_0} 1/G_{n+1} + \\ &\quad + \sup_{I_0}|F\circ f^n| \operatorname{Var}1/G_{n+1}|I_0\} \leqslant \sum_{n=0}^{\infty} \frac{s_n L}{G_{n+1}(x_0)} + M \sum_{n=0}^{\infty} \frac{\sup_{I_0}|F\circ f^n|}{G_{n+1}(x_0)} \\ &\quad \leqslant L\overline{M} \sum_{n=0}^{\infty} s_n + M\overline{M} \sum_{n=0}^{\infty} \big\{F\big(f^n(x_0)\big) + s_n\big\} < \infty,\end{aligned}$$

since $\big|F\big(f^n(x)\big)\big| \leqslant \big|F\big(f^n(x_0)\big)\big| + s_n$ for $x \in I_0$.

The function φ_0 fulfils equation (1) and $\varphi|I_0 \in BV[I_0]$. Then by Lemma 1 it follows that $\varphi_0 \in B_1V[J]$. Theorem 1 in [5] implies that

$1/G \in B_1 V[J]$, whence $\varphi \in B_1 V[J]$, where φ is given by formula (5). Series (7) and product (6) converge uniformly in I_0. Thus φ is continuous in I_0 and, by Lemma 1, φ is continuous in J.

THEOREM 2. *Let hypotheses* (i), (ii), (iii) *be fulfilled. If product* (6) *converges absolutely at a point* $x \in J$ *and series* (7) *converges at an* $x_0 \in J$, *then equation* (1) *has exactly a one-parameter family of solutions* $\varphi \in B_1 V[J]$. *These solutions are continuous in* J *and they are given by formula* (5).

Proof. The uniqueness of the solutions follows by an argument similar to that applied in the preceding theorem. Theorem 1 in [5] implies that $1/G \in B_1 V[J]$ and G is continuous in J. Thus it suffices to prove that the function φ_0 given by formula (7) belongs to $B_1 V[J]$ and is continuous in J.

Just as in the proof of Theorem 1 in [5] one can show that product (6) converges absolutely and uniformly in every compact set $K \subset J$. Let $x_0 \in J$ and suppose that series (7) converges at x_0. The sequence (2) is bounded at x_0. Therefore there exists a $D > 0$ that $G_n(x_0) \leqslant D$ for $n = 1, 2, \ldots$ Let $0 < \varepsilon < 1$. Then there exists an N such that for all $n \geqslant N$ and $m = 1, 2, \ldots$ we have

$$\Big|\sum_{i=n+1}^{n+m} \frac{F\big(f^i(x_0)\big)}{G_{i+1}(x_0)}\Big| < \varepsilon/2D \quad \text{and} \quad \sum_{i=n+1}^{n+m} \big|g\big(f^i(x_0)\big)-1\big| < \varepsilon.$$

Let $n \geqslant N$. Then by the well-known Abel transformation (see [2], Vol. 2, p. 264) we have

$$\Big|\sum_{i=n+1}^{n+m} F\big(f^i(x_0)\big)\Big| = \Big|\sum_{i=n+1}^{n+m} \frac{F\big(f^i(x_0)\big)}{G_{i+1}(x_0)} G_{i+1}(x_0)$$

$$= \Big|G_{n+m+1}(x_0) \sum_{i=n+1}^{n+m} \frac{F\big(f^i(x_0)\big)}{G_{i+1}(x_0)} - \sum_{i=n+1}^{n+m-1} \big(G_{i+2}(x_0)-G_{i+1}(x_0)\big) \sum_{k=n+1}^{i} \frac{F\big(f^k(x_0)\big)}{G_{k+1}(x_0)}\Big|$$

$$\leqslant G_{n+m+1}(x_0) \Big|\sum_{i=n+1}^{n+m} \frac{F\big(f^i(x_0)\big)}{G_{i+1}(x_0)}\Big| + \sum_{i=n+1}^{n+m-1} |G_{i+2}(x_0)-G_{i+1}(x_0)| \Big|\sum_{k=n+1}^{i} \frac{F\big(f^k(x_0)\big)}{G_{k+1}(x_0)}\Big|$$

$$\leqslant D\frac{\varepsilon}{2D} + \frac{\varepsilon}{2D} \sum_{i=n+1}^{n+m-1} \big|g\big(f^{i+1}(x_0)\big)-1\big| G_{i+1}(x_0) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} < \varepsilon.$$

The Cauchy criterion implies the convergence of series (11) at x_0. Hence, by Lemma 2, it follows that series (11) converges uniformly in $I_0 = \langle f(x_0), x_0\rangle$.

Let $x \in I_0$ and $\beta_n(x) \overset{\text{df}}{=} F\big(f^n(x)\big) - F\big(f^n(x_0)\big)$. Then $|\beta_n(x)| \leqslant s_n \overset{\text{df}}{=} \operatorname{Var} F|I_n$, where $I_n = \langle f^{n+1}(x_0), f^n(x_0)\rangle$. If $x \in I_k$, then $F\big(f^n(x)\big)$

$= F(f^{n+k}(x_0)) + \beta_{n+k}(x^*)$, where $f^k(x^*) = x$ and $x^* \in I_0$. Let $x \in I_k$. Then we have the following estimation

$$\Big|\sum_{i=0}^{m} F(f^i(x))\Big| \leqslant \Big|\sum_{i=0}^{n} F(f^{i+k}(x_0))\Big| + \sum_{i=0}^{n} \beta_{i+k}(x^*)\ \Big| \leqslant \Big|\sum_{i=k}^{n+k} F(f^i(x_0))\Big| + \sum_{i=k}^{n+k} s_i$$
$$\leqslant \Big|\sum_{i=k}^{n+k} F(f^i(x_0))\Big| + \operatorname{Var} F\,|J.$$

