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An immersion of a differential space in a Cartesian space

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Abstract. For an arbitrary set A, the set $\boldsymbol{R}^A$ of all real functions defined on A can be considered in a natural way as a differential space. The concept of a differential space has been introduced by R. Sikorski [2] and, independently, by S. Mac Lane [1]. In the present paper we prove that an arbitrary differential space can be induced from the natural differential space of the set $\boldsymbol{R}^A$, and in the case when the differential structure separates points it can be immersed in $\boldsymbol{R}^A$.

1. The natural structure of the set $\boldsymbol{R}^A$. For every $a \in A$ and for every function x of $\boldsymbol{R}^A$ we set $\hat{a}(x) = x(a)$. Then we have the projection

$$\hat{a}\colon \boldsymbol{R}^A \to \boldsymbol{R}. \tag{1}$$

The set $\boldsymbol{R}$ of all reals has the natural differential structure $C^\infty(\boldsymbol{R})$ consisting of all infinitely many differentiable real functions on $\boldsymbol{R}$. The smallest differential structure $\boldsymbol{R}(A)$ on the set $\boldsymbol{R}^A$, for which all projections of the form (1), $a \in A$, are smooth will be called the *natural differential structure of the set* $\boldsymbol{R}^A$. For any set C of real functions defined on a set M, sc C is the set of all real functions of the form

$$\omega\big(a_1(\cdot), \ldots, a_s(\cdot)\big),$$

where $a_1, \ldots, a_s$ are elements of C, ω is any infinitely differentiable real function on $\boldsymbol{R}^s$, s is an arbitrary positive integer. In general, by C_M we denote the set of all functions $\beta\colon M \to \boldsymbol{R}$ such that for every p of M there exist $a \in C$ and a set U such that $\beta|U = a|U$, U is open in the weakest topology τ_C on the set M for which all functions of C are continuous. For an arbitrary mapping $f\colon M \to N$ we define the mapping $f^*\colon \boldsymbol{R}^N \to \boldsymbol{R}^M$ by the formula: $f^*(\beta) = \beta \circ f$ for $\beta \in \boldsymbol{R}^N$ (see [4]).

We have $\hat{a}\colon \boldsymbol{R}^A \to \boldsymbol{R}$. Then

$$\hat{a}^*\colon \boldsymbol{R}^{\boldsymbol{R}} \to \boldsymbol{R}^{(\boldsymbol{R}^A)}$$

is the mapping defined by $\hat{a}^*(\gamma) = \gamma \circ \hat{a}$, where $\gamma\colon \boldsymbol{R}^A \to \boldsymbol{R}$. We mean by $\hat{a}^*[C^\infty(\boldsymbol{R})]$ the set of all functions $\gamma \circ \hat{a}$, where $\gamma \in C^\infty(\boldsymbol{R})$. Then we have

$$(2) \qquad \boldsymbol{R}(A) = \big(\mathrm{sc}\bigcup_{a\in A} \hat{a}^*[C^\infty(\boldsymbol{R})]\big)_{\boldsymbol{R}^A}.$$

We will prove

1.1. *The structure $\boldsymbol{R}(A)$ defined by* (2) *is the unique differential structure on the set $\boldsymbol{R}^A$ such that for every differential space (M, C) and for every function $f\colon M \to \boldsymbol{R}^A$ the mapping*

$$(3) \qquad f\colon (M, C) \to \big(\boldsymbol{R}^A, \boldsymbol{R}(A)\big)$$

is smooth iff all mappings

$$(4) \qquad \hat{a}\circ f\colon (M, C) \to \big(\boldsymbol{R}, C^\infty(\boldsymbol{R})\big)$$

are smooth for $a \in A$.

Proof. Assume that we have smooth mappings (4), $a \in A$. Let $\beta \in \bigcup_{a\in A} \hat{a}^*[C^\infty(\boldsymbol{R})]$. Then $\beta \in \hat{a}^*[C^\infty(\boldsymbol{R})]$ for some $a \in A$. Therefore, $\beta = \gamma\circ\hat{a}$, where $\gamma \in C^\infty(\boldsymbol{R})$. Hence it follows that $\beta\circ f = \gamma\circ(\hat{a}\circ f) \in C$. Thus, the mapping (3) is smooth (see [4]).