Series (11) converges at x_0. Therefore, there exists a $C > 0$ such that

$$\Big|\sum_{i=0}^{n} F(f^i(x))\Big| \leqslant C \quad \text{for } x \in I_k,\ k, n = 0, 1, \ldots \tag{12}$$

Thus the latter inequality holds for $x \in (a, x_0\rangle$. From hypothesis (i) it follows that there exists a $k > 0$ such that $f^k(x) \in (a, x_0\rangle$ for $x \in J$. The function F is bounded in J, so we can assume that the constant C is such that inequality (12) holds for all $x \in J$. Under the assumptions of our theorem inequality (10) holds in J, too.

Applying the Abel transformation once more we have

(13)
$$\sum_{i=0}^{n} \frac{F(f^i(x))}{G_{i+1}(x)} = \frac{1}{G_{n+1}(x)} \sum_{i=0}^{n} F(f^i(x)) - \sum_{i=0}^{n-1} \left(\frac{1}{G_{i+2}(x)} - \frac{1}{G_{i+1}(x)}\right) \sum_{k=0}^{i} F(f^k(x)).$$

Letting $n \to \infty$ in equality (13) we get

(14)
$$\sum_{i=0}^{\infty} \frac{F(f^i(x))}{G_{i+1}(x)} = \frac{1}{G(x)} \sum_{i=0}^{\infty} F(f^i(x)) - \sum_{i=0}^{\infty} \left(\frac{1}{G_{i+2}(x)} - \frac{1}{G_{i+1}(x)}\right) \sum_{k=0}^{i} F(f^k(x)).$$

The first series on the right-hand side of (14) converges uniformly in I_0 and the second one converges absolutely and uniformly in I_0, since from inequalities (10) and (12) we have

$$\Big|\sum_{i=0}^{\infty} \left(\frac{1}{G_{i+2}(x)} - \frac{1}{G_{i+1}(x)}\right) \sum_{k=0}^{i} F(f^k(x))\Big|$$
$$\leqslant C \sum_{i=0}^{\infty} \Big|\frac{1}{G_{i+2}(x)} - \frac{1}{G_{i+1}(x)}\Big| = C \sum_{i=0}^{\infty} \frac{1}{G_{i+2}(x)} |1 - g(f^{i+1}(x))|$$
$$\leqslant C\bar{M} \sum_{i=0}^{\infty} |1 - g(f^{i+1}(x))| \overset{\text{df}}{=} C\bar{M}U(x).$$

where C and $\bar{M}$ are some positive constants.

Lemma 2 implies that the last series converges uniformly in I_0. Hence series (14) converges uniformly in I_0. Moreover, we have

$$\sup_{I_n} U = \sup_{I_0} U\circ f^n = \sup_{I_0} \sum_{i=n+1}^{\infty} |1-g\circ f^i|.$$

Therefore $\lim_{n\to\infty}\sup_{I_n} U = 0$, thus $\lim_{x\to a^+} U(x) = 0$. By (14) we have

$$\Big|\sum_{i=0}^{\infty} \frac{F(f^i(x))}{G_{i+1}(x)}\Big| \leqslant \frac{1}{G(x)}\Big|\sum_{i=0}^{\infty} F(f^i(x))\Big| + \bar{M}CU(x).$$

From Theorem 5 in [4] it follows that $\lim_{x\to a^+} \sum_{i=0}^{\infty} F(f^i(x)) = 0$. Furthermore, $\lim_{x\to a^+} G(x) = 1$, whence

$$\lim_{x\to a^+} \sum_{i=0}^{\infty} \frac{F(f^i(x))}{G_{i+1}(x)} = 0;$$

thus $\lim_{x\to a^+} \varphi_0(x) = 0$. Series (7) converges uniformly in I_0, and so φ_0 is continuous in I_0.

Now we shall show that $\varphi|I_0 \in BV[I_0]$. Let

$$B(x) = \sum_{i=0}^{\infty}\left(\frac{1}{G_{i+2}(x)} - \frac{1}{G_{i+1}(x)}\right)\sum_{k=0}^{i} F(f^k(x)).$$

From Lemma 4 it follows that there exists an $M > 0$ such that $\mathrm{Var}\, 1/G_n|I_0 \leqslant M/G_n(x_0)$ for $n = 1, 2, \ldots$ Let $s_k = \mathrm{Var}\, F|I_k$ for $k = 0, 1, \ldots$ Then from inequalities (10) and (12) we obtain the following estimation:

$$\begin{aligned}
\mathrm{Var}\, B|I_0 &\leqslant \sum_{i=0}^{\infty} \mathrm{Var}(1/G_{i+2} - 1/G_{i+1})|I_0 \sup_{I_0}\Big|\sum_{k=0}^{i} F\circ f^k\Big| + \\
&\qquad + \sum_{i=}^{\infty} \sup_{I_0} |1/G_{i+2} - 1/G_{i+1}|\,\mathrm{Var}\sum_{k=0}^{i} F\circ f^k|I_0 \\
&\leqslant C\sum_{i=0}^{\infty} \mathrm{Var}\frac{1}{G_{i+2}}(1-g\circ f^{i+1})|I_0 + \sum_{i=0}^{\infty}\sup_{I_0}\frac{1}{G_{i+2}}|1-g\circ f^{i+1}|\sum_{k=0}^{i} s_k \\
&\leqslant C\sum_{i=0}^{\infty} \mathrm{Var}\frac{1}{G_{i+2}}|I_0 \sup_{I_0}|1-g\circ f^{i+1}| + C\sum_{i=0}^{\infty}\sup_{I_0} 1/G_{i+2}\,\mathrm{Var}(1-g\circ f^{i+1})|I_0 + \\
&\qquad + \bar{M}\,\mathrm{Var}\, F|(a, x_0\rangle \sum_{i=0}^{\infty}\sup_{I_0}|1-g\circ f^{i+1}|
\end{aligned}$$

$$\leqslant CM\bar{M}\sum_{i=0}^{\infty}\sup_{I_0}|1-g\circ f^{i+1}|+C\bar{M}\sum_{i=1}^{\infty}\operatorname{Var}g|I_i+$$

$$+\bar{M}\operatorname{Var}F|\langle a,x_0\rangle\sum_{i=0}^{\infty}\sup_{I_0}|1-g\circ f^{i+1}|<\infty,$$

where $C, M, \bar{M}>0$ are some constants. Since product (6) converges absolutely and uniformly in I_0, the last series is convergent. The function g is of bounded variation in J, so $\sum_{i=1}^{\infty}\operatorname{Var}g|I_i<\infty$. Thus $B|I_0\in BV[I_0]$. Theorem 1 in [5], Theorem 5 in [4] and equality (14) imply that $\varphi_0|I_0 \in BV[I_0]$. There exists the limit $\lim_{x\to a^+}\varphi_0(x)=0$. Now, Lemma 1 implies that $\varphi_0\in B_1V[J]$ and φ_0 is continuous in J.

From the proofs of Theorems 1 and 2 the following remark follows directly.

Remark 1. Let $a>-\infty$, $I=\langle a,b\rangle$, $f(a)=a$, $F(a)=0$, $g(a)=1$. If the assumptions of either Theorem 1 or 2 are fulfilled, then equation (1) has exactly a one-parameter family of solutions continuous in I. These solutions are given by formula (5)

2. In this section we assume that case (B) occurs.

Write

$$H_n(x)=G_n(x)\sum_{i=0}^{n-2}\frac{F(f^i(x))}{G_{i+1}(x)},$$

$$F_c(x)=F(x)+c(g(x)-1),$$

$$H_n^c(x)=G_n(x)\sum_{i=0}^{n-2}\frac{F_c(f^i(x))}{G_{i+1}(x)}.$$

Let a function φ fulfil equation (1). Then by induction we get

$$\varphi(f^n(x))=H_n(x)+F(f^{n-1}(x))+G_n(x)\varphi(x). \tag{15}$$

Let $\psi(x)=\varphi(x)-c$. Then ψ fulfils the equation

$$\psi(f(x))=g(x)\psi(x)+F_c(x)$$

and likewise we have

$$\psi(f^n(x))=H_n^c(x)+F_c(f^{n-1}(x))+G_n(x)\psi(x). \tag{16}$$

Since $\varphi(x)-\psi(x)\equiv c$, equation (15) and (16) imply

$$H_n^c(x)=H_n(x)-cg(f^{n-1}(x))+G_n(x)c. \tag{17}$$

We have the following

Remark 2. If assumptions (i), (ii) and (B) are fulfilled, then the following equivalences hold:

1° $\lim_{n\to\infty} H_n^c(x) = 0$ iff $\lim_{n\to\infty} H_n(x) = c$ for $x \in J$.

2° $\lim_{n\to\infty} H_n^c(x) = 0$ almost uniformly in J iff $\lim_{n\to\infty} H_n(x) = c$ almost uniformly in J.

Proof. On account of Theorem 0.4 in [3] and the monotonicity of f it follows that $\lim_{n\to\infty} f^n(x) = a$ uniformly in J. If $\lim_{n\to\infty} G_n(x_0) = 0$ for an $x_0 \in J$, then $\lim_{n\to\infty} G_n(x) = 0$ almost uniformly in J (see Theorem in [5]). Therefore (17) implies equivalences 1° and 2°.

Remark 3. If assumptions (i), (ii) and (B) are fulfilled and equation (1) has a solution φ such that there exists $\lim_{n\to\infty} \varphi(x) = c$, then $\lim_{n\to\infty} H_n(x) = c$ almost uniformly in J.

Proof. Theorem 3 in [5] implies that $\lim_{n\to\infty} G_n(x) = 0$ almost uniformly in J. Thus our assertion results from relation (2).

THEOREM 3. *Let hypotheses* (i), (ii), (iii) *and* (B) *be fulfilled. If, moreover, there exists an* $x_0 \in J$ *such that* $\lim_{n\to\infty} H_n(x_0) = c$ *and*

$$g(f^n(x_0)) < 1 \qquad \text{for } n = 0, 1, 2, \ldots, \tag{18}$$

then equation (1) *has a continuous solution* $B_1V[J]$ *depending on an arbitrary function. More precisely, for any* $y_0 \in J$ *and an arbitrary continuous function* $\varphi_0 \in BV[\langle f(y_0), y_0\rangle]$ *fulfilling the condition*

$$\varphi_0(f(y_0)) = g(y_0)\varphi_0(y_0) + F(y_0),$$

there exists exactly one solution φ *of equation* (1) *in* J *such that* $\varphi(x) = \varphi_0(x)$ *for* $x \in \langle f(y_0), y_0\rangle$. *This solution* φ *is continuous in* J, $\varphi \in B_1V[J]$ *and* $\lim_{x\to a} \varphi(x) = c$.