Now, suppose that D is a differential structure on $\boldsymbol{R}^A$ such that for any $f\colon M \to \boldsymbol{R}^A$ the mapping

$$(5) \qquad f\colon (M, C) \to (\boldsymbol{R}^A, D)$$

is smooth iff all mappings (4) are smooth for $a \in A$. We have to prove that $D = \boldsymbol{R}(A)$. The mapping (5) is smooth iff the mapping (3) is smooth. Setting $f = \mathrm{id}$ and $M = \boldsymbol{R}^A$ we have the smooth mapping

$$\mathrm{id}\colon \big(\boldsymbol{R}^A, \boldsymbol{R}(A)\big) \to \big(\boldsymbol{R}^A, \boldsymbol{R}(A)\big).$$

Hence we get the smooth mapping

$$\mathrm{id}\colon \big(\boldsymbol{R}^A, \boldsymbol{R}(A)\big) \to (\boldsymbol{R}^A, D).$$

Then $D \subset \boldsymbol{R}(A)$. Similarly, the smoothness of $\mathrm{id}\colon (\boldsymbol{R}^A, D) \to (\boldsymbol{R}^A, D)$ yields that the mapping

$$\mathrm{id}\colon (\boldsymbol{R}^A, D) \to \big(\boldsymbol{R}^A, \boldsymbol{R}(A)\big)$$

is smooth. Consequently, we have the inclusion $\boldsymbol{R}(A) \subset D$.

1.2. *If* $\mathrm{card}\,A = \mathrm{card}\,B$, *then the differential spaces $\big(\boldsymbol{R}^A, \boldsymbol{R}(A)\big)$ and $\big(\boldsymbol{R}^B, \boldsymbol{R}(B)\big)$ are diffeomorphic. More exactly, if φ is a one-to-one mapping A onto B, then the mapping $\varphi^*\colon \boldsymbol{R}^B \to \boldsymbol{R}^A$ gives a diffeomorphism $\big(\boldsymbol{R}^B, \boldsymbol{R}(B)\big)$ onto $\big(\boldsymbol{R}^A, \boldsymbol{R}(A)\big)$.*

Proof. Suppose $\varphi\colon A \to B$ is one-to-one. Then $\varphi^*\colon \boldsymbol{R}^B \to \boldsymbol{R}^A$ is one-to-one. Let $\beta \in \hat{a}^*[C^\infty(\boldsymbol{R})]$, where $a \in A$. Thus $\beta = \hat{a}^*(\gamma)$, $\gamma \in C^\infty(\boldsymbol{R})$. Hence it follows $\beta\circ\varphi^* = \hat{a}^*(\gamma)\circ\varphi^* = \gamma\circ\hat{a}\circ\varphi^* = \gamma\circ\hat{b} = \hat{b}^*(\gamma) \in \hat{b}^*[C^\infty(\boldsymbol{R})]$, where $b = \varphi(a)$, $\hat{b}(y) = y(b)$ for $y \in \boldsymbol{R}^B$. Therefore, $\beta\circ\varphi^* \in \boldsymbol{R}(B)$ when

$\beta \in \bigcup_{a\in A} \hat{a}^*[C^\infty(\boldsymbol{R})]$. Then we have the smooth mapping

$$\varphi^*\colon\ (\boldsymbol{R}^B, \boldsymbol{R}(B)) \to (\boldsymbol{R}^A, \boldsymbol{R}(A)). \tag{6}$$

Similarly, we prove that the mapping inverse to (6) is smooth.

Now, for any cardinal n we define the n-th power of $\boldsymbol{R}$, $\boldsymbol{R}^n$, as a differential space of the form $(\boldsymbol{R}^A, \boldsymbol{R}(A))$, where A is the set of all ordinals less than ν, ν being the smallest ordinal of cardinality n. From 1.2 it follows that every differential space $(\boldsymbol{R}^B, \boldsymbol{R}(B))$, where $\operatorname{card} B = n$, is diffeomorphic to $\boldsymbol{R}^n$.

2. The inducing of a differential space from $\boldsymbol{R}^n$. Let (M, C) be any differential space and let $n = \operatorname{card} C$. For every $p \in M$, $i(p)$ denotes the function defined by the formula

$$i(p)(a) = a(p) \quad \text{for } a \in C. \tag{7}$$

We have the mapping

$$i\colon\ M \to \boldsymbol{R}^C. \tag{8}$$

We will make use of the concept of the differential space induced by a mapping, which has introduced in [4]. First we prove the lemma:

2.1. *The differential space induced from $(\boldsymbol{R}^C, \boldsymbol{R}(C))$ by the mapping* (8) *coincides with* (M, C).

Proof. Let us denote by $\tilde{C}$ the differential structure induced from $(\boldsymbol{R}^C, \boldsymbol{R}(C))$ by (8). We have (see [4]) $\tilde{C} = (\mathrm{sc}\, i^*[\boldsymbol{R}(C)])_M$. We will prove that $\tilde{C} = C$. It is easy to see that $C \subset \tilde{C}$ and that the weakest topology $\tau_{\tilde{C}}$ for which all functions of $\tilde{C}$ are continuous consists of the counter-images of the members of the topology $\tau_{\boldsymbol{R}(C)}$ under mapping (8). In other words, a set is open in $\tau_{\tilde{C}}$ iff it is of the form $i^{-1}[U]$, where U is open in $\tau_{\boldsymbol{R}(C)}$. We set

$$C_0 = \bigcup_{a\in C} \hat{a}^*[C^\infty(\boldsymbol{R})].$$

Then we have (see [4]) $\tau_{\boldsymbol{R}(C)} = \tau_{C_0}$. We will prove that the topology τ_{C_0} coincides with the Tichonov topology of the product