Proof. By Theorem 3 in [5] it follows that condition (B) implies $\lim_{n\to\infty} G_n(x) = 0$ uniformly in $\langle f(x_0), x_0\rangle$. Remark 2 implies that $\lim_{n\to\infty} H_n^c(x_0) = 0$. We shall show that $\lim_{n\to\infty} H_n^c(x) = 0$ uniformly in $\langle f(x_0), x_0\rangle$.

Let

$$A_{n+1}(x) \overset{\text{df}}{=} G_{n+1}(x_0) \sum_{i=0}^{n-1} \frac{F_c(f^i(x))}{G_{i+1}(x_0)}. \tag{19}$$

First we show that A_n tends uniformly to zero in I_0. Write

$$d_n = \operatorname{Var} F_c | I_n, \qquad \text{where } I_n = \langle f^{n+1}(x_0), f^n(x_0)\rangle.$$

Then the condition $F_c \in BV|J$ implies that $\sum_{n=0}^{\infty} d_n < \infty$. Put $D_n = \sum_{i=0}^{n} d_i$ and $\lim_{n\to\infty} D_n = D$.

Applying the Abel transformation we get

$$\sum_{i=0}^{n-1}\frac{d_i}{G_{i+1}(x_0)}=\sum_{i=0}^{n-1}\frac{1}{G_{i+1}(x_0)}d_i=\frac{D_{n-1}}{G_n(x_0)}-\sum_{i=0}^{n-2}\left(\frac{1}{G_{i+2}(x_0)}-\frac{1}{G_{i+1}(x_0)}\right)D_i,$$

for $n\geqslant 2$; thus

$$G_{n+1}(x_0)\sum_{i=0}^{n-1}\frac{d_i}{G_{i+1}(x_0)}=g\big(f^n(x_0)\big)\left[D_{n-1}-G_n(x_0)\sum_{i=0}^{n-2}\left(\frac{1}{G_{i+2}(x_0)}-\frac{1}{G_{i+1}(x_0)}\right)D_i\right]$$

because $G_n(x)=\prod_{i=0}^{n-1}g\big(f^i(x)\big)$.

From (18) it follows that the sequence $1/G_n(x_0)$ is strictly increasing and $\lim_{n\to\infty}1/G_n(x_0)=\infty$; then by Stolz's theorem (see [2], Vol. 1, p. 53) we have

$$\lim_{n\to\infty}G_n(x_0)\sum_{i=0}^{n-2}\left(\frac{1}{G_{i+2}(x_0)}-\frac{1}{G_{i+1}(x_0)}\right)D_i=\lim_{n\to\infty}\frac{\sum_{i=0}^{n-2}\left(\frac{1}{G_{i+2}(x_0)}-\frac{1}{G_{i+1}(x_0)}\right)D_i}{\frac{1}{G_n(x_0)}}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{1}{G_{n+1}(x_0)}-\frac{1}{G_n(x_0)}\right)D_{n-1}}{\frac{1}{G_{n+1}(x_0)}-\frac{1}{G_n(x_0)}}=\lim_{n\to\infty}D_n=D,$$

then

$$\lim_{n\to\infty}G_{n+1}(x_0)\sum_{i=0}^{n-1}\frac{d_i}{G_{i+1}(x_0)}=0.$$

For $x\in I_0$ we have the inequality

$$G_{n+1}(x_0)\left|\frac{F_c(x)-F_c(x_0)}{G_1(x_0)}+\ldots+\frac{F_c\big(f^{n-1}(x)\big)-F_c\big(f^{n-1}(x_0)\big)}{G_n(x_0)}\right|$$
$$\leqslant G_{n+1}(x_0)\left(\frac{d_0}{G_i(x_0)}+\ldots+\frac{d_{n-1}}{G_n(x_0)}\right);$$

thus

$$\lim_{n\to\infty}G_{n+1}(x_0)\left(\frac{F_c(x)-F_c(x_0)}{G_1(x_0)}+\ldots+\frac{F_c\big(f^{n-1}(x)\big)-F_c\big(f^{n-1}(x_0)\big)}{G_n(x_0)}\right)=0$$

uniformly in I_0. Hence and in view of the condition $\lim_{n\to\infty}H_n^c(x_0)=0$ it follows that

$$\lim_{n\to\infty}G_{n+1}(x_0)\sum_{i=0}^{n-1}\frac{F_c\big(f^i(x)\big)}{G_{i+1}(x_0)}=0$$

uniformly in I_0.

In paper [5] (Lemma 3) we have proved that under our assumptions there exist a uniform limit $\lim_{n\to\infty} G_n(x)/G_n(x_0) = a(x)$ in I_0, $a \in B_1V[J]$ and a constant L_0 such that $0 < 1/L_0 < a(x) < L_0$ for $x \in I_0$. Hence there exist a constant $M > 1$ and an m such that

$$0 < m < \frac{G_n(x)}{G_n(x_0)} < M \quad \text{for } n = 0, 1, 2, \dots \text{ and } x \in I_0. \tag{20}$$

We shall transform H^c_{n+1}, again applying the Abel transformation. Put

$$Z_i(x) = \frac{G_{i+2}(x_0)}{G_{i+2}(x)} - \frac{G_{i+1}(x_0)}{G_{i+1}(x)}.$$

We have

$$\begin{aligned}
H^c_{n+1}(x) &= G_{n+1}(x)\sum_{i=0}^{n-1}\frac{F_c(f^i(x))}{G_{i+1}(x_0)}\,\frac{G_{i+1}(x_0)}{G_{i+1}(x)} \\
&= G_{n+1}(x)\frac{G_n(x_0)}{G_n(x)}\sum_{i=0}^{n-1}\frac{F_c(f^i(x))}{G_{i+1}(x_0)} - \sum_{i=0}^{n-2}Z_i(x)\sum_{k=0}^{i}\frac{F_c(f^k(x))}{G_{k+1}(x_0)} \\
&= \frac{g(f^n(x))}{g(f^n(x_0))}G_{n+1}(x_0)\sum_{i=0}^{n-1}\frac{F_c(f^i(x))}{G_{i+1}(x_0)} - \\
&\qquad - \sum_{i=0}^{n-2}Z_i(x)\frac{G_{n+1}(x)}{G_{i+2}(x_0)}G_{i+2}(x_0)\sum_{k=0}^{i}\frac{F_c(f^k(x))}{G_{k+1}(x_0)} \\
&= \frac{g(f^n(x))}{g(f^n(x_0))}A_{n+1}(x) - \sum_{i=0}^{n-2}Z_i(x)\frac{G_{n+1}(x)}{G_{i+2}(x_0)}A_{i+2}(x),
\end{aligned} \tag{21}$$

where $A_i(x)$ is defined by (19).

For $x \in I_0$ we have the inequality

$$\begin{aligned}
\sum_{i=0}^{\infty}|Z_i(x)| &= \sum_{i=0}^{\infty}\frac{G_{i+1}(x_0)}{G_{i+2}(x)}\left|g(f^{i+1}(x_0)) - g(f^{i+1}(x))\right| \\
&\leqslant \frac{1}{m\inf_J g}\sum_{i=1}^{\infty}\left|g(f^i(x_0)) - g(f^i(x))\right| \leqslant \frac{1}{m\inf_J g}\operatorname{Var} g\,|\,(a, x_0\rangle \leqslant C < \infty,
\end{aligned} \tag{22}$$

where C is a constant greater then unity.

Put $P \stackrel{\text{df}}{=} \sup\{g(f^n(x))/g(f^n(x_0)) \colon x \in I_0,\ n \in N\}$. The function g is bounded in J and $\inf_J g > 0$, so P is finite. Let $\varepsilon > 0$. From the uniform convergence of A_n to zero in I_0 it follows that there exists an N such that

$$(23)\qquad |A_n(x)| < \varepsilon/MC(P+2) < \varepsilon/P+2 \quad \text{for } n \geqslant N \text{ and } x \in I_0,$$

where M, C, P are constants defined previously.

Moreover, there exists an $T > 0$ such that

$$(24)\qquad |A_n(x)| \leqslant T \quad \text{for } n = 1, 2, \ldots\ x \in I_0.$$

From (18) we get $G_{n+1}(x_0)/G_{i+2}(x_0) < 1$ for $i \leqslant n-2$. Therefore

$$\frac{G_{n+1}(x)}{G_{i+2}(x_0)} = \frac{G_{n+1}(x)}{G_{n+1}(x_0)}\,\frac{G_{n+1}(x_0)}{G_{i+2}(x_0)} < \frac{G_{n+1}(x)}{G_{n+1}(x_0)} < M.$$

Consequently by (22) and (23) we have

$$(25)\qquad \left|\sum_{i=N}^{n-2} Z_i(x)\,\frac{G_{n+1}(x)}{G_{i+1}(x_0)}\,A_{i+2}(x)\right| \leqslant \sum_{i=N}^{n-2} |Z_i(x)|M\varepsilon/C(P+2)M < \varepsilon/P+2$$

for $n \geqslant N+2$, $x \in I_0$, since $i \geqslant N$.

If $i \leqslant N-1$, then $1/G_{i+1}(x_0) \leqslant 1/G_N(x_0)$. On account of assumption (B) it follows that there exists an $N_1 > N+2$ such that $G_{n+1}(x_0) < G_N(x_0)\varepsilon/MTC(P+2)$ for $n > N_1$. Consequently, in view of (24), we have for $n > N_1$ and $x \in I_0$

$$(26)\qquad \left|\sum_{i=0}^{N-1} Z_i(x)\,\frac{G_{n+1}(x)}{G_{i+2}(x_0)}\,A_{i+2}(x)\right| \leqslant \sum_{i=0}^{N-1} |Z_i(x)|\,\frac{G_{n+1}(x)}{G_{n+1}(x_0)}\,\frac{G_{n+1}(x_0)}{G_{i+2}(x_0)}\,T$$
$$\leqslant TM\,\frac{G_{n+1}(x_0)}{G_N(x_0)}\sum_{i=0}^{N-1}|Z_i(x)| \leqslant \frac{G_{n+1}(x_0)}{G_N(x_0)}\,TMC < \varepsilon/P+2.$$

Finally, from (21), (23), (24), (25) and (26) we get $|H_n^c(x)| < \varepsilon$ for $n > N_1$ and $x \in I_0$. Consequently $\lim_{n\to\infty} H_n^c(x) = 0$ uniformly in I_0. Hence, in view of Remark 2 and Theorem 7 in [1], it follows that equation (1) has a continuous solution φ in J depending on an arbitrary function, such that there exists a finite limit $\lim_{x\to a^+}\varphi(x)$. Lemma 1 implies that this solution is of finite variation in J.