$$\mathsf{X}_{a\in C} \boldsymbol{R}_a, \tag{9}$$

where, for each $a \in C$, $\boldsymbol{R}_a$ denotes the set $\boldsymbol{R}$ with the natural topology. Indeed, for any $a \in C$, any reals $a < b$ and for every real function $\beta \in \hat{a}^*[C^\infty(\boldsymbol{R})]$ we have

$$\beta^{-1}[(a; b)] = \hat{a}^{-1}[\gamma^{-1}[(a; b)]],$$

where $\gamma \in C^\infty(\boldsymbol{R})$. A subbase of the topology τ_{C_0} is then contained in the set of all sets of the form $\hat{a}^{-1}[H]$, where H is an open subset of $\boldsymbol{R}$, a is an arbitrary function of C. Thus, this subbase is contained in the Tichonov topology of product (9). Now, let us take any set of the form $\hat{a}^{-1}[(a; b)]$, where $a < b$ and $a \in C$. Then we have

$$\hat{a}^{-1}[(a; b)] = \big(\hat{a}^*(\mathrm{id}_R)\big)^{-1}[(a; b)],$$

$\hat{a}^*(\mathrm{id}_R) \in C_0$. Therefore, the subbase of the topology of the product (9) is contained in τ_{C_0}. Consequently these topologies coincide.

To prove the inclusion $\tilde{C} \subset C$, let us take any $a \in i^*[\boldsymbol{R}(C)]$. Then $a = \gamma \circ i$, $\gamma \in \boldsymbol{R}(C)$. Let $p \in M$. Then $i(p) \in \boldsymbol{R}^C$. There exists $U \in \tau_{C_0}$ such that $i(p) \in U$ and $\gamma | U = \gamma_0 | U$, where $\gamma_0 \in \mathrm{sc}\, C_0$. Therefore, for some $\gamma_1, \ldots, \gamma_s \in C_0$, $\gamma_0 = \omega\big(\gamma_1(\cdot), \ldots, \gamma_s(\cdot)\big)$, where $\omega \in C^\infty(\boldsymbol{R}^s)$. There exist $a_1, \ldots, a_s \in C$ such that $\gamma_j \in \hat{a}_j^*[C^\infty(\boldsymbol{R})]$, $j = 1, \ldots, s$.

Thus, $\gamma_j = \hat{a}_j^*(\eta_j) = \eta_j \circ \hat{a}_j$, where $\eta_j \in C^\infty(\boldsymbol{R})$. For any $x \in \boldsymbol{R}^C$ we get

$$\gamma_0(x) = \omega\Big(\eta_1\big(\hat{a}_1(x)\big), \ldots, \eta_s\big(\hat{a}_s(x)\big)\Big) = \theta\big(x(a_1), \ldots, x(a_s)\big),$$

where $\theta(u^1, \ldots, u^s) = \omega\big(\eta_1(u^1), \ldots, \eta_s(u^s)\big)$ for $(u^1, \ldots, u^s) \in \boldsymbol{R}^s$. Thus we have

$$\begin{aligned} a(q) &= i^*(\gamma)(q) = (\gamma \circ i)(q) = \gamma\big(i(q)\big) = \gamma_0\big(i(q)\big) \\ &= \theta\big(i(q)(a_1), \ldots, i(q)(a_s)\big) = \theta\big(a_1(q), \ldots, a_s(q)\big) \end{aligned}$$

for $q \in i^{-1}[U]$.

The set U being open in the topology of the product (9) is of the form $\mathsf{X}_{\delta \in C} U_\delta$, where the sets U_δ are open in $\boldsymbol{R}$ and there are only finitely many $\delta \in C$ for which $U_\delta \neq \boldsymbol{R}$. Call then $\delta_1, \ldots, \delta_r$. Then we get

$$p \in i^{-1}[U] = \bigcap_{h=1}^{r} \delta_h^{-1}[U_{\delta_h}] \in \tau_C.$$

The function $\theta\big(a_1(\cdot), \ldots, a_s(\cdot)\big)$ belongs to C. Hence it follows that $a \in C_M = C$.

As a corollary we get the following theorem.

2.2. *Every differential space has a differential structure induced from $\boldsymbol{R}^n$, n being a suitable cardinal, by some mapping. If the topology of the differential space is Hausdorff, we may require this mapping to be one-to-one.*

References

[1] S. MacLane, *Differentiable spaces*, Notes for Geometrical Mechanics, Winter 1970, p. 1–9.

[2] R. Sikorski, *Abstract covariant derivative*, Colloq. Math. 18 (1967), p. 251–272.

[3] — *Wstęp do geometrii różniczkowej* (Polish), Warszawa 1972.

[4] W. Waliszewski, *Regular and coregular mappings of differential spaces*, Ann. Polon. Math. 30 (1975), p. 263–281.

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