THEOREM 4. *Let hypotheses* (i), (ii), (iii) *and* (B) *be fulfilled. If, moreover, there exist an* $x_0 \in J$ *such that* (18) *holds and a finite limit* $\lim_{x\to a^+}\frac{F(x)}{1-g(x)} = c$, *then equation* (1) *has a continuous solution* $\varphi \in B_1V[J]$ *depending on an arbitrary function and such that* $\lim_{x\to a^+}\varphi(x) = c$.

Proof. The sequence $1/G_n(x_0)$ is strictly increasing and $\lim_{n\to\infty} 1/G_n(x_0) = \infty$. Then by Stolz's theorem we have

$$\lim_{n\to\infty} H_{n+1}(x_0) = \lim_{n\to\infty} g\big(f^n(x_0)\big) \frac{\sum_{i=0}^{n-1} \frac{F\big(f^i(x_0)\big)}{G_{i+1}(x_0)}}{1/G_n(x_0)}$$

$$= \lim_{n\to\infty} \frac{F\big(f^n(x_0)\big)/G_{n+1}(x_0)}{\frac{1}{G_{n+1}(x_0)} - \frac{1}{G_n(x_0)}} \lim_{n\to\infty} g\big(f^n(x_0)\big) = \lim_{n\to\infty} \frac{F\big(f^n(x_0)\big)}{1-g\big(f^n(x_0)\big)} = c.$$

Our assertion follows now immediately from Theorem 3.

THEOREM 5. *Let hypotheses* (i), (iii), (B) *be fulfilled; suppose that* $g \in BV[J]$, $\inf_J g > 0$, $g(x) < 1$, $F \in B_1 V(J)$, $\lim_{x\to a^+} F(x) = 0$ *and that there exists a finite limit* $\lim_{x\to a^+} \frac{F(x)}{1-g(x)} = c$. *Then there exists a continuous solution* $\varphi \in B_1 V[J]$ *depending on an arbitrary function.*

Proof. It is easy to check that Stolz's theorem remains true also for functional sequences and uniform convergence. We must show that $\lim_{n\to\infty} H_n(x) = c$ uniformly in I_0. This proof is analogous to the proof of Theorem 4. It suffices to observe only that $\lim_{n\to\infty} \frac{F\big(f^n(x)\big)}{1-g\big(f^n(x)\big)} = c$ uniformly in I_0.

The following remark follows directly from the proofs of Theorems 3, 4 and 5.

Remark 4. Let $a > -\infty$, $I = \langle a, b\rangle$, $f(a) = a$, $F(a) = 0$, $g(a) = 1$. If the assumption of one of Theorems 3 or 4 or 5 are fulfilled, then equation (1) has a continuous solutions in I depending on an arbitrary function.

3. Finally we are going to study case (C).

We shall prove

THEOREM 6. *Let assumptions* (i), (ii), (iii) *be fulfilled. Moreover, suppose that there exist an* $x_0 \in J$ *such that* $g\big(f^n(x_0)\big) > 1$ *for* $n = 0, 1, 2, \ldots$ *and* $\lim_{n\to\infty} G_n(x_0) = \infty$, *and a finite limit* $\lim_{x\to a^+} \frac{F(x)}{1-g(x)} = c$. *Then equation* (1) *has exactly one solution* $\varphi \in B_1 V[J]$. *This solution is given by the formula*

$$\varphi(x) = -\sum_{n=0}^{\infty} \frac{F\big(f^n(x)\big)}{G_{n+1}(x)}. \tag{27}$$

Furthermore, φ *is continuous in* J *nad* $\lim_{x\to a^+} \varphi(x) = c$.

Proof. Let $x_0 \in J$, $g(f^n(x_0)) > 1$ for $n = 0, 1, \ldots$ and $\lim_{n\to\infty} G_n(x_0) = \infty$. Then from Lemma 3 and relation (8) it follows that $\lim_{n\to\infty} G_n(x) = \infty$ almost uniformly in J.

We have the identity

$$\frac{g(f^i(x))-1}{G_{i+1}(x)} = \frac{g(f^i(x))}{G_{i+1}(x)} - \frac{1}{G_{i+1}(x)} = \frac{1}{G_i(x)} - \frac{1}{G_{i+1}(x)}.$$

Then

$$\sum_{i=0}^{\infty} \frac{g(f^i(x))-1}{G_{i+1}(x)} \equiv 1 \quad \text{for } x \in J. \tag{28}$$

This series converges uniformly in $I_0 = \langle f(x_0), x_0 \rangle$.

The assumption $\lim_{x\to a+} \frac{F(x)}{1-g(x)} = c$ implies that $\lim_{i\to\infty} \frac{F(f^i(x_0))}{g(f^i(x_0))-1} = -c$. From the conditions $g(f^i(x_0)) > 1$ for $i = 0, 1, \ldots$ follows the absolute convergence of series (28) at x_0. Then the series

$$\sum_{i=0}^{\infty} \frac{F(f^i(x_0))}{G_{i+1}(x_0)} = \sum_{i=0}^{\infty} \frac{g(f^i(x_0))-1}{G_{i+1}(x_0)} \, \frac{F(f^i(x_0))}{g(f^i(x_0))-1}$$

converges absolutely.

We have the identity

$$\frac{F_c(f^i(x_0))}{G_{i+1}(x_0)} = \frac{F(f^i(x_0))}{G_{i+1}(x_0)} + c\,\frac{g(f^i(x_0))-1}{G_{i+1}(x_0)}.$$

Hence and from the absolute convergence of series (27) and (28) at x_0 it follows the absolute convergence of the series

$$\psi(x) = \sum_{i=0}^{\infty} \frac{F_c(f^i(x))}{G_{i+1}(x)} \tag{29}$$

at x_0. We shall show that this series converges absolutely and uniformly in I_0.

Write

$$d_i = \operatorname{Var} F_c | I_i, \quad \text{where } I_i = \langle f^{i+1}(x_0), f^i(x_0) \rangle.$$

Let

$$a_i(x) \overset{\text{df}}{=} F_c(f^i(x)) - F_c(f^i(x_0)) \quad \text{for } x \in I_0,\ i = 0, 1, \ldots$$

Therefore $|a_i(x)| \leqslant d_i$ for $x \in I_0$ and consequently

$$(30)\qquad |F_c(f^i(x))| \leqslant |F_c(f^i(x_0))| + d_i \quad \text{for } x \in I_0,\ i = 0, 1, \ldots$$

Hence we have the inequality

$$\sum_{i=0}^{\infty} \frac{|F_c(f^i(x))|}{G_{i+1}(x_0)} \leqslant \sum_{i=0}^{\infty} \frac{|F_c(f^i(x_0))|}{G_{i+1}(x_0)} + \sum_{i=0}^{\infty} \frac{d_i}{G_{i+1}(x_0)}$$

$$\leqslant \sum_{i=0}^{\infty} \frac{|F_c(f^i(x_0))|}{G_{i+1}(x_0)} + \sum_{i=0}^{\infty} d_i < \infty \quad \text{for } x \in I_0.$$

Therefore the series $\sum_{i=0}^{\infty} \frac{|F_c(f^i(x))|}{G_{i+1}(x_0)}$ converges uniformly in I_0.

From Lemma 3 it follows that there exists an $L > 0$ such that

$$\frac{|F_c(f^i(x))|}{G_{i+1}(x)} \leqslant L \frac{|F_c(f^i(x))|}{G_{i+1}(x_0)} \quad \text{for } x \in I_0.$$

Consequently series (29) converges absolutely and uniformly in I_0.

Similarly as for series (29) one can show the absolute and uniform convergence of series (27) in I_0. From relation (8) it follows that series (27) and (29) converge absolutely in J.

It is easy to see that the function φ given by formula (27) fulfils equation (1) in J. Moreover, $\varphi | I_0$ is continuous, whence by Theorem 2.1 in [3] it follows that φ is continuous in J. We shall show that there exists the limit $\lim_{x \to a^+} \varphi(x) = c$.

From the definitions of φ and ψ and by relation (28) it follows that $\psi(x) = -\varphi(x) + c$. By (29) and (8) we have the following equality:

$$(31)\qquad \psi(f^k(x)) = G_k(x) \sum_{i=k}^{\infty} \frac{F_c(f^i(x))}{G_{i+1}(x)}, \quad k = 0, 1, \ldots,\ x \in J.$$

By Lemma 3 and by (31), in view of the absolute convergence of series (29), we have

$$|\psi(f^k(x))| \leqslant L^2 G_k(x_0) \sum_{i=k}^{\infty} \frac{|F_c(f^i(x_0))|}{G_{i+1}(x_0)} \quad \text{for } x \in I_0.$$

Hence on account of inequality (30) we get

$$|\psi(f^k(x))| \leqslant L^2 G_k(x_0) \sum_{i=k}^{\infty} \frac{|F_c(f^i(x_0))|}{G_{i+1}(x_0)} + L^2 \sum_{i=k}^{\infty} \frac{d_i G_k(x_0)}{G_{i+1}(x_0)} \quad \text{for } x \in I_0.$$

We have $G_k(x_0) \leqslant G_i(x_0)$ for $i \geqslant k$. Therefore

$$\sup_{I_k} |\psi| \leqslant L^2 G_k(x_0) \sum_{i=k}^{\infty} \frac{|F_c(f^i(x_0))|}{G_{i+1}(x_0)} + L^2 \sum_{i=k}^{\infty} d_i. \tag{32}$$

The assumption $\lim_{x \to a^+} \frac{F(x)}{1-g(x)} = c$ implies that $\lim_{x \to a^+} \frac{F_c(x)}{1-g(x)} = 0$. Then for any $\varepsilon > 0$ there exists an N_1 such that

$$\frac{|F_c(f^i(x_0))|}{g(f^i(x_0))-1} < \varepsilon/2L^2 \quad \text{for } i > N_1.$$

Therefore for $k > N_1$ we have

$$G_k(x_0) \sum_{i=k}^{\infty} \frac{|F_c(f^i(x_0))|}{G_{i+1}(x_0)} = G_k(x_0) \sum_{i=k}^{\infty} \frac{g(f^i(x_0))-1}{G_{i+1}(x_0)} \, \frac{F_c(f^i(x_0))}{g(f^i(x_0))-1}$$

$$\leqslant \frac{\varepsilon}{2L^2} G_k(x_0) \sum_{i=k}^{\infty} \frac{g(f^i(x_0))-1}{G_{i+1}(x_0)} = \frac{\varepsilon}{2L^2} G_k(x_0) \frac{1}{G_k(x_0)} = \varepsilon/2L^2.$$

Since $\sum_{i=0}^{\infty} d_i < \infty$, there exists an $N > N_1$ such that

$$L^2 \sum_{i=k}^{\infty} d_i < \varepsilon/2 \quad \text{for } k > N.$$

Hence by (32) we have $\sup_{I_k} |\psi| < \varepsilon$ for $k > N$ and consequently $\lim_{x \to a^+} \psi(x) = 0$. Thus $\lim_{x \to a^+} \varphi(x) = c$.

Now we shall show that $\psi | I_0 \in BV[I_0]$, where ψ is given by formula (29). From Lemma 4 it follows that there exists an $M > 0$ such that

$$\operatorname{Var} 1/G_n | I_0 \leqslant M/G_n(x_0) \quad \text{for } n = 1, 2, \ldots,$$

where $I_0 = \langle f(x_0), x_0 \rangle$.

Therefore we have

$$\operatorname{Var} \frac{F_c \circ f^i}{G_{i+1}} | I_0 \leqslant \operatorname{Var} 1/G_{i+1} | I_0 \sup_{I_0} |F_c \circ f^i| + \sup_{I_0} 1/G_{i+1} \operatorname{Var} F_c \circ f^i | I_0$$

$$\leqslant \frac{M}{G_{i+1}(x_0)} \sup_{I_0} |F_c \circ f^i| + \frac{d_i}{\inf_{I_0} G_{i+1}}.$$

From this, in view of inequalities (30) and (3), we get

$$\operatorname{Var} \frac{F_c \circ f^i}{G_{i+1}} | I_0 \leqslant \frac{M}{G_{i+1}(x_0)} \big(|F_c(f^i(x_0))| + d_i \big) + \frac{d_i L}{G_{i+1}(x_0)}$$

$$\leqslant \frac{M |F_c(f^i(x_0))|}{G_{i+1}(x_0)} + (L+M) d_i,$$

since $G_i(x_0) > 1$. Hence

$$\operatorname{Var}\psi|I_0 = \operatorname{Var}\sum_{i=0}^{\infty}\frac{F_c\circ f^i}{G_{i+1}}\Big|I_0 \leqslant \sum_{i=0}^{\infty}\operatorname{Var}\frac{F_c\circ f^i}{G_{i+1}}\Big|I_0$$
$$\leqslant M\sum_{i=0}^{\infty}\frac{|F_c(f^i(x_0))|}{G_{i+1}(x_0)}+(L+M)\sum_{i=0}^{\infty}d_i.$$

Since $\varphi(x) = -\psi(x)+c$, $\operatorname{Var}\varphi|I_0 < \infty$. Hence by Lemma 1 it follows that $\varphi \in B_1V[J]$.

The uniqueness of solutions follows directly from Theorem 2.6 in [3].

THEOREM 7. *Let hypotheses* (i), (ii), (iii) *be fulfilled and suppose that case* (C) *occurs. There exist a* $T > 0$ *such that* $G_n(f^k(x_0)) > T$ *for* $k, n = 1, 2, \ldots$ *and a* c *such that the series* $\sum_{i=0}^{\infty} F_c(f^i(x))$ *converges for an* $x \in J$. *Then equation* (1) *has exactly one solution* $\varphi \in B_1V[J]$. *This solution is given by the formula*

$$\varphi(x) = c-\sum_{i=0}^{\infty}\frac{F_c(f^i(x))}{G_{i+1}(x)}, \tag{33}$$

is continuous in J *and* $\lim_{x\to a^+}\varphi(x) = c$.

Proof. Under our assumptions inequality (3) is true. It implies

$$\frac{1}{L}G_n(f^k(x_0)) \leqslant G_n(x) \quad \text{for } x \in \langle f^{k+1}(x_0), f^k(x_0)\rangle,\ n, k = 0, 1, \ldots$$

Therefore

$$\frac{T}{L} \leqslant G_n(x) \quad \text{for } x \in (a, x_0\rangle.$$

Hence by relation (8) it follows that there exists a $D > 0$ such that

$$D \leqslant G_n(x) \quad \text{for } x \in J,\ n = 1, 2, \ldots$$

From Lemma 2 it follows that the series $\sum_{i=0}^{\infty}|F_c(f^i(x))|$ converges almost uniformly in J. We have the inequality

$$\Big|\sum_{i=0}^{\infty}\frac{F_c(f^i(x))}{G_{i+1}(x)}\Big| \leqslant \sum_{i=0}^{\infty}\frac{|F_c(f^i(x))|}{G_{i+1}(x)} \leqslant \frac{1}{D}\sum_{i=0}^{\infty}|F_c(f^i(x))| \overset{\mathrm{df}}{=} \gamma(x).$$

Therefore series (33) converges absolutely and almost uniformly in J. Moreover, we have the inequality $|\varphi(x)-c| \leqslant \gamma(x)$. From Theorem 5 in [4] it follows that $\lim_{x\to a^+}\gamma(x) = 0$. Thus $\lim_{x\to a^+}\varphi(x) = c$ and φ is continuous in J.

It is easy to verify that the function φ given by formula (33) fulfils equation (1).

The proof of the fact that $\varphi | I_0 \in BV[I_0]$ is analogous to the proof of this fact in Theorem 6. Thus $\varphi \in B_1 V[J]$.

The uniqueness follows directly from Theorem 2 in [1].

We have the following

Remark 5. Let $a > -\infty$, $I = \langle a, b \rangle$, $f(a) = a$, $F(a) = 0$, $g(a) = 1$. If the assumptions of Theorem 6 (Theorem 7) are fulfilled, then equation (1) has exactly one solution φ continuous in I. This solution φ is given by formula (27) ((33)).

From the proofs of Theorems 1–7 the following remark follows directly.

Remark 6. If in Theorems 1–7 hypothesis (iii) does not hold, then the assertions remain true with the only exception that the solution φ need not be continuous in J.

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